HL

Now with interactive e-book

WORKED SOLUTIONS

PEARSON BACCALAUREATE

HIGHER LEVEL

Mathematics

2012 edition

DEVELOPED SPECIFICALLY FOR THE

IB DIPLOMA

IBRAHIM WAZIR • TIM GARRY

PETER ASHBOURNE • PAUL BARCLAY • PETER FLYNN • KEVIN FREDERICK • MIKE WAKEFORD

ALWAYS LEARNING

PEARSON

Chapter 1

Exercise 1.1

1 $\mathbb{Z} \subset \mathbb{Q}$, because every integer can be written as a fraction with a denominator of 1 and there are fractions, such as $\frac{1}{3}, \frac{2}{13}, \frac{23}{225}$, and so on, that are not integers.

2 $\mathbb{N} \subset \mathbb{Q}$, because every natural number is also an integer.

3 $\mathbb{C} \supset \mathbb{R}$, because every real number can be written as a complex number with the imaginary part equal to zero.

4 $\mathbb{Z} \supset \mathbb{N}$ or $\mathbb{N} \subset \mathbb{Z}$, because every natural number is an integer and there are integers, such as -2, -5, and so on, that are not natural.

5 $\mathbb{Z} \supset \mathbb{Z}^+$ or $\mathbb{Z}^+ \subset \mathbb{Z}$, because the set of positive integers does not include zero and negative integers.

6 $\mathbb{N} \subset \mathbb{R}$, because every natural number is rational, and therefore real. (Real numbers consist of rational and irrational numbers.)

7 The decimal has a period of 2 so we need to multiply it by 10^2 and then subtract the equations.

$$\left.\begin{aligned} x &= 2.151\,515... \\ 100x &= 215.151\,515... \end{aligned}\right\} \Rightarrow 99x = 213 \Rightarrow x = \frac{213}{99} = \frac{71}{33}$$

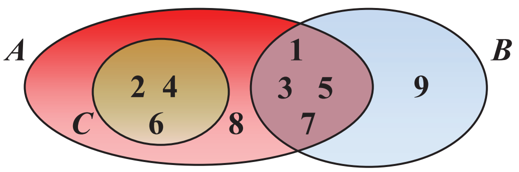
8 The whole decimal part is not periodic so we need to multiply by 100 first to obtain a periodic decimal. Then we need to multiply by 10 to obtain the same decimal part before subtracting the equations.

$$\left.\begin{aligned} 100x &= 1191.3333... \\ 1000x &= 11\,913.3333... \end{aligned}\right\} \Rightarrow 900x = 10\,722 \Rightarrow x = \frac{10\,722}{900} = \frac{1787}{150}$$

9 The decimal has a period of 6 so we need to multiply it by 10^6 and then subtract the equations.

$$\left.\begin{aligned} x &= 8.714\,285\,714\,285... \\ 1\,000\,000x &= 8\,714\,285.714\,285 \end{aligned}\right\} \Rightarrow 999\,999x = 8\,714\,277 \Rightarrow x = \frac{8\,714\,277}{999\,999} = \frac{61}{7}$$

The following Venn diagram represents the sets described and will help us to find the solutions to questions 10–15.



10 $A \cap B = \{1, 3, 5, 7\}$

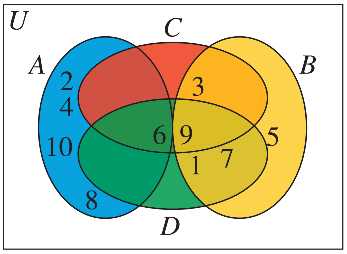
11 $A \cup B = \{1, 2, 3, 4, 5, 6, 7, 8, 9\}$

12 $B \cap C = \varnothing$

13 $A \cup C = \{1, 2, 3, 4, 5, 6, 7, 8\} = A$

14 $A \cap C = \{2, 4, 6\} = C$

15 $A \cup B \cup C = \{1, 2, 3, 4, 5, 6, 7, 8, 9\} = A \cup B$



16 **a)** A: all the even positive integers less than 11. B: all the odd positive integers less than 10. C: all the positive multiples of 3 less than 10.

The Venn diagram right represents the sets described above and will help us to find the solutions.

b) **i)** $A \cap B = \varnothing$

ii) $A \cup B = \{1, 2, 3, 4, 5, 6, 7, 8, 9, 10\} = U$ **iii)** $A' = \{1, 3, 5, 7, 9\} = B$

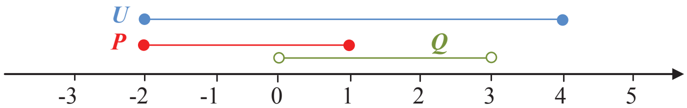
iv) $B' = \{2, 4, 6, 8, 10\} = A$ **v)** $A \cap D = \{6\}$

vi) $B \cap C = \{3, 9\}$ **vii)** $B \cap C \cap D = \{9\}$

viii) $(C \cup D)' = \{2, 4, 5, 8, 10\}$ **ix)** $A \cap (C \cap D)' = \{2, 4, 8, 10\}$

c) On the diagram above, ignore set B and the universal set U.

17 We can represent the sets on the following number line.



a) $P \cap Q =]0, 1]$ **b)** $P \cup Q = [-2, 3[$

c) $P' =]1, 4]$ **d)** $Q' = [-2, 0] \cup [3, 4]$

e) $(P \cup Q)' = [3, 4]$ **f)** $(P \cap Q)' = [-2, 0] \cup]1, 4]$

18 $\left(\frac{x}{5} > -2\right)\Big/ \times 5 \Rightarrow x > -10$

19 $(3 + 4x \leqslant -9)/-3 \Rightarrow (4x \leqslant -12)/\div 4 \Rightarrow x \leqslant -3$

20 $(7 - 3x < -3)/+3 + 3x \Rightarrow (10 < 3x)\Big/ \times \frac{1}{3} \Rightarrow x > \frac{10}{3}$

Note: When we swap the sides in an inequality, we need to change the sign of the inequality too.

21 $6(2 - x) < 2x + 15 \Rightarrow 12 - 6x < 2x + 15 \Rightarrow -3 < 8x \Rightarrow x > -\frac{3}{8}$

22 $(9 \leqslant 8x - 3 < 11)/+3 \Rightarrow (12 \leqslant 8x < 14)/\div 8 \Rightarrow \frac{3}{2} \leqslant x < \frac{7}{4}$

23 $(-4 \leqslant 1 - 5x \leqslant 16)/-1 \Rightarrow (-5 \leqslant -5x \leqslant 15)\Big/ \times \left(-\frac{1}{5}\right) \Rightarrow -3 \leqslant x \leqslant 1$

Note: When we multiply an inequality by a negative number, the inequality symbol is reversed.

24 False. Any $x < 0$; for example, $2 \times (-1) \geqslant -1 \Rightarrow -2 \geqslant -1$, which is false.

25 True

26 False. Any $x \leqslant 0$; for example, $(-2)^3 + (-2) > (-2)^3 \Rightarrow -10 > -8$, which is false.

27 False. Any $0 < x < 1$; for example, $\left(\frac{1}{2}\right)^2 \geqslant \frac{1}{2} \Rightarrow \frac{1}{4} \geqslant \frac{1}{2}$, which is false.

28 True

29 True

30 False. Any $x < 0$; for example, $-(-3) \leqslant 0 \Rightarrow 3 \leqslant 0$, which is false.

31 False. $\frac{1}{x} - x \leqslant 0 \Rightarrow \frac{1-x^2}{x} \leqslant 0$, any $x < -1$ or $0 < x < 1$; for example, $\frac{1}{-4} \leqslant -4 \Rightarrow -\frac{1}{4} \leqslant -4$, which is false, or $\frac{1}{\frac{1}{3}} \leqslant \frac{1}{3} \Rightarrow 3 \leqslant \frac{1}{3}$, which is also false.

32 $\left|-7 - \frac{15}{2}\right| = \frac{29}{2}$

33 $|-2-(-11)| = 9$

34 $|27.4 - 19.2| = 8.2$

35 $|\pi - 3| = \pi - 3$

36 $\left|-3\pi - \frac{2\pi}{3}\right| = \frac{11\pi}{3}$

37 $\left|\frac{61}{7} - \left(-\frac{23}{11}\right)\right| = \frac{832}{77}$

38 $-5 \leqslant x \leqslant 3$, closed and bounded

39 $-10 < x \leqslant -2$, half-open from the left and bounded

40 $x \geqslant 1$, half-open and unbounded from the right

41 $x \leqslant 4$, half-open from the left and unbounded

42 $0 \leqslant x < 2\pi$, half-open from the right and bounded

43 $a \leqslant x \leqslant b$, closed and bounded

44 $]-3, \infty[$

45 $]-4, 6[$

46 $]-\infty, 10]$

47 $[0, 12[$

48 $]-\infty, \pi[$

49 $[-3, 3]$

50 $x \geqslant 6$, $[6, \infty[$

51 $4 \leqslant x < 10$, $[4, 10[$

52 $x < 0$, $]-\infty, 0[$

53 $0 < x < 25$, $]0, 25[$

54 $|x| < 6$

55 $|x| \geqslant 4$

56 $|x| \leqslant \pi$

57 $|x| > 1$

58 $|-13| = 13$

59 $|7 - 11| = |-4| = 4$

60 $-5|-5| = -5 \times 5 = -25$

61 $|-3| - |-8| = 3 - 8 = -5$

62 $|\sqrt{3} - 3| = 3 - \sqrt{3}$

Note: $\sqrt{3} - 3 < 0$ and therefore its absolute value is the opposite number.

63 $\frac{-1}{|-1|} = \frac{-1}{1} = -1$

64 $x = 5$ or $x = -5$

65
$$\left.\begin{array}{c} x-3=4 \\ \text{or} \\ x-3=-4 \end{array}\right\} \Rightarrow \begin{array}{c} x=7 \\ \text{or} \\ x=-1 \end{array}$$

66 $|6-x|=10 \Rightarrow |x-6|=10 \Rightarrow \left.\begin{cases} x-6=10 \\ \text{or} \\ x-6=-10 \end{cases}\right\} \Rightarrow \begin{matrix} x=16 \\ \text{or} \\ x=-4 \end{matrix}$

Note: It is always easier to manipulate with the opposite expression if the coefficient of the variable is negative.

67 $x \in \varnothing$; it is impossible since the absolute value cannot be a negative number.

68 $|3x+5|=1 \Rightarrow \left.\begin{cases} 3x+5=1 \\ \text{or} \\ 3x+5=-1 \end{cases}\right\} \Rightarrow \left.\begin{matrix} 3x=-4 \\ \text{or} \\ 3x=-6 \end{matrix}\right\} \Rightarrow \begin{matrix} x=-\frac{4}{3} \\ \text{or} \\ x=-2 \end{matrix}$

69 The following problem can be approached in two different ways. One way is to immediately target the standard form of $|ax+b|=c$, and the second is to simplify the expression as much as possible by using the properties of absolute value.

Method I: $\frac{1}{2}\left|x-\frac{2}{3}\right|=5 \Big/ \times 2 \Rightarrow \left|x-\frac{2}{3}\right|=10 \Rightarrow \left.\begin{cases} x-\frac{2}{3}=10 \\ \text{or} \\ x-\frac{2}{3}=-10 \end{cases}\right\} \Rightarrow \begin{matrix} x=\frac{32}{3} \\ \text{or} \\ x=-\frac{28}{3} \end{matrix}$

Method II: $\frac{1}{2}\left|x-\frac{2}{3}\right|=5 \Big/ \times 6 \Rightarrow |3x-2|=30 \Rightarrow \left.\begin{cases} 3x-2=30 \\ \text{or} \\ 3x-2=-30 \end{cases}\right\} \Rightarrow \begin{matrix} x=\frac{32}{3} \\ \text{or} \\ x=-\frac{28}{3} \end{matrix}$

70 $\left|\frac{6-2x}{3}\right|+\frac{2}{5}=8 \Big/ -\frac{2}{5} \Rightarrow \left|\frac{6-2x}{3}\right|=\frac{38}{5} \Big/ \times\frac{15}{2} \Rightarrow |5x-15|=57 \Rightarrow \left.\begin{cases} 5x-15=57 \\ \text{or} \\ 5x-15=-57 \end{cases}\right\} \Rightarrow \begin{matrix} x=\frac{72}{5} \\ \text{or} \\ x=-\frac{42}{5} \end{matrix}$

71 $\cancel{2}\left|\frac{x+2}{\cancel{2}}\right|=2 \Rightarrow \left.\begin{cases} x+2=2 \\ \text{or} \\ x+2=-2 \end{cases}\right\} \Rightarrow \begin{matrix} x=0 \\ \text{or} \\ x=-4 \end{matrix}$

72 **a)** False. Any $(x<0 \text{ and } y>0)$ or $(x>0 \text{ and } y<0)$; for example,

$x=-1$ and $y=2 \Rightarrow |-1+2|=|-1|+|2| \Rightarrow |1|=1+2 \Rightarrow 1=3$, which is false.

b) False. Any $(x<0 \text{ and } y>0)$ or $(x>0 \text{ and } y<0)$; for example,

$x=3$ and $y=-2 \Rightarrow |3-(-2)|=|3|-|-2| \Rightarrow |5|=3-2 \Rightarrow 5=1$, which is false.

73 **a)** $(x<y)\Big/\times\frac{1}{x}, x>0 \Rightarrow \left(1<\frac{y}{x}\right)\Big/\times\frac{1}{y}, y>0 \Rightarrow \frac{1}{y}<\frac{1}{x}$

b) $(x<y)\Big/\times\frac{1}{x}, x<0 \Rightarrow \left(1>\frac{y}{x}\right)\Big/\times\frac{1}{y}, y>0 \Rightarrow \frac{1}{y}>\frac{1}{x}$

Note: When we multiply an inequality by a negative value, the inequality symbol is reversed.

Exercise 1.2

1 $\sqrt{h^2}\cdot\sqrt{h^2}=|h|\cdot|h|=|h|^2=h^2$

2 $\dfrac{\sqrt{45}}{\sqrt{5}}=\dfrac{3\sqrt{5}}{\sqrt{5}}=3$

3 $\sqrt{18}\cdot\sqrt{10}=3\sqrt{2}\cdot\sqrt{2\cdot 5}=6\sqrt{5}$

4 $\sqrt{\dfrac{28}{49}}=\sqrt{\dfrac{4}{7}}=\dfrac{2}{\sqrt{7}}=\dfrac{2\sqrt{7}}{7}$

5 $\sqrt[3]{4}\cdot\sqrt[3]{16}=\sqrt[3]{2^2}\cdot\sqrt[3]{2^4}=\sqrt[3]{2^6}=4$

6 $\sqrt{\dfrac{15}{20}}=\sqrt{\dfrac{3}{4}}=\dfrac{\sqrt{3}}{2}$

7 $\sqrt{5}\left(3+4\sqrt{5}\right)=\sqrt{5}\cdot 3+4\cdot 5=3\sqrt{5}+20$

8 $\left(2+\sqrt{6}\right)\left(2-\sqrt{6}\right)=2^2-\left(\sqrt{6}\right)^2=4-6=-2$

Notice that we have recognized the conjugate form of the factors and hence used the formula for the difference of two squares.

9 $\sqrt{98}=\sqrt{49\cdot 2}=7\sqrt{2}$

Note: In problems such as this, we need to factorize the radical into as many perfect square factors as possible.

10 $4\sqrt{1000}=4\sqrt{100\cdot 10}=40\sqrt{10}$

11 $\sqrt[3]{48}=\sqrt[3]{8\cdot 6}=2\sqrt[3]{6}$

Note: In problems such as this, we need to factorize the radical into as many perfect cube factors as possible.

12 $\sqrt{12x^3y^3}=\sqrt{4\cdot 3\cdot x^2\cdot x\cdot y^2\cdot y}=2xy\sqrt{3xy}$

13 $\sqrt[5]{m^5}=m$

14 $\sqrt{\dfrac{27}{6}}=\sqrt{\dfrac{9}{2}}=\dfrac{3}{\sqrt{2}}=\dfrac{3\sqrt{2}}{2}$

15 $\sqrt{x^{16}\left(1+x\right)^2}=x^8\left|1+x\right|$

16 $13\sqrt{7}-10\sqrt{7}=3\sqrt{7}$

Notice that we simply combine like terms.

17 $\sqrt{72}-8\sqrt{3}+3\sqrt{48}=\sqrt{36\times 2}-8\sqrt{3}+3\sqrt{16\times 3}=6\sqrt{2}-8\sqrt{3}+12\sqrt{3}=6\sqrt{2}+4\sqrt{3}$

18 $\sqrt{500}+5\sqrt{20}-\sqrt{45}=\sqrt{5\times 100}+5\sqrt{4\times 5}-\sqrt{9\times 5}=10\sqrt{5}+10\sqrt{5}-3\sqrt{5}=17\sqrt{5}$

19 $\dfrac{1}{\sqrt{5}}=\dfrac{1}{\sqrt{5}}\cdot\dfrac{\sqrt{5}}{\sqrt{5}}=\dfrac{\sqrt{5}}{5}$

20 $\dfrac{2}{5\sqrt{2}}=\dfrac{2}{5\sqrt{2}}\cdot\dfrac{\sqrt{2}}{\sqrt{2}}=\dfrac{\cancel{2}\sqrt{2}}{5\cdot\cancel{2}}=\dfrac{\sqrt{2}}{5}$

21 $\dfrac{6\sqrt{7}}{\sqrt{3}}=\dfrac{6\sqrt{7}}{\sqrt{3}}\cdot\dfrac{\sqrt{3}}{\sqrt{3}}=\dfrac{6\sqrt{21}}{3}=2\sqrt{21}$

22 $\dfrac{4}{\sqrt{32}}=\dfrac{\cancel{4}}{\cancel{4}\sqrt{2}}\cdot\dfrac{\sqrt{2}}{\sqrt{2}}=\dfrac{\sqrt{2}}{2}$

Note: In the first four problems of rationalization, questions 19–22, we were multiplying by the same surd expression in its simplest form.

In questions 23–30, we rationalize the denominator by multiplying the numerator and denominator by the conjugate of the denominator.

23 $\dfrac{2}{1+\sqrt{5}} = \dfrac{2}{1+\sqrt{5}} \times \dfrac{\sqrt{5}-1}{\sqrt{5}-1} = \dfrac{\cancel{2}\left(\sqrt{5}-1\right)}{\cancel{4}\,2} = \dfrac{\sqrt{5}-1}{2}$

24 $\dfrac{1}{3+2\sqrt{5}} = \dfrac{1}{3+2\sqrt{5}} \times \dfrac{2\sqrt{5}-3}{2\sqrt{5}-3} = \dfrac{2\sqrt{5}-3}{\left(2\sqrt{5}\right)^2 - 3^2} = \dfrac{2\sqrt{5}-3}{11}$

25 $\dfrac{\sqrt{3}}{2-\sqrt{3}} = \dfrac{\sqrt{3}}{2-\sqrt{3}} \times \dfrac{2+\sqrt{3}}{2+\sqrt{3}} = \dfrac{\sqrt{3}\left(2+\sqrt{3}\right)}{2^2 - \left(\sqrt{3}\right)^2} = 2\sqrt{3}+3$

26 $\dfrac{4}{\sqrt{2}+\sqrt{5}} = \dfrac{4}{\sqrt{2}+\sqrt{5}} \times \dfrac{\sqrt{5}-\sqrt{2}}{\sqrt{5}-\sqrt{2}} = \dfrac{4\left(\sqrt{5}-\sqrt{2}\right)}{\left(\sqrt{5}\right)^2 - \left(\sqrt{2}\right)^2} = \dfrac{4\left(\sqrt{5}-\sqrt{2}\right)}{3}$

27 $\dfrac{x-y}{\sqrt{x}+\sqrt{y}} = \dfrac{x-y}{\sqrt{x}+\sqrt{y}} \times \dfrac{\sqrt{x}-\sqrt{y}}{\sqrt{x}-\sqrt{y}} = \dfrac{\cancel{(x-y)}\left(\sqrt{x}-\sqrt{y}\right)}{\cancel{(x-y)}} = \sqrt{x}-\sqrt{y}$

28 $\dfrac{1+\sqrt{3}}{2+\sqrt{3}} = \dfrac{1+\sqrt{3}}{2+\sqrt{3}} \times \dfrac{2-\sqrt{3}}{2-\sqrt{3}} = \dfrac{2+2\sqrt{3}-\sqrt{3}-3}{4-3} = \sqrt{3}-1$

29 $\sqrt{\dfrac{1}{x^2}-1} = \sqrt{\dfrac{1-x^2}{x^2}} = \dfrac{\sqrt{1-x^2}}{|x|}$

30 $\dfrac{h}{\sqrt{x+h}-\sqrt{x}} = \dfrac{h}{\sqrt{x+h}-\sqrt{x}} \times \dfrac{\sqrt{x+h}+\sqrt{x}}{\sqrt{x+h}+\sqrt{x}} = \dfrac{h\left(\sqrt{x+h}+\sqrt{x}\right)}{x+h-x} = \sqrt{x+h}+\sqrt{x}$

In questions 31–33, we rationalize the numerator by multiplying the numerator and denominator by the conjugate of the numerator.

31 $\dfrac{\sqrt{a}-3}{a-9} = \dfrac{\sqrt{a}-3}{a-9} \times \dfrac{\sqrt{a}+3}{\sqrt{a}+3} = \dfrac{\cancel{(a-9)}}{\cancel{(a-9)}\left(\sqrt{a}+3\right)} = \dfrac{1}{\sqrt{a}+3}$

32 $\dfrac{\sqrt{x}-\sqrt{y}}{x-y} = \dfrac{\sqrt{x}-\sqrt{y}}{x-y} \times \dfrac{\sqrt{x}+\sqrt{y}}{\sqrt{x}+\sqrt{y}} = \dfrac{\cancel{(x-y)}}{\cancel{(x-y)}\left(\sqrt{x}+\sqrt{y}\right)} = \dfrac{1}{\sqrt{x}+\sqrt{y}}$

33 $\dfrac{\sqrt{m}-\sqrt{7}}{7-x} \times \dfrac{\sqrt{m}+\sqrt{7}}{\sqrt{m}+\sqrt{7}} = \dfrac{m-7}{(7-x)\left(\sqrt{m}+\sqrt{7}\right)}$

Exercise 1.3

1 $16^{\frac{1}{4}} = \left(2^4\right)^{\frac{1}{4}} = 2$

2 $9^{\frac{3}{2}} = \left(3^2\right)^{\frac{3}{2}} = 3^3 = 27$

3 $64^{\frac{2}{3}} = \left(4^3\right)^{\frac{2}{3}} = 4^2 = 16$

4 $8^{\frac{4}{3}} = \left(2^3\right)^{\frac{4}{3}} = 2^4 = 16$

5 $32^{\frac{3}{5}} = \left(2^5\right)^{\frac{3}{5}} = 2^3 = 8$

6 $\left(\sqrt{2}\right)^6 = \left(2^{\frac{1}{2}}\right)^6 = 2^3 = 8$

7 $\left(\dfrac{8}{27}\right)^{\frac{2}{3}} = \left(\left(\dfrac{2}{3}\right)^3\right)^{\frac{2}{3}} = \left(\dfrac{2}{3}\right)^2 = \dfrac{4}{9}$

8 $\left(\dfrac{9}{16}\right)^{\frac{1}{2}} = \left(\left(\dfrac{3}{4}\right)^2\right)^{\frac{1}{2}} = \dfrac{3}{4}$

9 $\left(\dfrac{25}{4}\right)^{\frac{3}{2}} = \left(\left(\dfrac{5}{2}\right)^2\right)^{\frac{3}{2}} = \left(\dfrac{5}{2}\right)^3 = \dfrac{125}{8}$

10 $(-3)^{-2} = \dfrac{1}{(-3)^2} = \dfrac{1}{9}$

11 $(13)^0 = 1$

12 $\dfrac{4\times3^{-2}}{2^{-2}\times3^{-1}}=\dfrac{4\times2^{2}\times\cancel{3}}{3^{\cancel{2}}}=\dfrac{16}{3}$

Note: In this first approach, the negative exponents change sign when moved from numerator to denominator, and vice versa.

In the second approach, we use the general power properties and leave the answer with no negative exponent.

$\dfrac{4\times3^{-2}}{2^{-2}\times3^{-1}}=2^{2-(-2)}\times3^{-2-(-1)}=2^{4}\times3^{-1}=\dfrac{16}{3}$

13 $\left(-\dfrac{3}{4}\right)^{-3}=\dfrac{4^{3}}{(-3)^{3}}=-\dfrac{64}{27}$

14 $\left(-xy^{3}\right)^{2}=x^{2}y^{6}$

15 $-\left(xy^{3}\right)^{2}=-x^{2}y^{6}$

16 $\left(-2xy^{3}\right)^{3}=-8x^{3}y^{9}$

17 $\left(2x^{3}y^{-5}\right)\left(2x^{-1}y^{3}\right)^{4}=2x^{3}y^{-5}\times16x^{-4}y^{12}=\dfrac{32y^{7}}{x}$

18 $\left(4m^{2}\right)^{-3}=\dfrac{1}{64m^{6}}$

19 $\dfrac{3k^{3}p^{4}}{\left(3k^{3}\right)^{2}p^{2}}=\dfrac{\cancel{3}\,\cancel{k^{3}}\,p^{\cancel{4}\,2}}{3\,\cancel{9}\,k^{\cancel{6}\,3}\,\cancel{p^{2}}}=\dfrac{p^{2}}{3k^{3}}$

20 $(-32)^{\frac{3}{5}}=\left(-2^{5}\right)^{\frac{3}{5}}=(-2)^{3}=-8$

21 $(125)^{\frac{2}{3}}=\left(5^{3}\right)^{\frac{2}{3}}=5^{2}=25$

22 $\dfrac{x\sqrt{x}}{\sqrt[3]{x}}=\dfrac{x^{\frac{3}{2}}}{x^{\frac{1}{3}}}=x^{\frac{3}{2}-\frac{1}{3}}=x^{\frac{7}{6}}=\sqrt[6]{x^{7}}=x\sqrt[6]{x}$

23 $\dfrac{4a^{3}b^{5}}{\left(2a^{2}b\right)^{4}}\times\dfrac{b^{-1}}{a^{-3}}=\dfrac{\cancel{4}\,\cancel{a^{3}}\,\cancel{b^{5}}}{4\,\cancel{16}\,a^{\cancel{8}\,2}\,\cancel{b^{4}}}\times\dfrac{\cancel{a^{3}}}{\cancel{b}}=\dfrac{1}{4a^{2}}$

24 $\dfrac{\left(\sqrt[3]{x}\right)\left(\sqrt[3]{x^{4}}\right)}{\left(\sqrt[3]{x^{2}}\right)}=\dfrac{x^{\frac{1}{3}}\times x^{\frac{4}{3}}}{x^{\frac{2}{3}}}=x^{\frac{1+4-2}{3}}=x$

25 $\dfrac{6\,(a-b)^{2}}{3a-3b}=\dfrac{2\,\cancel{6}\,(a-b)^{\cancel{2}}}{\cancel{3}\,\cancel{(a-b)}}=2(a-b)$

26 $\dfrac{(x+4y)^{\frac{1}{2}}}{2\,(x+4y)^{-1}}=\dfrac{(x+4y)^{\frac{1}{2}-(-1)}}{2}=\dfrac{(x+4y)^{\frac{3}{2}}}{2}=\dfrac{(x+4y)\sqrt{x+4y}}{2}$

27 $\dfrac{p^{2}+q^{2}}{\sqrt{p^{2}+q^{2}}}=\dfrac{p^{2}+q^{2}}{\sqrt{p^{2}+q^{2}}}\cdot\dfrac{\sqrt{p^{2}+q^{2}}}{\sqrt{p^{2}+q^{2}}}=\dfrac{\cancel{\left(p^{2}+q^{2}\right)}\sqrt{p^{2}+q^{2}}}{\cancel{p^{2}+q^{2}}}=\sqrt{p^{2}+q^{2}}$

28 $\dfrac{5^{3x+1}}{25}=\dfrac{5^{3x+1}}{5^{2}}=5^{3x-1}$

29 $\dfrac{x^{\frac{1}{3}}+x^{\frac{1}{4}}}{x^{\frac{1}{2}}}=\dfrac{x^{\frac{1}{3}}}{x^{\frac{1}{2}}}+\dfrac{x^{\frac{1}{4}}}{x^{\frac{1}{2}}}=x^{-\frac{1}{6}}+x^{-\frac{1}{4}}=\dfrac{1}{x^{\frac{1}{6}}}+\dfrac{1}{x^{\frac{1}{4}}}=\dfrac{1}{\sqrt[6]{x}}+\dfrac{1}{\sqrt[4]{x}}$

30 $3^{n+1}-3^{n-2}=3^{n-2}\left(3^{3}-1\right)=26\times3^{n-2}$

31 $\dfrac{8^{k+2}}{2^{3k+2}}=\dfrac{2^{3k+6}}{2^{3k+2}}=2^{4}=16$

32 $\sqrt[3]{24x^{6}y^{12}}=\sqrt[3]{8\cdot3\cdot\left(x^{2}\right)^{3}\left(y^{4}\right)^{3}}=2\sqrt[3]{3}\,x^{2}y^{4}$

33 $\dfrac{1}{n}\sqrt{n^{2}+n^{4}}=\sqrt{\dfrac{1}{n^{2}}\left(n^{2}+n^{4}\right)}=\sqrt{1+n^{2}}$

34 $\dfrac{x+\sqrt{x}}{1+\sqrt{x}}=\dfrac{\sqrt{x}\,\cancel{\left(\sqrt{x}+1\right)}}{\cancel{1+\sqrt{x}}}=\sqrt{x}$

Exercise 1.4

The answers to questions 1–8 are given correct to three significant figures (3 s.f.). The first significant figure is the first non-zero digit counting from the left. To round to three significant figures, look at the fourth significant figure. If the digit is less than 5, round down. If the digit is 5 or more, round up.

1 $253.8 = 2.54 \cdot 10^2$

2 $0.007\,81 = 7.81 \times 10^{-3}$

3 $7\,405\,239 = 7.41 \cdot 10^6$

4 $0.000\,001\,0448 = 1.04 \times 10^{-6}$

5 $4.9812 = 4.98 \cdot 10^0 = 4.98$

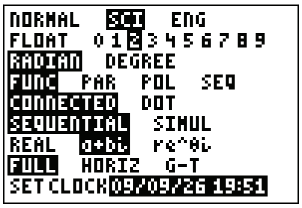
6 $0.001\,991 = 1.99 \times 10^{-3}$

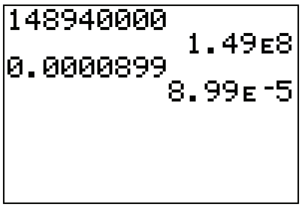
7 $148\,940\,000\ \text{m}^2 = 1.49 \cdot 10^8\ \text{km}^2$

8 $0.000\,0899$ grams per $\text{cm}^3 = 8.99 \times 10^{-5}\ \dfrac{\text{g}}{\text{cm}^3}$ (or $8.99 \times 10^{-5}\ \text{g cm}^{-3}$)

Note: Questions 1–8 can be answered by using a calculator. First select the numerical representation mode. In the first row we need to select the scientific form, whilst in the second row we need to select two decimal places.

The GDC screens for questions 7 and 8 are shown below.





9 $149\,597\,870.691\ \text{km} = 1.50 \cdot 10^8\ \text{km}$

10 $0.000\,000\,000\,000\,000\,000\,000\,000\,000\,000\,910\,938\,15 = 9.11 \times 10^{-31}$

11 $2.7 \times 10^{-3} = 0.0027$

12 $5 \cdot 10^7 = 50\,000\,000$

13 $9.035 \times 10^{-8} = 0.000\,000\,090\,35$

14 $4.18 \cdot 10^{12} = 4\,180\,000\,000\,000$

15 $(2.5 \times 10^{-3})(10 \times 10^5) = 2.5 \times 10^{-3} \times 10^6 = 2.5 \times 10^3$

16 $\dfrac{3.2 \times 10^6}{1.6 \times 10^2} = \dfrac{\cancel{3.2}\,2}{\cancel{1.6}\,1} \times 10^{6-2} = 2.0 \times 10^4$

17 $\dfrac{(1 \times 10^{-3})(3.28 \times 10^6)}{4 \times 10^7} = \dfrac{32.8 \times 10^5}{4 \times 10^{7+3}} = \dfrac{\cancel{32.8}\ 8.2}{\cancel{4}\,1} \times 10^{5-10} = 8.2 \times 10^{-5}$

18 $(2 \cdot 10^3)^4 (3.5 \cdot 10^5) = (16 \cdot 10^{12})(3.5 \cdot 10^5) = (1.6 \cdot 10^{13})(3.5 \cdot 10^5) = 1.6 \cdot 3.5 \cdot 10^{13+5} = 5.6 \cdot 10^{18}$

19 $(0.000\,000\,03)(6\,000\,000\,000\,000) = 3 \times 10^{-8} \times 6 \times 10^{12} = 1.8 \times 10^5$

20 $\dfrac{(1\,000\,000)^2\sqrt{0.000\,000\,04}}{(8\,000\,000\,000)^{\frac{2}{3}}} = \dfrac{(10^6)^2 \times (4 \times 10^{-8})^{\frac{1}{2}}}{(8 \times 10^9)^{\frac{2}{3}}} = \dfrac{10^{12} \times 2 \times 10^{-4}}{4 \times 10^6} = \dfrac{10^2}{2} = 5 \times 10^1$

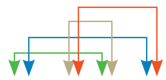
21 $\dfrac{4 \times 10^4}{(6.4 \times 10^2)(2.5 \times 10^{-5})} = \dfrac{4 \times 10^4}{1.6 \times 10^{-2}} = \dfrac{\cancel{4} \times 10^6}{\cancel{4} \times (4 \times 10^{-1})} = \dfrac{1}{4} \times 10^7 = 2.5 \times 10^6$

22 $(5.4 \times 10^2)^5 (-1.1 \times 10^{-6})^2 = 2^5 \times 2.7^5 \times 10^{10} \times 1.21 \times 10^{-12} = 5.555\,896\,79 \times 10^1 \approx 5.56 \times 10^1$ (to 3 s.f.)

Exercise 1.5

In the process of expanding and simplifying polynomials, we apply the distributive property and then combine like terms. Sometimes we can use slightly different methods in order to simplify and speed up the working.

In the case of a product of two binomial expressions, the application of the distributive property follows **FOIL**: first multiply **F**irst terms, then **O**uter terms and **I**nner terms, and finally the **L**ast terms.



1 $(x-4)(x+5) = x^2 - 4x + 5x - 20 = x^2 + x - 20$

2 $(3h-1)(2h-3) = 6h^2 - 2h - 9h + 3 = 6h^2 - 11h + 3$

Sometimes we use the **box method**, which involves using a geometrical representation of the algebraic problem. The dimensions of the box are the given binomial expressions, and the area can be split into four smaller boxes. Each box on the leading diagonal represents the quadratic and constant term respectively; whilst the boxes on the opposite diagonal represent the linear terms that we add.

	$3h$	-1
$2h$	$6h^2$	$-2h$
-3	$-9h$	3

$+$

$-2h - 9h = -11h$

$(3h-1)(2h-3) = 6h^2 - 11h + 3$

3 In this case we apply the difference of two squares formula.
$(y+9)(y-9) = y^2 - 9^2 = y^2 - 81$

4 We apply the formula for the square of a sum.
$(4x+2)^2 = (4x)^2 + 2 \cdot 4x \cdot 2 + 2^2 = 16x^2 + 16x + 4$

5 We apply the formula for the square of a difference.
$(2n-5)^2 = (2n)^2 - 2 \times 2n \times 5 + 5^2 = 4n^2 - 20n + 25$

6 Apart from using the distributive property twice, we can apply the formula for the cube of a difference.
$(2y-5)^3 = (2y)^3 - 3 \times (2y)^2 \times 5 + 3 \times 2y \times 5^2 - 5^3 = 8y^3 - 60y^2 + 150y - 125$

7 We notice that we have a product of two conjugate expressions and therefore we can apply the formula for the difference of two squares.
$(6a-7b)(6a+7b) = 36a^2 - 49b^2$

8 By grouping the first two terms in both brackets, we obtain two conjugate expressions and hence we can again use the formula for the difference of two squares.
$(2x+3+y)(2x+3-y) = ((2x+3)+y)((2x+3)-y) = (2x+3)^2 - y^2 = 4x^2 + 12x + 9 - y^2$

9 Applying the formula for the cube of a sum:
$(ax+b)^3 = a^3x^3 + 3a^2bx^2 + 3ab^2x + b^3$

10 Applying the binomial theorem for the fourth power:
$(ax+b)^4 = a^4x^4 + 4a^3bx^3 + 6a^2b^2x^2 + 4ab^3x + b^4$

11 $\left(2 + x\sqrt{5}\right)\left(2 - x\sqrt{5}\right) = 2^2 - \left(\sqrt{5}x\right)^2 = 4 - 5x^2$

12 Applying the formula for the difference of two cubes:
$(2x-1)\left(4x^2 + 2x + 1\right) = (2x)^3 - 1^3 = 8x^3 - 1$

13 In this question we can apply the formula for the square of a trinomial.
$(x + y - z)^2 = x^2 + y^2 + z^2 + 2xy - 2xz - 2yz$

14 $(x + yi)(x - yi) = x^2 + y^2$
Notice that this is the product of two conjugate complex numbers which is the square of the modulus of that complex number.

15 $(m + 3)(3 - m) = 9 - m^2$
Notice that it is the negative term that changes its sign.

16 $\left(1 - \sqrt{x^2 + 1}\right)^2 = 1 - 2\sqrt{x^2 + 1} + x^2 + 1 = 2 - 2\sqrt{x^2 + 1} + x^2$

17 So far we have been expanding polynomials. Now we are going to use the same formulae but in a different order. For this question, we will again use the formula for the difference of two squares. This time we recognize that the expression is a difference of two squares, and therefore we can write it as a product of two conjugate binomials.
$12x^2 - 48 = 12\left(x^2 - 4\right) = 12\left(x^2 - 2^2\right) = 12(x + 2)(x - 2)$

18 Notice that the common factor of different powers is always the smaller power that appears in all of the terms.
$x^3 - 6x^2 = x^2(x - 6)$

19 In factorizing quadratic expressions of the form $x^2 + px + q$, we need to look at the factors of the coefficient q. Then we need to find two such factors that will add up to p.
$x^2 + x - 12 = x^2 + 4x - 3x - 12 = x(x + 4) - 3(x + 4) = (x - 3)(x + 4)$

20 We are going to apply the same formula as used in question 19, but working backwards since the quadratic term is the last term and negative.
$7 - 6m - m^2 = 7 - 7m + m - m^2 = 7(1 - m) + m(1 - m) = (1 - m)(7 + m)$

21 $x^2 - 10x + 16 = x^2 - 2x - 8x + 16 = x(x - 2) - 8(x - 2) = (x - 2)(x - 8)$

22 $y^2 + 7y + 6 = y^2 + 6y + y + 6 = y(y + 6) + 1 \cdot (y + 6) = (y + 6)(y + 1)$

23 In this problem we notice that all the coefficients are divisible by 3, and therefore we factor out 3 and proceed as we did in the previous questions.
$3n^2 - 21n + 30 = 3\left(n^2 - 7n + 10\right) = 3\left(n^2 - 2n - 5n + 10\right) = 3[n(n - 2) - 5(n - 2)] = 3(n - 2)(n - 5)$

24 In this problem the common factor is $2x$, so we need to factor it out first and then proceed with the method.
$$2x^3 + 20x^2 + 18x = 2x\left(x^2 + 10x + 9\right) = 2x\left(x^2 + x + 9x + 9\right)$$
$$= 2x[x(x + 1) + 9(x + 1)] = 2x(x + 1)(x + 9)$$

25 In this problem we recognize the difference of two squares, so we apply the formula.
$a^2 - 16 = a^2 - 4^2 = (a + 4)(a - 4)$

26 To factorize a quadratic expression of the form $mx^2 + px + q$, we need to apply more general expressions.

Notice that: $(ax + b)(cx + d) = acx^2 + (ad + bc)x + bd$ and, by comparing the corresponding coefficients, we see that: $mq = acbd = (ad)(bc)$.

By equating the coefficients of x we get the sum of the factors: $p = ad + bc$.

So, in this problem, we need to find the factors of -15 that will add up to -14. Very quickly we can see that those two factors are -15 and 1.
$3y^2 - 14y - 5 = 3y^2 - 15y + y - 5 = 3y(y - 5) + (y - 5) = (y - 5)(3y + 1)$

27 Although this expression contains terms of a higher power, the exponent is even, so it can always be seen as a square; therefore, we recognize the difference of two squares.
$25n^4 - 4 = \left(5n^2\right)^2 - 2^2 = \left(5n^2 + 2\right)\left(5n^2 - 2\right)$

28 We notice a common factor of a in all of the terms. After writing a in front of the parenthesis, we recognize the square of the sum expansion.
$ax^2 + 6ax + 9a = a\left(x^2 + 6x + 9\right) = a\left(x^2 + 2 \cdot x \cdot 3 + 3^2\right) = a(x+3)^2$

29 If we spot that $(m+1)^2$ is a common factor, this problem becomes much easier.
$2n(m+1)^2 - (m+1)^2 = (m+1)^2(2n-1)$

30 In this case we will apply the formula for the difference of two squares twice.
$x^4 - 1 = \left(x^2\right)^2 - 1^2 = \left(x^2 - 1\right)\left(x^2 + 1\right) = (x-1)(x+1)\left(x^2+1\right)$

31 This problem can be solved in two ways. The first way is by factorizing the difference of two squares.
$9 - (y-3)^2 = 3^2 - (y-3)^2 = [3-(y-3)][3+(y-3)] = (6-y)\,y$

The second way involves expanding the square of a difference first, simplifying that expression and then factorizing.
$9 - (y-3)^2 = 9 - \left(y^2 - 6y + 9\right) = -y^2 + 6y = y(-y+6) = y(6-y)$

32 Firstly, we notice that all the terms have a common factor of $2y^2$, so we need to factor it out and then, if possible, proceed with the method described in question 26.
$4y^4 - 10y^3 - 96y^2 = 2y^2\left(2y^2 - 5y - 48\right)$

We notice that -96 does not have factors that will add up to -5. The closest we get is with the pair of factors -12 and 8, but they add up to -4, not -5. So, the above factorization is the final one.

33 In this question we recognize the formula for the square of a difference.
$4x^2 - 20x + 25 = (2x)^2 - 2 \times 2x \times 5 + 5^2 = (2x-5)^2$

34 When using the distributive property, the greatest common factor is the one with the lowest power. So, when we have negative powers, in this question -2 and -3, the greatest common factor will be the one with exponent -3.
$(2x+3)^{-2} + 2x(2x+3)^{-3} = (2x+3)^{-3}[(2x+3)+2x] = \dfrac{4x+3}{(2x+3)^3}$
Note that we have left the answer with no negative exponent.

35 $(n-2)^4 - (n-2)^3(2n-3) = (n-2)^3((n-2)-(2n-3)) = (n-2)^3(1-n)$

36 $m^3 - \dfrac{4}{3}m^2 + \dfrac{4}{9}m = m\left(m^2 - 2 \times m \times \dfrac{2}{3} + \left(\dfrac{2}{3}\right)^2\right) = m\left(m - \dfrac{2}{3}\right)^2$

In the following ten questions (37–46), the strategy that we are going to use relies on the fact that simplification of fractions can only be done if the numerator and denominator have a common factor.

37 $\dfrac{x+4}{x^2+5x+4} = \dfrac{x+4}{x^2+x+4x+4} = \dfrac{x+4}{x(x+1)+4(x+1)} = \dfrac{\cancel{x+4}\;1}{(x+1)\cancel{(x+4)}\;1} = \dfrac{1}{x+1}$

38 $\dfrac{3n-3}{6n^2-6n} = \dfrac{1\cancel{3}\cancel{(n-1)}}{2\cancel{6}n\cancel{(n-1)}} = \dfrac{1}{2n}$

39 $\dfrac{a^2-b^2}{5a-5b} = \dfrac{(a+b)\cancel{(a-b)}}{5\cancel{(a-b)}} = \dfrac{a+b}{5}$

40 $\dfrac{x^2+4x+4}{x+2}=\dfrac{(x+2)^{\not 2}}{\not{x+2}}=x+2$

41 Notice that the numerator and denominator are opposite expressions, and therefore their quotient is -1.

$\dfrac{2a-5}{5-2a}=-1$

42 $\dfrac{(2x+h)^2-4x^2}{h}=\dfrac{(2x+h)^2-(2x)^2}{h}=\dfrac{[(2x+h)-2x][(2x+h)+2x]}{h}=\dfrac{{}_1\not h\,(4x+h)}{\not h_1}=4x+h$

Note: We could have expanded the numerator first and then simplified and factorized. This alternative method would lead us to the same result.

43 $\dfrac{(x+1)^3(3x-5)-(x+1)^2(8x+3)}{(x-4)(x+1)^3}=\dfrac{(x+1)^2(3x^2-2x-5-8x-3)}{(x-4)(x+1)^3}$

$=\dfrac{3x^2-10x-8}{(x-4)(x+1)}=\dfrac{(3x+2)(x-4)}{(x-4)(x+1)}=\dfrac{3x+2}{x+1}$

44 $\dfrac{3y(y+3)-2(2y+1)}{(y+2)^2}=\dfrac{3y^2+9y-4y-2}{(y+2)^2}=\dfrac{3y^2+5y-2}{(y+2)^2}=\dfrac{(3y-1)\not{(y+2)}}{(y+2)^{\not 2}}=\dfrac{3y-1}{y+2}$

45 $\dfrac{a-\frac{a^2}{b}}{\frac{a^2}{b}-a}=\dfrac{\not a\left(1-\frac{a}{b}\right)}{\not a\left(\frac{a}{b}-1\right)}=-1$

46 $\dfrac{1+\dfrac{1}{1+\dfrac{1}{x-1}}}{1-\dfrac{1}{x-1}}=\dfrac{1+\dfrac{1}{\frac{x\not{-1}\not{+1}}{x-1}}}{\frac{x-1-1}{x-1}}=\dfrac{1+\frac{x-1}{x}}{\frac{x-2}{x-1}}=\dfrac{\frac{x+x-1}{x}}{\frac{x-2}{x-1}}=\dfrac{(2x-1)(x-1)}{x(x-2)}$

47 $\dfrac{1}{n}-1=\dfrac{1-n}{n}$

48 $\dfrac{2}{2x-1}-4=\dfrac{2-4(2x-1)}{2x-1}=\dfrac{6-8x}{2x-1}$

49 $\dfrac{x}{5}-\dfrac{x-1}{3}=\dfrac{3x-5(x-1)}{15}=\dfrac{5-2x}{15}$

50 $\dfrac{1}{a}-\dfrac{1}{b}=\dfrac{b-a}{ab}$

51 $\dfrac{1}{(x-3)^2}-\dfrac{3}{x-3}=\dfrac{1-3(x-3)}{(x-3)^2}=\dfrac{10-3x}{(x-3)^2}$

52 $\dfrac{x}{x+3}+\dfrac{1}{x}=\dfrac{x^2+x+3}{x(x+3)}\left(\text{or }\dfrac{x^2+x+3}{x^2+3x}\right)$

53 $\dfrac{1}{x+y}+\dfrac{1}{x-y}=\dfrac{x\not{-y}+x\not{+y}}{(x+y)(x-y)}=\dfrac{2x}{x^2-y^2}$

54 $\dfrac{3}{x-2}+\dfrac{5}{2-x}=\dfrac{-3+5}{2-x}=\dfrac{2}{2-x}\left(\text{or }\dfrac{-2}{x-2}\right)$

55 $\dfrac{2x-6}{\not x}\times\dfrac{3\not x}{x-3}=\dfrac{2\not{(x-3)}}{1}\times\dfrac{3}{\not{x-3}}=6$

56 $\dfrac{2x+6}{7}\times\dfrac{1}{x^2-9}=\dfrac{2\not{(x+3)}}{7}\times\dfrac{1}{(x-3)\not{(x+3)}}=\dfrac{2}{7x-21}$

57 $\dfrac{a+b}{a}\times\dfrac{1}{a^2-b^2}=\dfrac{\not{a+b}}{a}\times\dfrac{1}{(a-b)\not{(a+b)}}=\dfrac{1}{a(a-b)}\left(\text{or }\dfrac{1}{a^2-ab}\right)$

58 $\frac{3x^2-3}{6x}\times\frac{5x^2}{1-x}=\frac{\cancel{3}\left(x^2-1\right)}{{}_2\cancel{6}\cancel{x}}\times\frac{5x^{\cancel{2}}}{1-x}=\frac{\cancel{(x-1)}\,(x+1)}{2}\times\frac{5x}{(-1)\,\cancel{(x-1)}}=-\frac{5x\,(x+1)}{2}$

59 $\frac{3}{y+2}+\frac{5}{y^2-3y-10}=\frac{3}{y+2}+\frac{5}{(y+2)(y-5)}=\frac{3(y-5)+5}{(y+2)(y-5)}=\frac{3y-10}{(y-5)(y+2)}$

60 $\frac{8}{9-x^2}\div\frac{2x}{x^3-x^2-6x}=\frac{4\cancel{8}}{(-1)\,\cancel{(3-x)}(3+x)}\times\frac{\cancel{x}\,\cancel{(x-3)}(x+2)}{\cancel{2}\,\cancel{x}}=-\frac{4(x+2)}{3+x}$

In the following four questions (61–64), we will again multiply both the numerator and denominator by the corresponding conjugate expression.

61 $\frac{1}{x-\sqrt{2}}=\frac{1}{x-\sqrt{2}}\times\frac{x+\sqrt{2}}{x+\sqrt{2}}=\frac{x+\sqrt{2}}{x^2-2}$

62 $\frac{5}{2+x\sqrt{3}}=\frac{5}{2+x\sqrt{3}}\times\frac{2-x\sqrt{3}}{2-x\sqrt{3}}=\frac{5\left(2-x\sqrt{3}\right)}{4-3x^2}$

63 $\frac{\sqrt{x}+\sqrt{y}}{\sqrt{x}-\sqrt{y}}=\frac{\sqrt{x}+\sqrt{y}}{\sqrt{x}-\sqrt{y}}\times\frac{\sqrt{x}+\sqrt{y}}{\sqrt{x}+\sqrt{y}}=\frac{x+2\sqrt{xy}+y}{x-y}$

64 $\frac{1}{\sqrt{x+h}+\sqrt{x}}=\frac{1}{\sqrt{x+h}+\sqrt{x}}\times\frac{\sqrt{x+h}-\sqrt{x}}{\sqrt{x+h}-\sqrt{x}}=\frac{\sqrt{x+h}-\sqrt{x}}{h}$

Exercise 1.6

1 $m(h-x)=n\Rightarrow h-x=\frac{n}{m}\Rightarrow h-\frac{n}{m}=x \text{ or } x=\frac{hm-n}{m}$

2 $v=\sqrt{ab-t}\Rightarrow v^2=ab-t\Rightarrow v^2+t=ab\Rightarrow a=\frac{v^2+t}{b}$

3 $A=\frac{h}{2}(b_1+b_2)\Rightarrow\frac{2A}{h}=b_1+b_2\Rightarrow\frac{2A}{h}-b_2=b_1 \text{ or } b_1=\frac{2A-b_2h}{h}$

4 $A=\frac{1}{2}r^2\theta\Rightarrow\frac{2A}{\theta}=r^2\Rightarrow r=\pm\sqrt{\frac{2A}{\theta}}$

5 $\frac{f}{g}=\frac{h}{k}\Rightarrow f\times k=g\times h\Rightarrow k=\frac{gh}{f}$

6 $at=x-bt\Rightarrow at+bt=x\Rightarrow t(a+b)=x\Rightarrow t=\frac{x}{a+b}$

7 $V=\frac{1}{3}\pi r^3h\Rightarrow\frac{3V}{\pi h}=r^3\Rightarrow r=\sqrt[3]{\frac{3V}{\pi h}}$

8 $F=\frac{g}{m_1k+m_2k}\Rightarrow(m_1+m_2)\,k=\frac{g}{F}\Rightarrow k=\frac{g}{F\,(m_1+m_2)}$

In the following four questions (9–12), we can use many different methods to find the slope-intercept (explicit) form of a straight line. We will explore a few methods.

9 Firstly, we find the value of the slope that passes through the two points:

$$m=\frac{y_2-y_1}{x_2-x_1}\Rightarrow m=\frac{-7-1}{3-(-9)}=\frac{-\cancel{8}\,2}{\cancel{12}\,3}=-\frac{2}{3}$$

Then, using the equation of the line and one point, we can find the y-intercept.

$$y = -\frac{2}{3}x + c \Rightarrow c = -7 + \frac{2}{\cancel{3}} \times \cancel{3} = -5$$

Therefore, the equation of the line is $y = -\frac{2}{3}x - 5$.

10 We notice that both points have equal y-coordinates; therefore, the line passing through them is horizontal and its equation is $y = -4$.

11 It is possible to derive a formula for the equation of a straight line that passes through two given points, $P_1 = (x_1, y_1)$ and $P_2 = (x_2, y_2)$. We know that the formula for the slope between two points is $m = \frac{y_2 - y_1}{x_2 - x_1}$ and then we take any point on the line, $P = (x, y)$, and again use the slope between P and P_1.

$$\frac{y - y_1}{x - x_1} = \frac{y_2 - y_1}{x_2 - x_1} \Rightarrow y - y_1 = \frac{y_2 - y_1}{x_2 - x_1}(x - x_1)$$

So, the final formula that we use is: $y = \frac{y_2 - y_1}{x_2 - x_1}(x - x_1) + y_1$.

When we apply it in this question we get:

$$y = \frac{11 - (-9)}{4 - (-12)}(x - (-12)) + (-9) \Rightarrow y = \frac{5}{4}(x + 12) - 9 \Rightarrow y = \frac{5}{4}x + 15 - 9 \Rightarrow y = \frac{5}{4}x + 6$$

Note: The equation of a line can also be written as $y - y_1 = m(x - x_1)$ (point-slope form).

12 We notice that both points have equal x-coordinates; therefore, the line passing through them is vertical and its equation is $x = \frac{7}{3}$.

13 We know that two parallel lines have equal slopes. Given that the slope of the line $4x + y - 3 = 0 \Rightarrow y = -4x + 3$ is -4, the slope of the parallel line is -4. We can now use the slope and the point to find the y-intercept: $y = -4x + c \Rightarrow -17 = -4 \times 7 + c \Rightarrow c = -17 + 28 = 11$

So, the equation of the parallel line is $y = -4x + 11$.

14 We know that two perpendicular lines have slopes that are mutually opposite and reciprocal. Given that the slope of the line $2x - 5y - 35 = 0 \Rightarrow y = \frac{2}{5}x - 7$ is $\frac{2}{5}$, the slope of the perpendicular line is $-\frac{5}{2}$.

We can now use the slope and the point to find the y-intercept:

$$y = -\frac{5}{2}x + c \Rightarrow \frac{11}{2} = -\frac{5}{2} \times (-5) + c \Rightarrow c = \frac{11}{2} - \frac{25}{2} = -7$$

So, the equation of the perpendicular line is $y = -\frac{5}{2}x - 7$.

Note: Two non-vertical lines are perpendicular if $m_1 = -\frac{1}{m_2}$, which is equivalent to $m_1 \cdot m_2 = -1$.

In the following four questions (15–18), we are going to use the distance formula between two points, $P_1 = (x_1, y_1)$ and $P_2 = (x_2, y_2)$: $P_1P_2 = \sqrt{(x_2 - x_1)^2 + (y_2 - y_1)^2}$, where P_1P_2 represents the distance between the two points.

We are also going to use the midpoint formula: $x_M = \frac{x_1 + x_2}{2}$, $y_M = \frac{y_1 + y_2}{2}$.

15 **a)** $P_1P_2 = \sqrt{(4 - (-4))^2 + (-5 - 10)^2} = \sqrt{64 + 225} = \sqrt{289} = 17$

b) $x_M = \frac{-4 + 4}{2} = 0$, $y_M = \frac{10 + (-5)}{2} = \frac{5}{2}$; so the midpoint is $M = \left(0, \frac{5}{2}\right)$.

16 **a)** $P_1P_2 = \sqrt{(5-(-1))^2 + (4-2)^2} = \sqrt{36+4} = \sqrt{40} = 2\sqrt{10}$

b) $x_M = \dfrac{-1+5}{2} = 2,\ y_M = \dfrac{2+4}{2} = 3$; so the midpoint is $M = (2, 3)$.

17 **a)** $P_1P_2 = \sqrt{\left(-\dfrac{5}{2}-\dfrac{1}{2}\right)^2 + \left(\dfrac{4}{3}-1\right)^2} = \sqrt{9+\dfrac{1}{9}} = \sqrt{\dfrac{82}{9}} = \dfrac{\sqrt{82}}{3}$

b) $x_M = \dfrac{\dfrac{1}{2}+\left(-\dfrac{5}{2}\right)}{2} = -1,\ y_M = \dfrac{1+\dfrac{4}{3}}{2} = \dfrac{7}{6}$; so the midpoint is $M = \left(-1, \dfrac{7}{6}\right)$.

18 **a)** $P_1P_2 = \sqrt{(-10-12)^2 + (9-2)^2} = \sqrt{484+49} = \sqrt{533}$

b) $x_M = \dfrac{12+(-10)}{2} = 1,\ y_M = \dfrac{2+9}{2} = \dfrac{11}{2}$; so the midpoint is $M = \left(1, \dfrac{11}{2}\right)$.

The following two questions (19–20) are going to be solved with and without the use of a calculator.

Solution Paper 1 type

19 $\sqrt{(k-5)^2 + (2-(-1))^2} = 5 \Rightarrow k^2 - 10k + 25 + 9 = 25 \Rightarrow k^2 - 10k + 9 = 0$

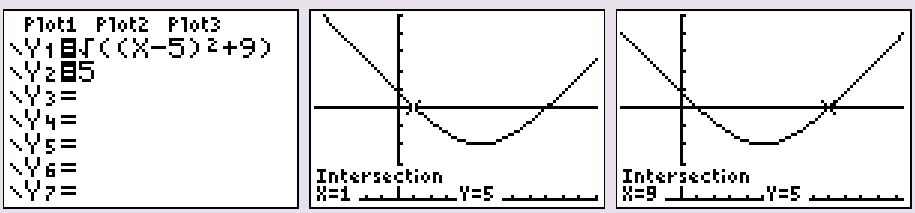
We can always use the quadratic formula to solve equations of this type, but, in this case, factorization might be slightly faster since the coefficients are 'nice'.

$(k-9)(k-1) = 0 \Rightarrow k_1 = 9, k_2 = 1$

So, $k = 1$ or $k = 9$.

Solution Paper 2 type

19 In this case, we can use the distance formula as a function of the parameter k. When using a calculator, the variable k will be denoted by x, since the function mode does not accept other variables. We then have to find the points of intersection of the function with the horizontal line $y = 5$.



The calculator gives both the x- and y-coordinates of the points of intersection with the x-axis. We only use the x-coordinate, so $k = 1$ or $k = 9$.

Solution Paper 1 type

20 $\sqrt{(1-(-2))^2 + (k-(-7))^2} = 5 \Rightarrow 9 + k^2 + 14k + 49 = 25 \Rightarrow k^2 + 14k + 33 = 0$

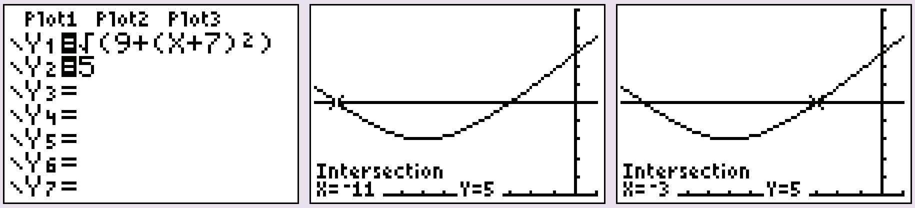
This time we will use the quadratic formula to solve the equation.

$k_{1,2} = \dfrac{-14 \pm \sqrt{196 - 4 \times 33}}{2} = \dfrac{-14 \pm \sqrt{64}}{2} = \dfrac{-14 \pm 8}{2} \Rightarrow k_1 = -11, k_2 = -3$

So, $k = -11$ or $k = -3$.

Solution Paper 2 type

20 We will use the distance formula as a function of the parameter k. Then we have to find the points of intersection of the function with the horizontal line $y = 5$.



Again we are only interested in the x-coordinates of the points of intersection. So, $k = -11$ or $k = -3$.

21 If the triangle is right angled, then two sides should be perpendicular. Therefore, the two slopes between the pairs of vertices should be opposite and reciprocal.

Let's denote the vertices by $A(4, 0)$, $B(2, 1)$ and $C(-1, -5)$.

We need to calculate the slopes between pairs of points:

$$\left.\begin{aligned} m_{AB} &= \frac{1-0}{2-4} = -\frac{1}{2} \\ m_{BC} &= \frac{-5-1}{-1-2} = 2 \end{aligned}\right\} \Rightarrow m_{AB} \times m_{BC} = -\frac{1}{2} \times 2 = -1 \Rightarrow (AB) \perp (BC)$$

Another way of solving the same question would be by using the converse of Pythagoras' theorem. For that, we need to calculate all the distances between the points.

$$\left.\begin{aligned} AB &= \sqrt{(4-2)^2 + (0-1)^2} = \sqrt{4+1} = \sqrt{5} \\ BC &= \sqrt{(2-(-1))^2 + (1-(-5))^2} = \sqrt{9+36} = \sqrt{45} = 3\sqrt{5} \\ CA &= \sqrt{(-1-4)^2 + (-5-0)^2} = \sqrt{25+25} = \sqrt{50} = 5\sqrt{2} \end{aligned}\right\} \Rightarrow$$

$$\left(\sqrt{5}\right)^2 + \left(3\sqrt{5}\right)^2 = 5 + 45 = 50 = \left(5\sqrt{2}\right)^2 \Rightarrow AB^2 + BC^2 = CA^2$$

Therefore, there is a right angle at vertex B.

22 If the triangle is isosceles, then two sides have the same length.

Let's denote the vertices by $A(1, -3)$, $B(3, 2)$ and $C(-2, 4)$.

We need to calculate the distances between the points:

$$\left.\begin{aligned} AB &= \sqrt{(1-3)^2 + (-3-2)^2} = \sqrt{4+25} = \sqrt{29} \\ BC &= \sqrt{(3-(-2))^2 + (2-4)^2} = \sqrt{25+4} = \sqrt{29} \\ CA &= \sqrt{(-2-1)^2 + (4-(-3))^2} = \sqrt{9+49} = \sqrt{58} \end{aligned}\right\} \Rightarrow AB = BC$$

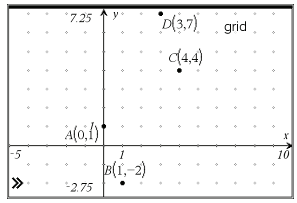
Therefore, the triangle is isosceles.

Note: We did not need to calculate the distance CA, but we wanted to see whether the triangle was not just isosceles but possibly equilateral.

23 If the points are vertices of a parallelogram, then pairs of opposite sides are parallel and their slopes are equal.

Let's denote the vertices by $A(0, 1)$, $B(1, -2)$, $C(4, 4)$ and $D(3, 7)$.

If we plot the points in the coordinate system, we would have a better idea as to which sides should be parallel.



$$\left.\begin{aligned} m_{AD} &= \frac{7-1}{3-0} = 2 \\ m_{BC} &= \frac{4+2}{4-1} = 2 \end{aligned}\right\} \Rightarrow (AD) \parallel (BC), \quad \left.\begin{aligned} m_{AB} &= \frac{-2-1}{1-0} = -3 \\ m_{CD} &= \frac{4-7}{4-3} = -3 \end{aligned}\right\} \Rightarrow (AB) \parallel (CD)$$

Therefore, the quadrilateral *ABCD* is a parallelogram.

Another way of solving the same problem would be to verify that the lengths of the opposite sides are equal; so we calculate:

$$AB = \sqrt{(0-1)^2 + (1-(-2))^2} = \sqrt{1+9} = \sqrt{10}$$
$$CD = \sqrt{(4-3)^2 + (4-7)^2} = \sqrt{1+9} = \sqrt{10}$$
$$AD = \sqrt{(0-3)^2 + (1-7)^2} = \sqrt{9+36} = \sqrt{45} = 3\sqrt{5}$$
$$BC = \sqrt{(1-4)^2 + (-2-4)^2} = \sqrt{9+36} = \sqrt{45} = 3\sqrt{5}$$

Therefore, the quadrilateral *ABCD* is a parallelogram.

Note: A third way of solving the problem would be to prove that one pair of opposite sides was equal and parallel.

24 For the elimination method, one of the variables must have either equal or opposite coefficients. In this case, the coefficients of x are both equal to 1. When the coefficients are equal, we subtract the equations; whilst if they are opposite, we add the equations.

$$\left.\begin{aligned} x + 3y &= 8 \\ x - 2y &= 3 \end{aligned}\right) - $$
$$\overline{3y - (-2y) = 8 - 3} \Rightarrow 5y = 5 \Rightarrow y = 1$$

Now we need to find x by using either of the equations. We will use the second equation, so:
$x - 2 \times 1 = 3 \Rightarrow x = 5$

The solution can be written as $x = 5,\ y = 1$, or $(5, 1)$.

25 If none of the coefficients are equal or opposite, we multiply or divide the equations by a number(s) to make them so.

$$\left.\begin{aligned} x - 6y &= 1 \\ (3x + 2y &= 13) / \times 3 \end{aligned}\right\} \Rightarrow \frac{\left.\begin{aligned} x - 6y &= 1 \\ 9x + 6y &= 39 \end{aligned}\right) +}{10x = 40} \Rightarrow x = 4$$

To find y we will use the second equation, so: $3 \times 4 + 2y = 13 \Rightarrow 2y = 1 \Rightarrow y = \frac{1}{2}$

The solution is $x = 4,\ y = \frac{1}{2}$, or $\left(4, \frac{1}{2}\right)$.

26 $$\left.\begin{aligned} (6x + 3y = 6) &\Big/ \times \frac{4}{3} \\ 5x + 4y &= -1 \end{aligned}\right\} \Rightarrow \frac{\left.\begin{aligned} 8x + 4y &= 8 \\ 5x + 4y &= -1 \end{aligned}\right) -}{3x = 9} \Rightarrow x = 3$$

To find y we will use the first equation divided by 3, so: $2 \times 3 + y = 2 \Rightarrow y = -4$

The solution is $x = 3,\ y = -4$, or $(3, -4)$.

27 $$\begin{array}{l}\left.\begin{array}{r}\not{x}+3y=-1\\ \not{x}-2y=7\end{array}\right\}-\\ \hline 3y-(-2y)=-1-7\end{array}\Rightarrow 5y=-8\Rightarrow y=-\frac{8}{5}$$

To find y we will use the second equation, so: $x-2\left(-\frac{8}{5}\right)=7\Rightarrow x=7-\frac{16}{5}\Rightarrow y=\frac{19}{5}$

The solution is $x=\frac{19}{5}$, $y=-\frac{8}{5}$, or $\left(\frac{19}{5},-\frac{8}{5}\right)$, or (3.8, −1.6).

28 $$\left.\begin{array}{r}(8x-12y=4)/\div 4\\ -2x+3y=2\end{array}\right\}\Rightarrow\begin{array}{c}\left.\begin{array}{r}\not{2x}\not{-3y}=1\\ \not{-2x}\not{+3y}=2\end{array}\right\}+\\ \hline 0=3\end{array}\Rightarrow x\notin\mathbb{R},\ y\notin\mathbb{R};\text{ no solution.}$$

Note: If the graphs of these two straight lines were drawn, they would not intersect as they are parallel.

29 $$\left.\begin{array}{r}(5x+7y=9)/\times 11\\ (-11x-5y=1)/\times 5\end{array}\right\}\Rightarrow\begin{array}{c}\left.\begin{array}{r}\not{55x}+77y=99\\ \not{-55x}-25y=5\end{array}\right\}+\\ \hline 52y=104\end{array}\Rightarrow y=2$$

To find x we will use the first equation, so: $5x+7\times 2=9\Rightarrow 5x=-5\Rightarrow x=-1$

The solution is $x=-1$, $y=2$, or $(-1, 2)$.

30 For the substitution method, we select the variable with the smallest coefficient from either of the equations and make it the subject.

$$\left.\begin{array}{r}2x+y=1\\ 3x+2y=3\end{array}\right\}\Rightarrow\left.\begin{array}{r}y=1-2x\\ 3x+2(1-2x)=3\end{array}\right\}\Rightarrow\left.\begin{array}{r}y=1-2x\\ 3x+2-4x=3\end{array}\right\}\Rightarrow\left.\begin{array}{r}y=1-2x\\ -x=3-2\end{array}\right\}\Rightarrow$$

$$\left.\begin{array}{r}y=1-2\times(-1)\\ x=-1\end{array}\right\}\Rightarrow\begin{array}{r}y=3\\ x=-1\end{array}$$

The solution is $x=-1$, $y=3$, or $(-1, 3)$.

31 $$\left.\begin{array}{r}3x-2y=7\\ 5x-y=-7\end{array}\right\}\Rightarrow\left.\begin{array}{r}3x-2y=7\\ 5x+7=y\end{array}\right\}\Rightarrow\left.\begin{array}{r}3x-2(5x+7)=7\\ y=5x+7\end{array}\right\}\Rightarrow\left.\begin{array}{r}3x-10x-14=7\\ y=5x+7\end{array}\right\}\Rightarrow$$

$$\left.\begin{array}{r}-7x=21\\ y=5x+7\end{array}\right\}\Rightarrow\left.\begin{array}{r}x=-3\\ y=5\times(-3)+7\end{array}\right\}\Rightarrow\left.\begin{array}{r}x=-3\\ y=-15+7\end{array}\right\}\Rightarrow\begin{array}{r}x=-3\\ y=-8\end{array}$$

The solution is $x=-3$, $y=-8$, or $(-3, -8)$.

32 $$\left.\begin{array}{r}(2x+8y=-6)/\div 2\\ (-5x-20y=15)/\div(-5)\end{array}\right\}\Rightarrow\left.\begin{array}{r}x+4y=-3\\ x+4y=-3\end{array}\right\}$$

We can see that the equations coincide and hence we have an infinite number of solutions: all the pairs of (x, y) that satisfy the equation $x+4y=-3$ or $\left(x,-\frac{1}{4}x-\frac{3}{4}\right)$, $x\in\mathbb{R}$ or $(-3-4y, y)$, $y\in\mathbb{R}$.

Geometrically, the solution of this problem would be one straight line: $y=-\frac{1}{4}x-\frac{3}{4}$.

33 $\left.\begin{aligned}\left(\frac{x}{5}+\frac{y}{2}=8\right)/\times 10\\ x+y=20\end{aligned}\right\}\Rightarrow\left.\begin{aligned}2x+5y=80\\ x=20-y\end{aligned}\right\}\Rightarrow\left.\begin{aligned}2(20-y)+5y=80\\ x=20-y\end{aligned}\right\}\Rightarrow\left.\begin{aligned}40-2y+5y=80\\ x=20-y\end{aligned}\right\}\Rightarrow$

$\left.\begin{aligned}3y=40\\ x=20-y\end{aligned}\right\}\Rightarrow\left.\begin{aligned}y=\frac{40}{3}\\ x=20-\frac{40}{3}\end{aligned}\right\}\Rightarrow\begin{aligned}y=\frac{40}{3}\\ x=\frac{20}{3}\end{aligned}$

The solution is $x=\frac{20}{3}$, $y=\frac{40}{3}$, or $\left(\frac{20}{3},\frac{40}{3}\right)$.

34 $\left.\begin{aligned}2x-y=-2\\ 4x+y=5\end{aligned}\right\}\Rightarrow\left.\begin{aligned}2x+2=y\\ 4x+2x+2=5\end{aligned}\right\}\Rightarrow\left.\begin{aligned}y=2x+2\\ 6x=3\end{aligned}\right\}\Rightarrow\left.\begin{aligned}y=2x+2\\ x=\frac{1}{2}\end{aligned}\right\}\Rightarrow\left.\begin{aligned}y=2\times\frac{1}{2}+2\\ x=\frac{1}{2}\end{aligned}\right\}\Rightarrow\begin{aligned}y=3\\ x=\frac{1}{2}\end{aligned}$

The solution is $x=\frac{1}{2}$, $y=3$, or $\left(\frac{1}{2},3\right)$.

35 $\left.\begin{aligned}(0.4x+0.3y=1)/\times 10\\ (0.25x+0.1y=-0.25)/\times 10\end{aligned}\right\}\Rightarrow\left.\begin{aligned}4x+3y=10\\ 2.5x+y=-2.5\end{aligned}\right\}\Rightarrow\left.\begin{aligned}4x+3(-2.5x-2.5)=10\\ y=-2.5x-2.5\end{aligned}\right\}\Rightarrow$

$\left.\begin{aligned}4x-7.5x-7.5=10\\ y=-2.5x-2.5\end{aligned}\right\}\Rightarrow\left.\begin{aligned}-3.5x=17.5\\ y=-2.5x-2.5\end{aligned}\right\}\Rightarrow\left.\begin{aligned}x=\frac{17.5}{-3.5}=-5\\ y=-2.5\times(-5)-2.5\end{aligned}\right\}\Rightarrow\begin{aligned}x=-5\\ y=10\end{aligned}$

The solution is $x=-5$, $y=10$, or $(-5, 10)$.

36 There is another algebraic method, Cramer's method, for solving two linear simultaneous equations in two variables. In this method we use determinants.

Given the simultaneous equations $\begin{cases}ax+by=e\\ cx+dy=f\end{cases}$, we define three determinants of order 2×2,

$\Delta=\begin{vmatrix}a & b\\ c & d\end{vmatrix}$, $\Delta_x=\begin{vmatrix}e & b\\ f & d\end{vmatrix}$, and $\Delta_y=\begin{vmatrix}a & e\\ c & f\end{vmatrix}$, and the solution is $x=\frac{\Delta_x}{\Delta}$, $y=\frac{\Delta_y}{\Delta}$.

In the question $\begin{cases}3x+2y=9\\ 7x+11y=2\end{cases}$, the values of the determinants are:

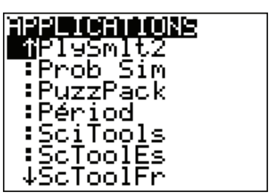
$\Delta=\begin{vmatrix}3 & 2\\ 7 & 11\end{vmatrix}=3\times 11-2\times 7=19$, $\Delta_x=\begin{vmatrix}9 & 2\\ 2 & 11\end{vmatrix}=9\times 11-2\times 2=99-4=95$, and

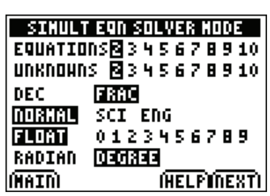
$\Delta_y=\begin{vmatrix}3 & 9\\ 7 & 2\end{vmatrix}=3\times 2-9\times 7=6-63=-57$

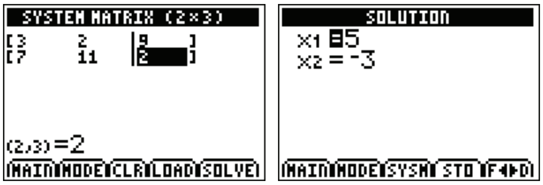
Therefore, the solution is $x=\frac{95}{19}=5$, $y=\frac{-57}{19}=-3$, or $(5,-3)$.

In questions 36–38, we are going to demonstrate the different features of a graphic display calculator which can be used to solve simultaneous equations.

36 The first feature is an application called PolySmlt (older version) or PolySmlt2 (new version) on the TI-84 Plus calculator.

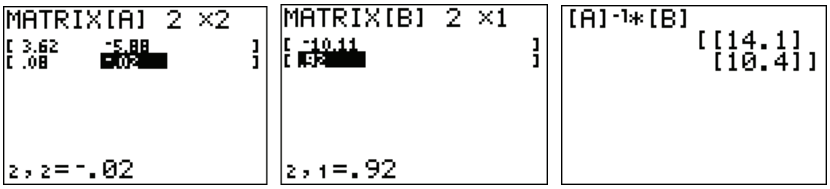






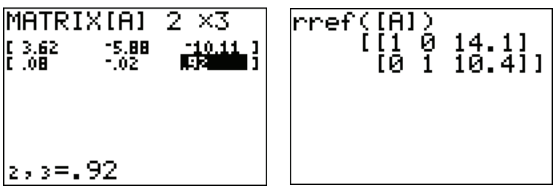
Note that x_1 gives the value of x and x_2 the value of y.

37 In this problem we are going to use matrices on the calculator. There are two different features that we can use in order to solve the simultaneous equations. The first method involves the use of matrix algebra.



The solution is $x = 14.1$, $y = 10.4$, or $(14.1, 10.4)$.

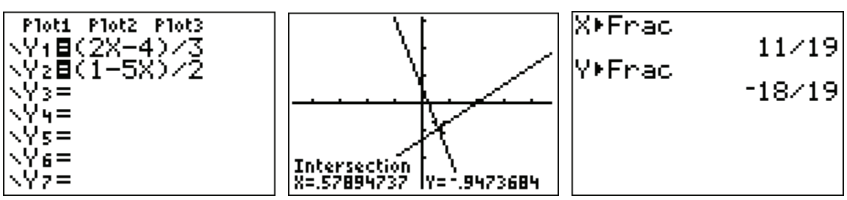
For the second method, we are going to use the augmented matrix of the system and Gauss' method, which can be found as a reduced row echelon form (rref) feature on the calculator.



Again the solution is $x = 14.1$, $y = 10.4$, or $(14.1, 10.4)$.

38 The last method which we are going to demonstrate here is a graphical method. A linear equation in two variables can be represented geometrically by a straight line and, as such, solving simultaneous equations is equivalent to finding the point of intersection of those two lines. Therefore, we need to rewrite the equations in explicit form.

$$\left.\begin{array}{l} 2x - 3y = 4 \\ 5x + 2y = 1 \end{array}\right\} \Rightarrow \left.\begin{array}{l} 2x - 4 = 3y \\ 2y = 1 - 5x \end{array}\right\} \Rightarrow \left.\begin{array}{l} y = \dfrac{2x-4}{3} \\ y = \dfrac{1-5x}{2} \end{array}\right\}$$



The solution is $x = \dfrac{11}{19}$, $y = -\dfrac{18}{19}$, or $\left(\dfrac{11}{19}, -\dfrac{18}{19}\right)$.

Chapter 2

Exercise 2.1

1 **a)** The equation $y = 2x$ represents a straight line that passes through the origin and has a slope of 2; therefore, the correct graph is G.

b) It is a linear function. The graph passes the vertical line test.

2 **a)** The equation $y = -3$ represents a horizontal line that passes through the point $(0, -3)$; therefore, the correct graph is L.

b) It is a constant function. The graph passes the vertical line test.

3 **a)** The equation $x - y = 2 \Rightarrow y = x - 2$ represents a straight line that passes through the point $(0, -2)$ and has a slope of 1; therefore, the correct graph is H.

b) It is a linear function. The graph passes the vertical line test.

4 **a)** The equation $x^2 + y^2 = 4$ represents a circle of radius 2 with the centre at the origin; therefore, the correct graph is K.

b) It is not a function. Any vertical line between -2 and 2 intersects the circle at two points.

5 **a)** The equation $y = 2 - x$ represents a straight line that passes through the point $(0, 2)$ and has a slope of -1; therefore, the correct graph is J.

b) It is a linear function. The graph passes the vertical line test.

6 **a)** The equation $y = x^2 + 2$ represents a normal parabola that opens upwards $(a > 0)$ and has a vertex at the point $(0, 2)$; therefore, the correct graph is C.

b) It is a quadratic function. The graph passes the vertical line test.

7 **a)** The equation $y^3 = x \Rightarrow y = \sqrt[3]{x}$ represents a cubic root graph that passes through the origin; therefore, the correct graph is A.

b) It is a cubic root function. The graph passes the vertical line test.

8 **a)** The equation $y = \dfrac{2}{x}$ represents a rectangular hyperbola that is symmetrical with respect to the line $y = x$; therefore, the correct graph is I.

b) It is a reciprocal function. The graph passes the vertical line test.

9 **a)** The equation $x^2 + y = 2 \Rightarrow y = 2 - x^2$ represents a normal parabola that opens downwards $(a < 0)$ and has a vertex at the point $(0, 2)$; therefore, the correct graph is F.

b) It is a quadratic function. The graph passes the vertical line test.

The remaining three graphs represent the following:

Graph B is a vertical line that passes through the point $(-3, 0)$; therefore, the equation is $x = -3$ and it is not a function.

Graph D is a rectangular hyperbola that is symmetrical with respect to the line $y = -x$ and it looks like it is passing through the points $(-1, 1)$ and $(1, -1)$; therefore, the equation is $y = -\frac{1}{x}, x \neq 0$, and it is a function.

Graph E looks like a cubic graph which passes through the origin and the points $(-2, -2)$ and $(2, 2)$; therefore, the equation is $y = \frac{1}{4}x^3$ and it is a function.

10 We know the formulae for the area A and the circumference C of a circle with radius r:

$A = r^2\pi, C = 2\pi r$. Since both formulae contain the same factor r, we are going to make r the subject of the circumference formula and then substitute it into the area formula.

$$r = \frac{C}{2\pi} \Rightarrow A = \left(\frac{C}{2\pi}\right)^2 \pi = \frac{C^2}{4\pi^{\not 2}} \not\pi = \frac{C^2}{4\pi}$$

11 The area A of any triangle is calculated using the formula $A = \frac{l \cdot h}{2}$, where l represents the length of a side and h represents the length of the height to the side l. The height of an equilateral triangle is also a median and therefore it divides the triangle into two right-angled triangles.

In one such triangle, we calculate the height by using Pythagoras' theorem:

$$l^2 = h^2 + \left(\frac{l}{2}\right)^2 \Rightarrow h = \sqrt{l^2 - \frac{l^2}{4}} = \sqrt{\frac{3l^2}{4}} = l\frac{\sqrt{3}}{2}$$

Now we substitute that into the area formula:

$$A = \frac{l \cdot h}{2} = \frac{l \cdot l\frac{\sqrt{3}}{2}}{2} = \frac{l^2\sqrt{3}}{4}$$

12 $A(x) = 2 \cdot 18 \cdot x + 2 \cdot 12 \cdot x + 4x^2 = 60x + 4x^2$

13 $h(x) = \sqrt{x^2 + x^2} = x\sqrt{2}$

14 **a)** $k = \frac{PV}{T} \Rightarrow k = \frac{23.5 \times 150}{375} = 9.4$

b) $V = \frac{kT}{P} \Rightarrow V = \frac{9.4 \times 375}{P} = \frac{3525}{P}$

15 **a)** $F = kx$

b) $25 = k(16 - 12) \Rightarrow k = \frac{25}{4} = 6.25$

c) $F = \frac{25}{4}(18 - 12) = \frac{75}{2} = 37.5$ N

16 $D(f) = \{-6.2, -1.5, 0.7, 3.2, 3.8\}$

Notice that we have simply listed the first component from each ordered pair. The second components are values from the range.

17 $D(f) = \{r \in \mathbb{R} | r > 0\}$

Note: A radius cannot be a negative number or zero.

18 $D(f) = \mathbb{R}$

Note: The domain of a linear function is always the set of real numbers.

19 $D(f) = \mathbb{R}$

Note: The domain of a quadratic function is always the set of real numbers.

20 An irrational function with an even surd exponent must always have a non-negative radical (the expression under the root). Hence: $3 - t \geqslant 0 \Rightarrow D(f) = \{t \in \mathbb{R} | t \leqslant 3\}$.

21 An irrational function with an odd surd exponent can accept any radical. Hence: $D(f) = \mathbb{R}$.

22 The denominator cannot be equal to zero. Hence: $x^2 - 9 \neq 0 \Rightarrow x^2 \neq 9 \Rightarrow D(f) = \{x \in \mathbb{R} | x \neq \pm 3\}$.

23 In this question we have a combination of a surd with an even exponent and a denominator which cannot be equal to zero. Hence: $x \neq 0,\ \frac{1}{x^2} - 1 \geqslant 0 \Rightarrow 1 - x^2 \geqslant 0 \Rightarrow D(f) = \{x \in \mathbb{R} | -1 \leqslant x \leqslant 1, x \neq 0\}$.

24 A linear equation $ax + by = c$ represents a function only if $b \neq 0$, because, in that case, the equation represents a vertical line, that is, not a graph of a function (does not pass a vertical line test).

25 **a)** **i)** $h(21) = \sqrt{21 - 4} = \sqrt{17}$ **ii)** $h(53) = \sqrt{53 - 4} = \sqrt{49} = 7$

iii) $h(4) = \sqrt{4 - 4} = 0$

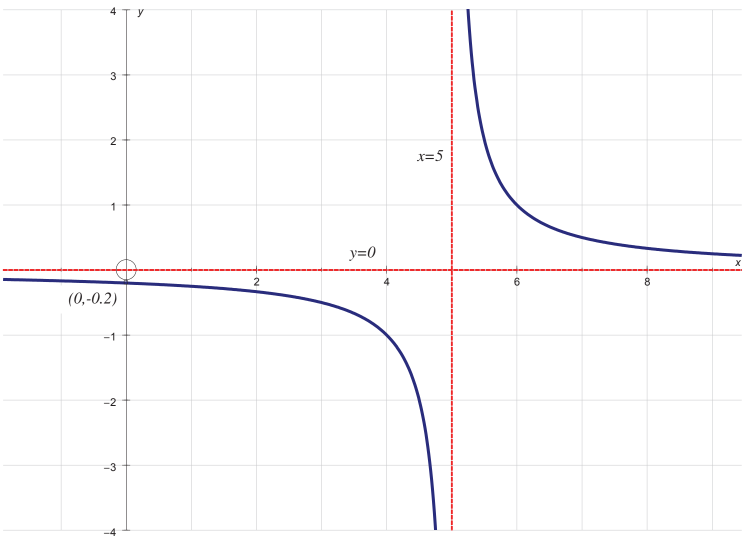
b) Since the radical must be non-negative, the function is undefined for negative values of the radical. Hence: $x - 4 < 0 \Rightarrow x < 4$.

c) $D(h) = \{x \in \mathbb{R} | x \geqslant 4\}$, as we can see in the previous part. Since the determinant of the surd is positive, the range is $R(h) = \{y \in \mathbb{R} | y \geqslant 0\}$.

In questions 26–30, the asymptotes are drawn in red and the graphs in blue. The values from the domain can always be found on the x-axis, whilst the values from the range are on the y-axis.

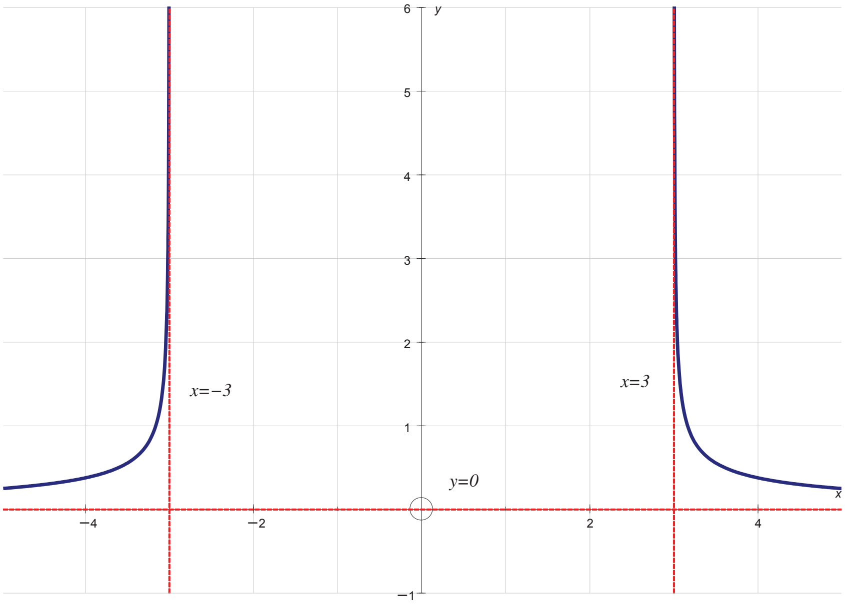
26 **a)** $D(f) = \{x \in \mathbb{R} | x \neq 5\}$, $R(f) = \{y \in \mathbb{R} | y \neq 0\}$

b)



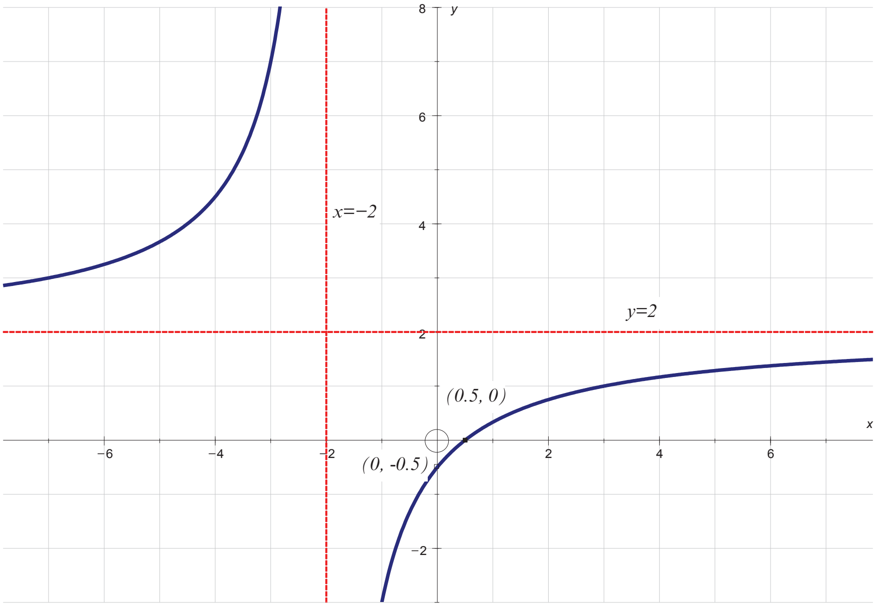
27 a) $D(g) = \{x \in \mathbb{R} | x < -3 \text{ or } x > 3\}$, $R(g) = \{y \in \mathbb{R} | y > 0\}$

b)



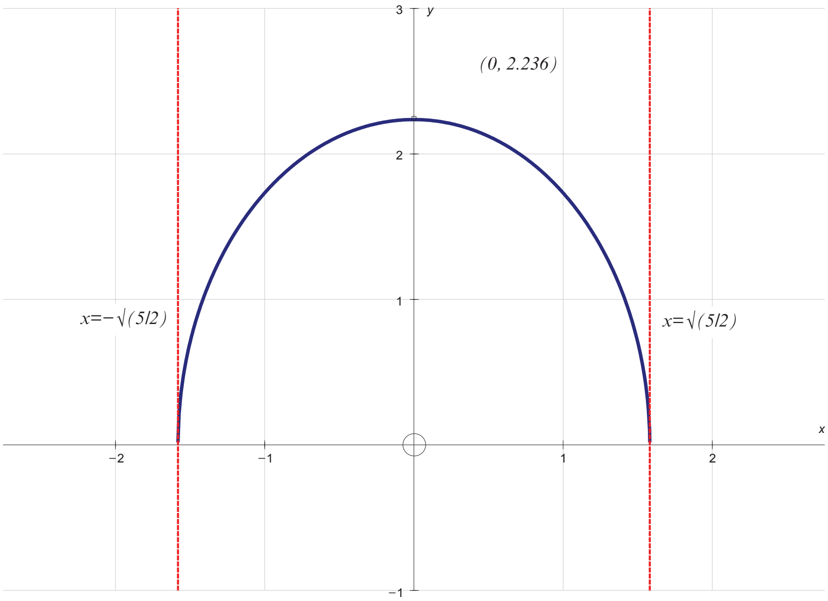
28 a) $D(h) = \{x \in \mathbb{R} | x \neq -2\}$, $R(h) = \{y \in \mathbb{R} | y \neq 2\}$

b)



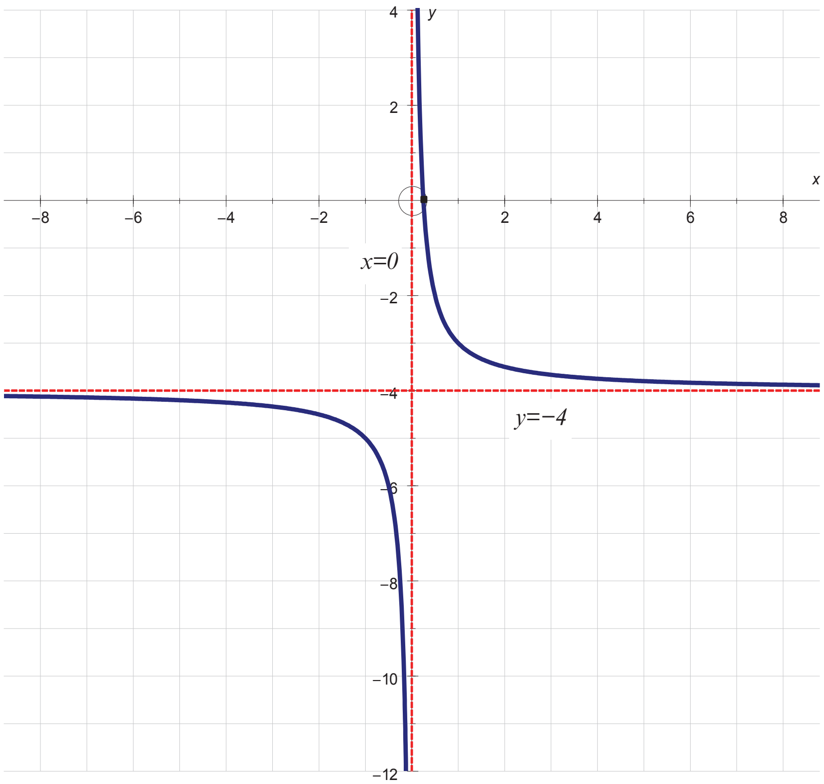
29 **a)** $D(p)=\left\{x\in\mathbb{R}\middle|-\sqrt{\frac{5}{2}}<x<\sqrt{\frac{5}{2}}\right\}$, $R(p)=\left\{y\in\mathbb{R}\middle|0\leqslant y\leqslant\sqrt{5}\right\}$

b)



30 **a)** $D(f)=\{x\in\mathbb{R}|\ x\neq 0\}$, $R(f)=\{y\in\mathbb{R}|\ y\neq -4\}$

b)



Exercise 2.2

Solution Paper 1 type

1 Let $f(x) = 2x$ and $g(x) = \dfrac{1}{x-3}$, $x \neq 3$.

a i $(f \circ g)(5) = f(g(5)) = f\left(\dfrac{1}{5-3}\right) = f\left(\dfrac{1}{2}\right) = 2 \times \dfrac{1}{2} = 1$

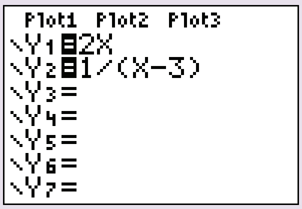
ii $(g \circ f)(5) = g(f(5)) = g(2 \times 5) = g(10) = \dfrac{1}{10-3} = \dfrac{1}{7}$

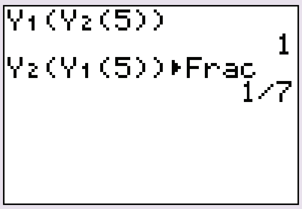
b i $(f \circ g)(x) = f(g(x)) = f\left(\dfrac{1}{x-3}\right) = 2 \times \dfrac{1}{x-3} = \dfrac{2}{x-3}$, $x \neq 3$

ii $(g \circ f)(x) = g(f(x)) = g(2x) = \dfrac{1}{2x-3}$, $x \neq \dfrac{3}{2}$

Solution Paper 2 type

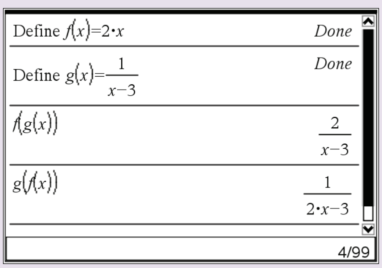
1 We can use a calculator to solve part **a**.





Note: Part **b** can also be solved by using a calculator which has symbolic manipulation features, but, as yet, these calculators are not permitted for the IB exams, although they are allowed for the Internal Assessment.

The following demonstration of the solution is done using a TI-Nspire CAS calculator.



Solution Paper 1 type

2 Let $f(x) = 2x - 3$ and $g(x) = 2 - x^2$.

a $(f \circ g)(0) = f(g(0)) = f\left(2 - 0^2\right) = f(2) = 2 \times 2 - 3 = 1$

b $(g \circ f)(0) = g(f(0)) = g(2 \times 0 - 3) = g(-3) = 2 - (-3)^2 = 2 - 9 = -7$

c $(f \circ f)(4) = f(f(4)) = f(2 \times 4 - 3) = f(5) = 2 \times 5 - 3 = 7$

d $(g \circ g)(-3) = g(g(-3)) = g\left(2 - (-3)^2\right) = g(-7) = 2 - (-7)^2 = -47$

e $(f \circ g)(-1) = f(g(-1)) = f\left(2 - (-1)^2\right) = f(1) = 2 \times 1 - 3 = -1$

f $(g \circ f)(-3) = g(f(-3)) = g(2 \times (-3) - 3) = g(-9) = 2 - (-9)^2 = 2 - 81 = -79$

g $(f \circ g)(x) = f(g(x)) = f(2 - x^2) = 2 \times (2 - x^2) - 3 = 4 - 2x^2 - 3 = 1 - 2x^2$

h $(g \circ f)(x) = g(f(x)) = g(2x - 3) = 2 - (2x - 3)^2 = 2 - (4x^2 - 2 \times 2x \times 3 + 3^2)$

$= 2 - 4x^2 + 12x - 9 = -4x^2 + 12x - 7$

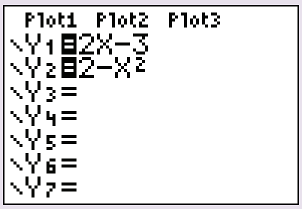
i $(f \circ f)(x) = f(f(x)) = f(2x - 3) = 2(2x - 3) - 3 = 4x - 6 - 3 = 4x - 9$

j $(g \circ g)(x) = g(g(x)) = g(2 - x^2) = 2 - (2 - x^2)^2 = 2 - (2^2 - 2 \times 2 \times x^2 + (x^2)^2)$

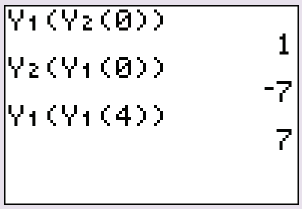
$= 2 - 4 + 4x^2 - x^4 = -2 + 4x^2 - x^4 = -x^4 + 4x^2 - 2$

Solution Paper 2 type

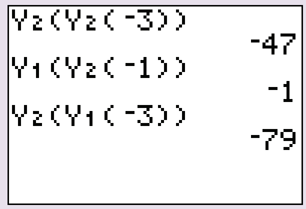
2 For parts **a**–**f**, we are going to show the solutions on an allowed calculator model. Parts **g**–**j** are done using a model that is, at present, not permitted for the IB courses.



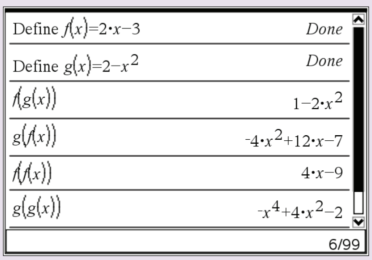
a–c



d–f



g–j



Note: Not permitted for IB exams.

3 $f(x) = 4x - 1, \; g(x) = 2 + 3x$

$(f \circ g)(x) = f(g(x)) = f(2 + 3x) = 4 \times (2 + 3x) - 1 = 12x + 7, \; D(f \circ g) = \mathbb{R}$

$(g \circ f)(x) = g(f(x)) = g(4x - 1) = 2 + 3 \times (4x - 1) = 12x - 1, \; D(g \circ f) = \mathbb{R}$

4 $f(x) = x^2 + 1, \; g(x) = -2x$

$(f \circ g)(x) = f(g(x)) = f(-2x) = (-2x)^2 + 1 = 4x^2 + 1, \; D(f \circ g) = \mathbb{R}$

$(g \circ f)(x) = g(f(x)) = g(x^2 + 1) = -2 \times (x^2 + 1) = -2x^2 - 2, \; D(g \circ f) = \mathbb{R}$

5 $f(x) = \sqrt{x + 1}, \; x \geqslant -1; \; g(x) = 1 + x^2$

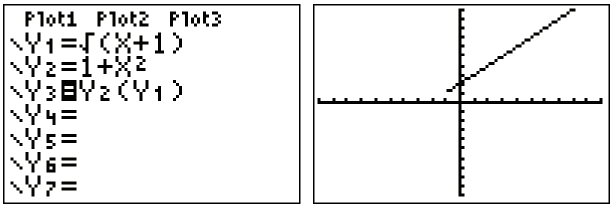
$(f \circ g)(x) = f(g(x)) = f(1 + x^2) = \sqrt{(1 + x^2) + 1} = \sqrt{x^2 + 2}$

Since the expression under the square root always has a positive value, $x^2 + 2 > 0 \Rightarrow D(f \circ g) = \mathbb{R}$.

$(g \circ f)(x) = g(f(x)) = g(\sqrt{x + 1}) = 1 + (\sqrt{x + 1})^2 = 1 + x + 1 = x + 2, \; x \geqslant -1 \Rightarrow$

$D(g \circ f) = \{x \in \mathbb{R} : x \geqslant -1\}$

Note: In order to have the composition, we need to be able to calculate the value of *f*(*x*) first; therefore, even though the composite function is linear, we cannot include the points where the function *f* is not defined. To verify this we are going to graph the composite function.



6 $f(x)=\dfrac{2}{x+4}, x\neq -4; g(x)=x-1$

$(f\circ g)(x)=f(g(x))=f(x-1)=\dfrac{2}{(x-1)+4}=\dfrac{2}{x+3}$

Since the denominator cannot be equal to zero, we get

$x+3\neq 0 \Rightarrow x\neq -3 \Rightarrow D(f\circ g)=\{x\in\mathbb{R}: x\neq -3\}$.

$(g\circ f)(x)=g(f(x))=g\left(\dfrac{2}{x+4}\right)=\dfrac{2}{x+4}-1=\dfrac{2-x-4}{x+4}=\dfrac{-x-2}{x+4}=-\dfrac{x+2}{x+4}$

The denominator, just as in the domain of the function *f*, cannot be equal to zero; therefore, $D(g\circ f)=\{x\in\mathbb{R}: x\neq -4\}$.

7 $f(x)=3x+5, g(x)=\dfrac{x-5}{3}$

$(f\circ g)(x)=f(g(x))=f\left(\dfrac{x-5}{3}\right)=\cancel{3}\times\dfrac{x-5}{\cancel{3}}+5=x-5+5=x, D(f\circ g)=\mathbb{R}$

$(g\circ f)(x)=g(f(x))=g(3x+5)=\dfrac{(3x+\cancel{5})-\cancel{5}}{3}=\dfrac{\cancel{3}x}{\cancel{3}}=x, D(g\circ f)=\mathbb{R}$

8 $f(x)=x^2-2x, g(x)=-x^2-2x \Rightarrow (f\circ g)(x)=f(g(x))=f(-x^2-2x)=(-x^2-2x)^2-2(-x^2-2x)$

$=x^4+4x^3+4x^2+2x^2+4x=x^4+4x^3+6x^2+4x \Rightarrow D(f\circ g)=\mathbb{R}$

$(g\circ f)(x)=g(f(x))=g(x^2-2x)=-(x^2-2x)^2-2(x^2-2x)$

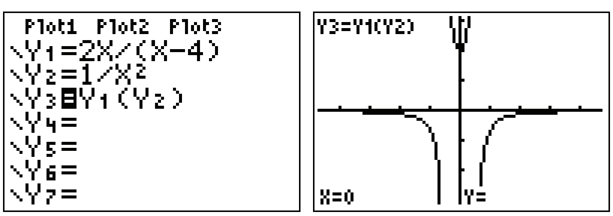
$=-x^4+4x^3-4x^2-2x^2+4x=-x^4+4x^3-6x^2+4x \Rightarrow D(g\circ f)=\mathbb{R}$

Note that in both cases we have polynomial functions, and therefore there are no restrictions on the domains.

9 $f(x)=\dfrac{2x}{4-x}, x\neq 4; g(x)=\dfrac{1}{x^2}, x\neq 0 \Rightarrow$

$(f\circ g)(x)=f(g(x))=f\left(\dfrac{1}{x^2}\right)=\dfrac{\frac{2}{x^2}}{4-\frac{1}{x^2}}=\dfrac{2}{4x^2-1}\Rightarrow D(f\circ g)=\left\{x\in\mathbb{R}|\, x\neq 0, \pm\dfrac{1}{2}\right\}$

Note that we had to exclude 0 because function *g* is not defined at 0.



$(g\circ f)(x)=g(f(x))=g\left(\dfrac{2x}{4-x}\right)=\dfrac{(4-x)^2}{4x^2}\Rightarrow D(g\circ f)=\{x\in\mathbb{R}|\, x\neq 0, 4\}$

Note that we had to exclude 4 because function *f* is not defined at 4. Try to trace the value at 4 on the composite graph yourself.

10 $f(x) = 2 - x^3,\ g(x) = \sqrt[3]{1-x^2} \Rightarrow$

$(f \circ g)(x) = f(g(x)) = f\left(\sqrt[3]{1-x^2}\right) = 2 - \left(\sqrt[3]{1-x^2}\right)^3 = 1 + x^2 \Rightarrow D(f \circ g) = \mathbb{R}$

$(g \circ f)(x) = g(f(x)) = g\left(2 - x^3\right) = \sqrt[3]{1-\left(2-x^3\right)^2} = \sqrt[3]{4x^3 - 3 - x^6} \Rightarrow D(g \circ f) = \mathbb{R}$

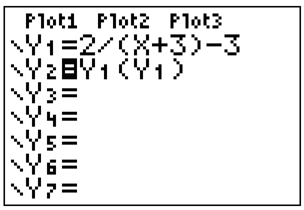
Again in this question, we did not have a problem with the domain since the odd root of any real number can be taken.

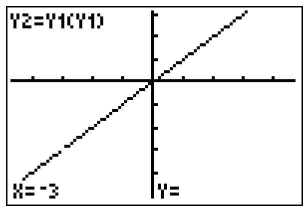
11 $f(x) = g(x) = \dfrac{2}{x+3} - 3,\ x \neq -3 \Rightarrow$

$(f \circ f)(x) = f\left(\dfrac{2}{x+3} - 3\right) = \dfrac{2}{\left(\dfrac{2}{x+3} - 3\right) + 3} - 3 = \dfrac{2}{\dfrac{2}{x+3}} - 3 = x + 3 - 3 = x \Rightarrow$

$D(f \circ f) = \{x \in \mathbb{R} | \ x \neq -3\}$

Since the functions are identical, it is sufficient to simply do this one composition. Again note that even though the composite function is the identity function, the domain has to be restricted by the value where the function itself is not defined.





12 $f(x) = \dfrac{x}{x-1},\ x \neq 1;\ g(x) = x^2 - 1 \Rightarrow$

$(f \circ g)(x) = f(g(x)) = f\left(x^2 - 1\right) = \dfrac{x^2-1}{x^2-1-1} = \dfrac{x^2-1}{x^2-2} \Rightarrow D(f \circ g) = \left\{x \in \mathbb{R} | \ x \neq \pm\sqrt{2}\right\}$

$(g \circ f)(x) = g(f(x)) = g\left(\dfrac{x}{x-1}\right) = \left(\dfrac{x}{x-1}\right)^2 - 1 = \dfrac{x^2 - x^2 + 2x - 1}{(x-1)^2} = \dfrac{2x-1}{(x-1)^2} \Rightarrow$

$D(g \circ f) = \{x \in \mathbb{R} | \ x \neq 1\}$

Note: The composition of a polynomial function and the rational function will always have a domain that is equal to the domain of the rational function, whilst the domain changes when we have the other composition.

13 $g(x) = \sqrt{x-1},\ x \geqslant 1;\ h(x) = 10 - x^2 \Rightarrow$

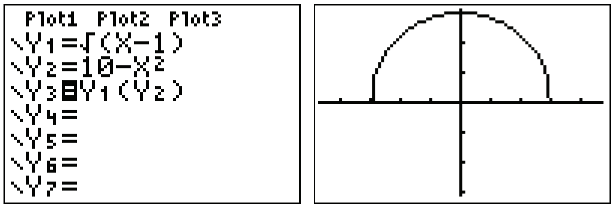
a) $(g \circ h)(x) = g(h(x)) = g\left(10 - x^2\right) = \sqrt{10 - x^2 - 1} = \sqrt{9 - x^2}$

To find the domain, we need to look at all the non-negative values under the square root.

$9 - x^2 \geqslant 0 \Rightarrow -3 \leqslant x \leqslant 3 \Rightarrow D(g \circ h) = [-3, 3]$

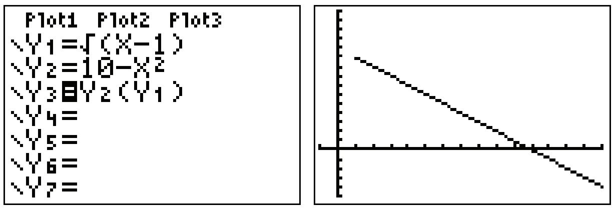
Regarding the range, we notice that the determinant of the square root is positive and that the quadratic expression achieves a minimum and a maximum value on the domain. The minimum value occurs at $x = \pm 3 \Rightarrow y = \sqrt{9 - (\pm 3)^2} = 0$, whilst the maximum occurs at the midpoint where $x = 0 \Rightarrow y = \sqrt{9 - 0} = 3$. So, the range is $R(g \circ h) = [0, 3]$.

To support our solutions, let's look at the graph.



b) $(h \circ g)(x) = h(g(x)) = h\left(\sqrt{x-1}\right) = 10 - \left(\sqrt{x-1}\right)^2 = 10 - (x-1) = 11 - x$

In this problem, since the inner function is irrational, we will have the same domain, and therefore $D(h \circ g) = \{x \in \mathbb{R} | \, x \geqslant 1\}$. Since we are not taking all the real values, just those that are greater than or equal to 1, the range will not be the whole set of real numbers, but just that part we get by taking those values from the domain. Therefore, $R(h \circ g) = \left]-\infty, 10\right]$.



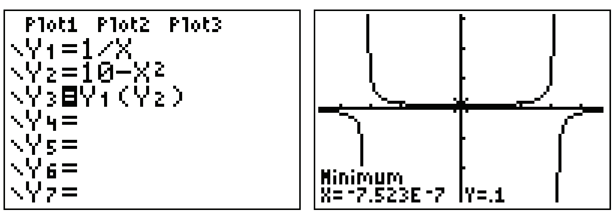
14 $f(x) = \dfrac{1}{x}, x \neq 0; \; g(x) = 10 - x^2 \Rightarrow$

a) $(f \circ g)(x) = f(g(x)) = f\left(10 - x^2\right) = \dfrac{1}{10 - x^2}$

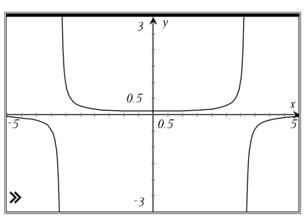
To find the domain, we need to look at all the non-zero values of the denominator.

$10 - x^2 \neq 0 \Rightarrow x \neq \pm\sqrt{10} \Rightarrow D(f \circ g) = \left\{x \in \mathbb{R} | \, x \neq \pm\sqrt{10}\right\}$

Regarding the range, we notice that the quadratic expression can be positive, $\left(-\sqrt{10} < x < \sqrt{10}\right)$, or negative, $\left(x < -\sqrt{10} \text{ or } x > \sqrt{10}\right)$. For the positive values, the expression achieves a maximum value on the domain; therefore, its reciprocal value will be a minimum of the composite function. The minimum value occurs at the point where $x = 0 \Rightarrow y = \dfrac{1}{10}$. All the negative values will be achieved, so the range is $R(f \circ g) = \left]-\infty, 0\right[\cup \left[\dfrac{1}{10}, +\infty\right[$.

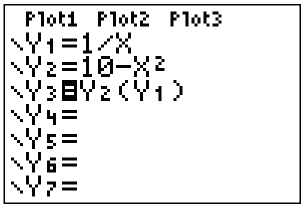


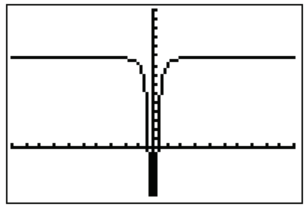
Notice that the calculator, due to numerical approximation, does not calculate the exact value of x at the minimum as 0 but -7.523×10^{-7}, which is very close to it. If we use a better window, zooming in on the region around the origin, we will get a better picture.



b) $(g \circ f)(x) = g(f(x)) = g\left(\dfrac{1}{x}\right) = 10 - \left(\dfrac{1}{x}\right)^2 = 10 - \dfrac{1}{x^2}$

In this problem, since the inner function is rational, we will have the same domain, and therefore $D(g \circ f) = \{x \in \mathbb{R} | \ x \neq 0\}$. Regarding the range, we notice that we square the value of x and, as such, we will always subtract a positive value from 10; therefore, we will never exceed 10. On the other hand, when we take extremely large values of x, the value of the function approaches 10; therefore, we can say that $y = 10$ is the horizontal asymptote. Hence: $R(h \circ g) =]-\infty, 10[$.





In questions 15–22, we have to recognize the inner function and the outer function. Sometimes this process will not give us a unique answer, especially in those cases where the composition can be found with more than two functions.

15 $h(x) = x + 3, \ g(x) = x^2 \Rightarrow f(x) = g(h(x)) = (x+3)^2$

16 $h(x) = x - 5, \ g(x) = \sqrt{x} \Rightarrow f(x) = g(h(x)) = \sqrt{x-5}$

17 $h(x) = \sqrt{x}, \ g(x) = 7 - x \Rightarrow f(x) = g(h(x)) = 7 - \sqrt{x}$

18 $h(x) = x + 3, \ g(x) = \dfrac{1}{x} \Rightarrow f(x) = g(h(x)) = \dfrac{1}{x+3}$

19 $h(x) = x + 1, \ g(x) = 10^x \Rightarrow f(x) = g(h(x)) = 10^{x+1}$

20 $h(x) = x - 9, \ g(x) = \sqrt[3]{x} \Rightarrow f(x) = g(h(x)) = \sqrt[3]{x-9}$

21 $h(x) = x^2 - 9, \ g(x) = |x| \Rightarrow f(x) = g(h(x)) = |x^2 - 9|$

22 $h(x) = x - 5, \ g(x) = \dfrac{1}{\sqrt{x}} \Rightarrow f(x) = g(h(x)) = \dfrac{1}{\sqrt{x-5}}$

Notice that this last composition could have been found as the composition of three functions:

$$h(x) = x - 5, \ g(x) = \sqrt{x}, \ f(x) = \frac{1}{x} \Rightarrow f(g(h(x))) = \frac{1}{\sqrt{x-5}}$$

$\left(\text{Alternative solution: } h(x) = \sqrt{x-5}, \ g(x) = \dfrac{1}{x}\right)$

23
$$\left.\begin{array}{l} f(x) = \sqrt{x} \Rightarrow D(f) = \{x \in \mathbb{R} | \ x \geqslant 0\} \\ g(x) = x^2 + 1 \Rightarrow D(g) = \mathbb{R} \end{array}\right\} \Rightarrow f(g(x)) = f(x^2+1) = \sqrt{x^2+1} \Rightarrow D(f \circ g) = \mathbb{R}$$

24
$$\left.\begin{array}{l} f(x) = \dfrac{1}{x} \Rightarrow D(f) = \{x \in \mathbb{R} | \ x \neq 0\} \\ g(x) = x + 3 \Rightarrow D(g) = \mathbb{R} \end{array}\right\} \Rightarrow f(g(x)) = f(x+3) = \frac{1}{x+3} \Rightarrow D(f \circ g) = \{x \in \mathbb{R} | \ x \neq -3\}$$

25
$$\left.\begin{array}{l} f(x) = \dfrac{3}{x^2 - 1} \Rightarrow D(f) = \{x \in \mathbb{R} | \ x \neq \pm 1\} \\ g(x) = x + 1 \Rightarrow D(g) = \mathbb{R} \end{array}\right\} \Rightarrow f(g(x)) = f(x+1) = \frac{3}{(x+1)^2 - 1} = \frac{3}{x(x+2)}$$

$$\Rightarrow D(f \circ g) = \{x \in \mathbb{R} | \ x \neq 0, -2\}$$

Notice that in the denominator we had a difference of two squares which we factorized by using the difference of two squares formula.

26 $\left.\begin{aligned} f(x) = 2x + 3 \Rightarrow D(f) = \mathbb{R} \\ g(x) = \frac{x}{2} \Rightarrow D(g) = \mathbb{R} \end{aligned}\right\} \Rightarrow f(g(x)) = f\left(\frac{x}{2}\right) = \cancel{2}\left(\frac{x}{\cancel{2}}\right) + 3 = x + 3 \Rightarrow D(f \circ g) = \mathbb{R}$

Exercise 2.3

1 **a)** $f(2) = -5 \Rightarrow f^{-1}(-5) = 2$ **b)** $f^{-1}(6) = 10 \Rightarrow f(10) = 6$

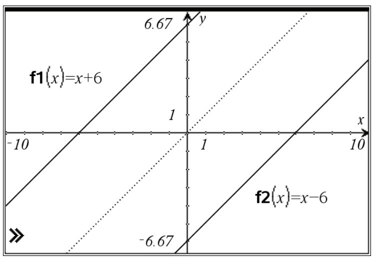
2 **a)** $f(-1) = 13 \Rightarrow f^{-1}(13) = -1$ **b)** $f^{-1}(b) = a \Rightarrow f(a) = b$

3 $g(x) = 5 \Rightarrow 3x - 7 = 5 \Rightarrow 3x = 12 \Rightarrow x = 4 \Rightarrow g^{-1}(5) = 4$

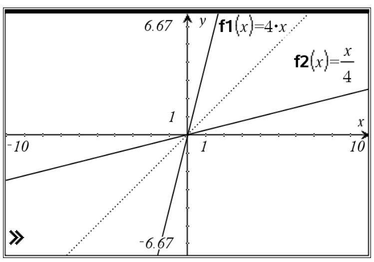
4 $h(x) = -12 \Rightarrow x^2 - 8x = -12 \Rightarrow x^2 - 8x + 12 = 0 \Rightarrow$ ~~$x_1 = 2$~~, $x_2 = 6 \Rightarrow h^{-1}(-12) = 6$

Since the domain of h was given as $x \geqslant 4$ we discounted one of the solutions.

5 $(f \circ g)(x) = f(g(x)) = f(x - 6) = (x - 6) + 6 = x$ **and**
$(g \circ f)(x) = g(f(x)) = g(x + 6) = (x + 6) - 6 = x.$

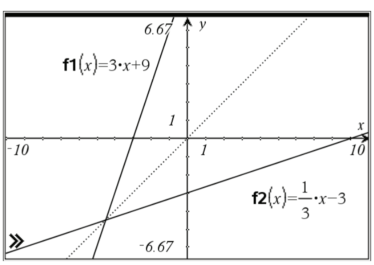


6 $(f \circ g)(x) = f(g(x)) = f\left(\frac{x}{4}\right) = 4 \times \frac{x}{4} = x$ **and** $(g \circ f)(x) = g(f(x)) = g(4x) = \frac{4x}{4} = x.$

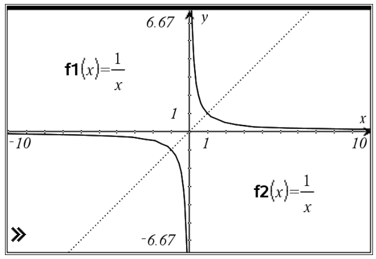


7 $(f \circ g)(x) = f(g(x)) = f\left(\frac{1}{3}x - 3\right) = 3\left(\frac{1}{3}x - 3\right) + 9 = x - 9 + 9 = x$ **and**

$(g \circ f)(x) = g(f(x)) = g(3x + 9) = \frac{1}{3}(3x + 9) - 3 = x + 3 - 3 = x.$

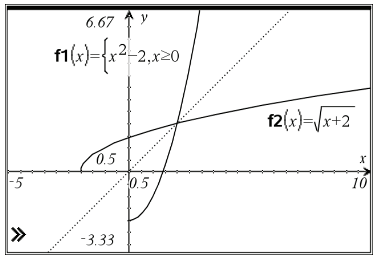


8 $(f \circ g)(x) = f(g(x)) = f\left(\frac{1}{x}\right) = \frac{1}{\frac{1}{x}} = x$ **and** $(g \circ f)(x) = g(f(x)) = g\left(\frac{1}{x}\right) = \frac{1}{\frac{1}{x}} = x, x \neq 0.$



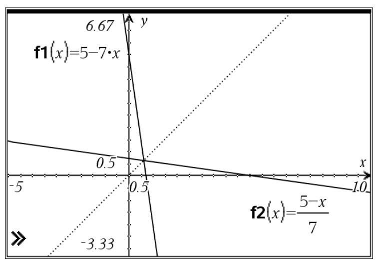
9 $(f \circ g)(x) = f(g(x)) = f\left(\sqrt{x+2}\right) = \left(\sqrt{x+2}\right)^2 - 2 = x + 2 - 2 = x$ **and**

$(g \circ f)(x) = g(f(x)) = g\left(x^2 - 2\right) = \sqrt{\left(x^2 - 2\right) + 2} = |x| = x$, since $x \geqslant 0$.



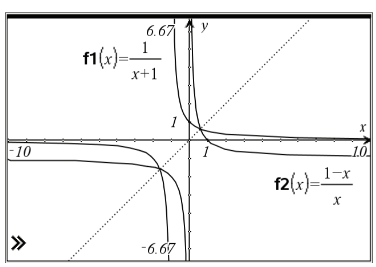
10 $(f \circ g)(x) = f(g(x)) = f\left(\dfrac{5-x}{7}\right) = 5 - 7 \times \dfrac{5-x}{7} = 5 - 5 + x = x$ **and**

$(g \circ f)(x) = g(f(x)) = g(5 - 7x) = \dfrac{5 - (5 - 7x)}{7} = \dfrac{7x}{7} = x$.



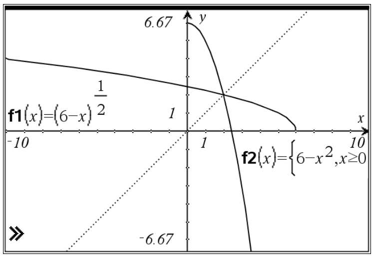
11 $(f \circ g)(x) = f(g(x)) = f\left(\dfrac{1-x}{x}\right) = \dfrac{1}{1 + \dfrac{1-x}{x}} = \dfrac{1}{\dfrac{x + 1 - x}{x}} = \dfrac{x}{1} = x$ **and**

$(g \circ f)(x) = g(f(x)) = g\left(\dfrac{1}{1+x}\right) = \dfrac{1 - \dfrac{1}{1+x}}{\dfrac{1}{1+x}} = \dfrac{\dfrac{1+x-1}{\cancel{1+x}}}{\dfrac{1}{\cancel{1+x}}} = x,\ x \neq -1$ and $x \neq 0$.



12 $(f \circ g)(x) = f(g(x)) = f\left(6 - x^2\right) = \left(6 - \left(6 - x^2\right)\right)^{\frac{1}{2}} = |x| = x$, since $x \geqslant 0$ **and**

$(g \circ f)(x) = g(f(x)) = g\left((6 - x)^{\frac{1}{2}}\right) = 6 - \left((6 - x)^{\frac{1}{2}}\right)^2 = 6 - 6 + x = x,\ x \leqslant 6$.

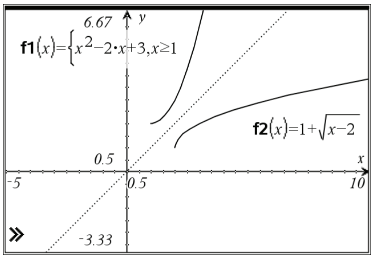


13 $(f\circ g)(x) = f(g(x)) = f\left(1+\sqrt{x-2}\right) = \left(1+\sqrt{x-2}\right)^2 - 2\left(1+\sqrt{x-2}\right)+3$

$= 1 + \cancel{2\sqrt{x-2}} + x - 2 - 2 - \cancel{2\sqrt{x-2}} + 3 = x$ **and**

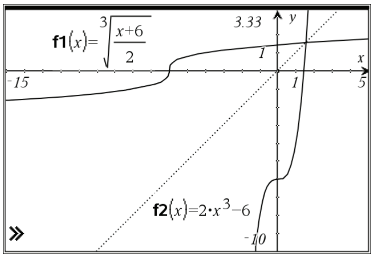
$(g\circ f)(x) = g(f(x)) = g\left(x^2-2x+3\right) = 1+\sqrt{\left(x^2-2x+3\right)-2} = 1+\sqrt{\left(x^2-1\right)^2}$

$= 1 + |x-1| = 1 + x - 1 = x$, since $x \geqslant 1$.



14 $(f\circ g)(x) = f(g(x)) = f\left(2x^3-6\right) = \sqrt[3]{\dfrac{2x^3-6+6}{2}} = \sqrt[3]{x^3} = x$ **and**

$(g\circ f)(x) = g(f(x)) = g\left(\sqrt[3]{\dfrac{x+6}{2}}\right) = 2\left(\sqrt[3]{\dfrac{x+6}{2}}\right)^3 - 6 = 2\times\dfrac{x+6}{2} - 6 = x+6-6 = x.$



15 $x = 2y-3 \Rightarrow x+3 = 2y \Rightarrow f^{-1}(x) = \dfrac{x+3}{2}, D(f^{-1}) = \mathbb{R}$

16 $x = \dfrac{y+7}{4} \Rightarrow 4x = y+7 \Rightarrow f^{-1}(x) = 4x-7, D(f^{-1}) = \mathbb{R}$

17 $x = \sqrt{y} \Rightarrow f^{-1}(x) = x^2, D(f^{-1}) = \{x\in\mathbb{R} | x \geqslant 0\}$

18 $x = \dfrac{1}{y+2} \Rightarrow y+2 = \dfrac{1}{x} \Rightarrow f^{-1}(x) = \dfrac{1}{x} - 2, D(f^{-1}) = \mathbb{R} - \{0\}$

19 $x = 4 - y^2 \Rightarrow x - 4 = y^2 \Rightarrow f^{-1}(x) = \sqrt{x-4}, D(f^{-1}) = \{x\in\mathbb{R} | x \geqslant 4\}$

20 $x = \sqrt{y-5} \Rightarrow x^2 = y-5 \Rightarrow f^{-1}(x) = x^2+5, D(f^{-1}) = \{x\in\mathbb{R} | x \geqslant 0\}$

21 $x = ay+b \Rightarrow x-b = ay \Rightarrow f^{-1}(x) = \dfrac{x-b}{a}, D(f^{-1}) = \mathbb{R}$

22 $x+1=y^2+2y+1 \Rightarrow (y+1)^2 = x+1 \Rightarrow y+1=\sqrt{x+1} \Rightarrow y=\sqrt{x+1}-1 \Rightarrow$

$f^{-1}(x)=\sqrt{x+1}-1, D(f^{-1})=\{x\in\mathbb{R}|\, x\geqslant -1\}$

23 $x=\dfrac{y^2-1}{y^2+1} \Rightarrow xy^2+x=y^2-1 \Rightarrow x+1=y^2(1-x) \Rightarrow y=\sqrt{\dfrac{1+x}{1-x}} \Rightarrow$

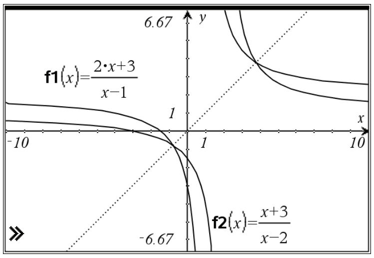
$f^{-1}(x)=\sqrt{\dfrac{1+x}{1-x}}, D(f^{-1})=[-1,1[$

24 $x=y^3+1 \Rightarrow y^3=x-1 \Rightarrow f^{-1}(x)=\sqrt[3]{x-1}, D(f^{-1})=\mathbb{R}$

25 The function *f* is bijective on the whole of its domain, $D(f)=\mathbb{R}-\{1\}$, and therefore we can find the inverse function.

$x=\dfrac{2y+3}{y-1} \Rightarrow xy-x=2y+3 \Rightarrow xy-2y=x+3 \Rightarrow y=\dfrac{x+3}{x-2} \Rightarrow$

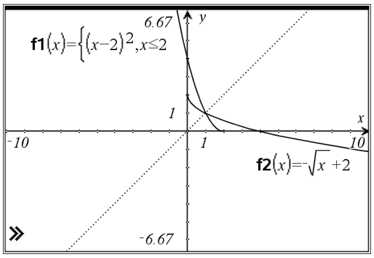
$f^{-1}(x)=\dfrac{x+3}{x-2}, D(f^{-1})=\mathbb{R}-\{2\}$

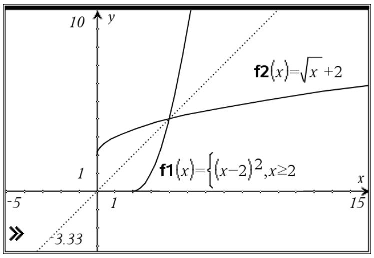


26 The function *f* is not injective so we need to restrict it either on $x\leqslant 2$ or $x\geqslant 2$ and then we can find the inverse functions for each restriction.

$x=(y-2)^2 \Rightarrow y-2=\pm\sqrt{x} \Rightarrow y=\pm\sqrt{x}+2$

For the first restriction, $x\leqslant 2$, the inverse function is $f^{-1}(x)=-\sqrt{x}+2, D(f^{-1})=\{x\in\mathbb{R}|\, x\geqslant 0\}$; whilst for the second restriction, $x\geqslant 2$, the inverse function is $f^{-1}(x)=\sqrt{x}+2, D(f^{-1})=\{x\in\mathbb{R}|\, x\geqslant 0\}$.

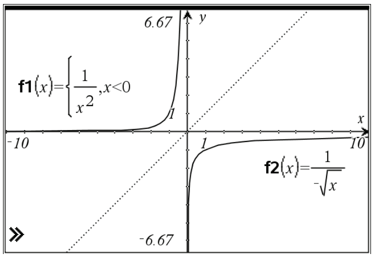


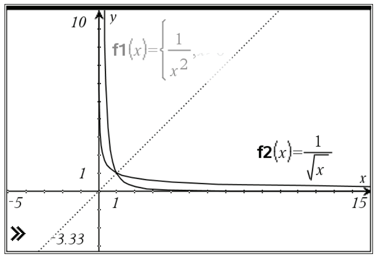


27 The function *f* is not injective so we need to restrict it either on $x<0$ or $x>0$ and then we can find the inverse functions for each restriction.

$x=\dfrac{1}{y^2} \Rightarrow y^2=\dfrac{1}{x} \Rightarrow y=\pm\dfrac{1}{\sqrt{x}}$

For the first restriction, $x<0$, the inverse function is $f^{-1}(x)=-\dfrac{1}{\sqrt{x}}, D(f^{-1})=\{x\in\mathbb{R}|\, x>0\}$; whilst for the second restriction, $x>0$, the inverse function is $f^{-1}(x)=\dfrac{1}{\sqrt{x}}, D(f^{-1})=\{x\in\mathbb{R}|\, x>0\}$.

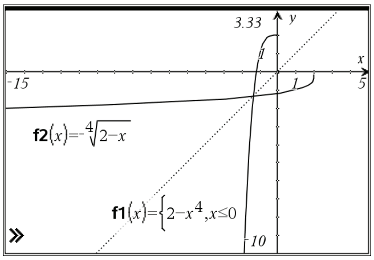


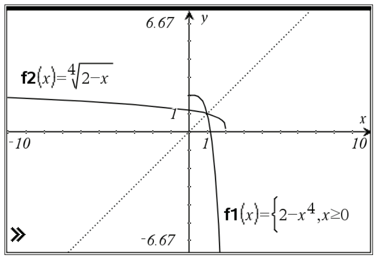


28 The function f is not injective so we need to restrict it either on $x \leqslant 0$ or $x \geqslant 0$ and then we can find the inverse functions for each restriction.

$x = 2 - y^4 \Rightarrow y^4 = 2 - x \Rightarrow y = \pm\sqrt[4]{2-x}$

For the first restriction, $x \leqslant 0$, the inverse function is $f^{-1}(x) = -\sqrt[4]{2-x}$, $D(f^{-1}) = \{x \in \mathbb{R} | x \leqslant 2\}$; whilst for the second restriction, $x \geqslant 0$, the inverse function is $f^{-1}(x) = \sqrt[4]{2-x}$, $D(f^{-1}) = \{x \in \mathbb{R} | x \leqslant 2\}$.





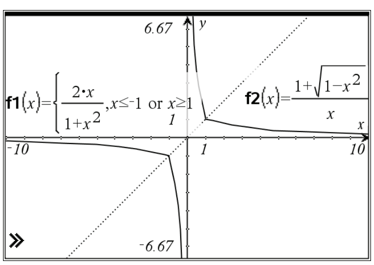
29 We will divide the whole set of real numbers into three intervals, but the domain of function f will actually have two restrictions on which the function is injective. The first restriction is $x \leqslant -1$ or $x \geqslant 1$ and the second is $-1 \leqslant x \leqslant 1$.

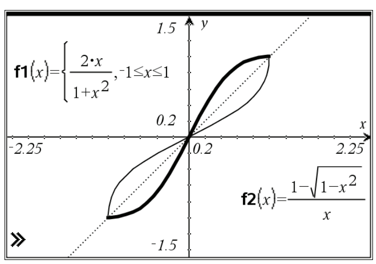
$$x = \frac{2y}{1+y^2} \Rightarrow xy^2 + x = 2y \Rightarrow y^2 - 2\frac{y}{x} = -1 \Rightarrow y^2 - 2\frac{y}{x} + \frac{1}{x^2} = \frac{1}{x^2} - 1 \Rightarrow \left(y - \frac{1}{x}\right)^2 = \frac{1}{x^2} - 1 \Rightarrow$$

$$y - \frac{1}{x} = \pm\sqrt{\frac{1}{x^2} - 1} \Rightarrow y = \frac{1}{x} \pm \sqrt{\frac{1-x^2}{x^2}}$$

Hence, the inverse functions are as follows:

For the first restriction, $x \leqslant -1$ or $x \geqslant 1$, $f^{-1}(x) = \dfrac{1+\sqrt{1-x^2}}{x}$, $D(f^{-1}) = [-1, 1] - \{0\}$; whilst for the second restriction, $-1 \leqslant x \leqslant 1$, $f^{-1}(x) = \dfrac{1-\sqrt{1-x^2}}{x}$, $D(f^{-1}) = [-1, 1] - \{0\}$.





In questions 30–35, we will find the inverse functions.

$x = y + 3 \Rightarrow x - 3 = y \Rightarrow g^{-1}(x) = x - 3$

$x = 2y - 4 \Rightarrow x + 4 = 2y \Rightarrow \dfrac{x+4}{2} = y \Rightarrow h^{-1}(x) = \dfrac{1}{2}x + 2$

30 $(g^{-1}\circ h^{-1})(5) = g^{-1}\left(h^{-1}(5)\right) = g^{-1}\left(\frac{9}{2}\right) = \frac{3}{2}$

31 $(h^{-1}\circ g^{-1})(9) = h^{-1}\left(g^{-1}(9)\right) = h^{-1}(6) = 5$

32 $(g^{-1}\circ g^{-1})(2) = g^{-1}\left(g^{-1}(2)\right) = g^{-1}(-1) = -4$

33 $(h^{-1}\circ h^{-1})(2) = h^{-1}\left(h^{-1}(2)\right) = h^{-1}(3) = \frac{7}{2}$

34 $(g^{-1}\circ h^{-1})(x) = g^{-1}\left(h^{-1}(x)\right) = g^{-1}\left(\frac{1}{2}x+2\right) = \frac{1}{2}x+2-3 = \frac{1}{2}x-1$

35 $(h^{-1}\circ g^{-1})(x) = h^{-1}\left(g^{-1}(x)\right) = h^{-1}(x-3) = \frac{1}{2}(x-3)+2 = \frac{1}{2}x+\frac{1}{2}$

36 $(g\circ h)(x) = g(h(x)) = g(2x-4) = 2x-4+3 = 2x-1 \Rightarrow$

$x = 2y-1 \Rightarrow x+1 = 2y \Rightarrow \frac{x+1}{2} = y \Rightarrow (g\circ h)^{-1}(x) = \frac{x+1}{2}$

37 $(h\circ g)(x) = h\left(g(x)\right) = h(x+3) = 2x+6-4 = 2x+2 \Rightarrow$

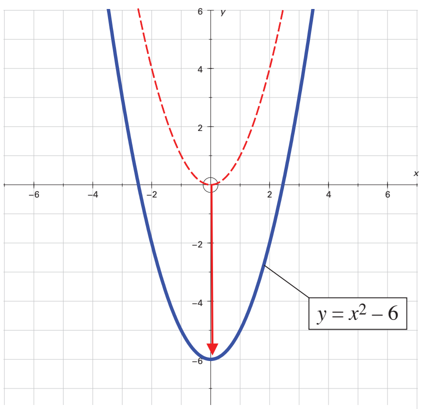
$x = 2y+2 \Rightarrow x-2 = 2y \Rightarrow \frac{x-2}{2} = y \Rightarrow (h\circ g)^{-1}(x) = \frac{x-2}{2} = \frac{x}{2}-1$

38 $(f\circ f)(x) = f\left(\frac{a}{x+b}-b\right) = \frac{a}{\frac{a}{x+b}\cancel{-b}\cancel{+b}} - b = \frac{\cancel{a}(x+b)}{\cancel{a}} - b = x+b-b = x$

Since the composition of the function by itself gives the identity function, we can conclude that the function is its own inverse.

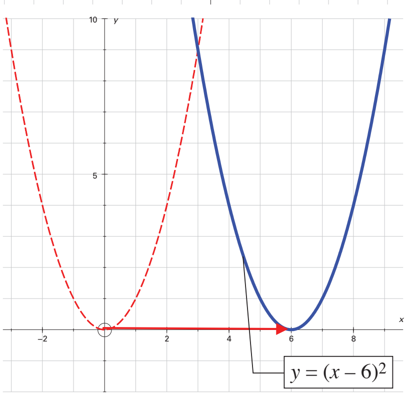
Exercise 2.4

1



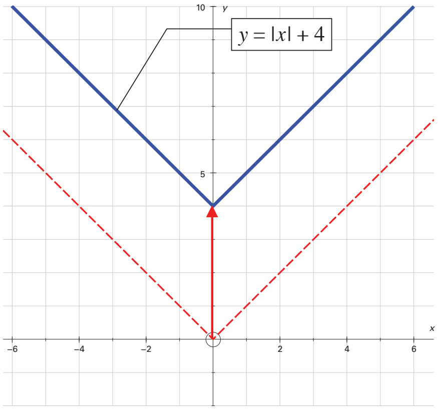
The graph of a normal parabola $y = x^2$ has been translated down six units.

2



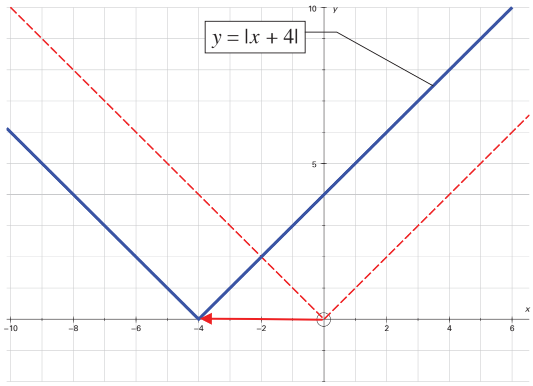
The graph of a normal parabola $y = x^2$ has been translated six units to the right.

3



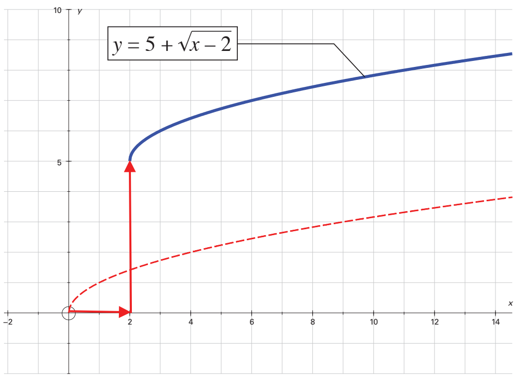
The graph of an absolute value $y = |x|$ has been translated up four units.

4



The graph of an absolute value $y = |x|$ has been translated four units to the left.

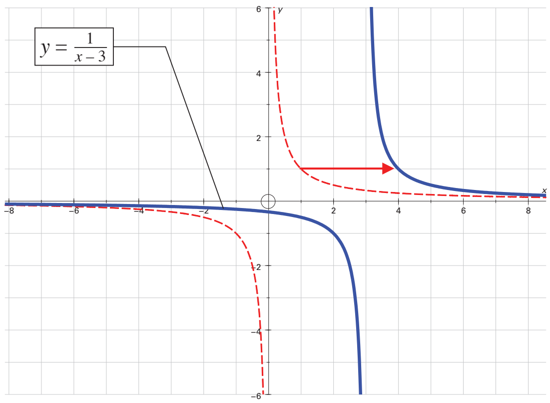
5



In this question we have two transformations: the graph of a square root $y = \sqrt{x}$ has been translated two units to the right and then translated up five units.

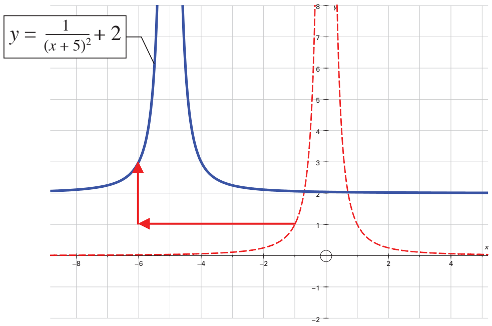
Note: We can swap these two transformations and still obtain the same graph.

6



The graph of an inverse function $y = \frac{1}{x}$ has been translated three units to the right.

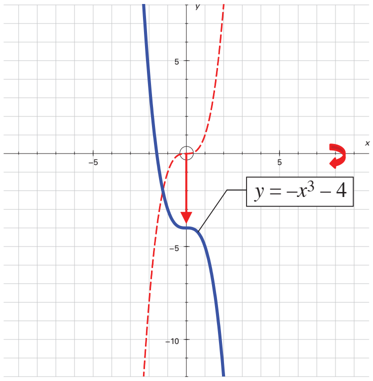
7



In this question we have two transformations: the graph of an inverse square function $y = \frac{1}{x^2}$ has been translated five units to the left and then translated up two units.

Note: We can swap these two transformations and still obtain the same graph.

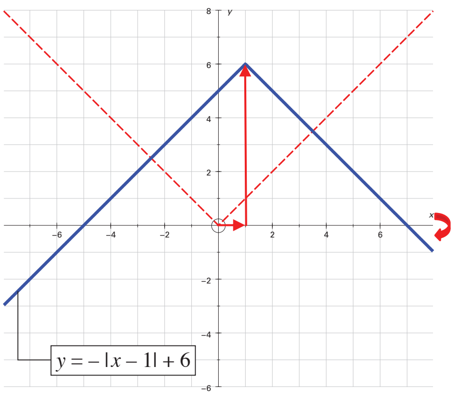
8



In this question we have two transformations: the graph of a cubic function $y = x^3$ has been reflected in the x-axis and then translated down four units.

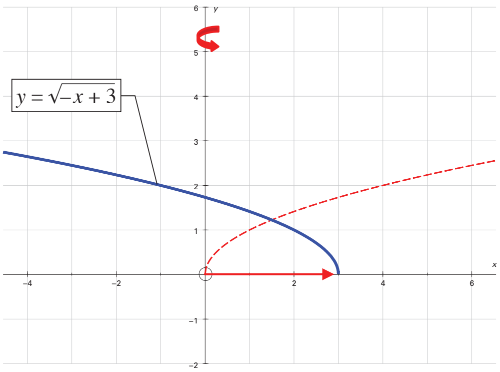
In this case we cannot swap the order of the transformations, since the translated graph is not symmetrical with respect to the origin.

9



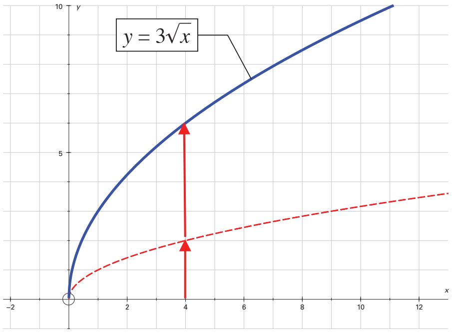
In this question we have three transformations: the graph of an absolute value $y = |x|$ has been reflected in the x-axis, then translated one unit to the right and then translated up six units.

10



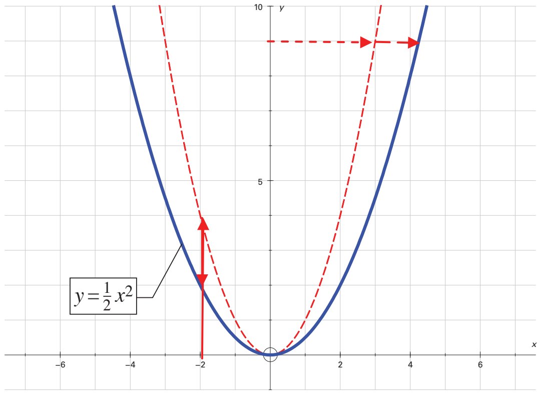
In this question we have two transformations: the graph of a square root $y = \sqrt{x}$ has been reflected in the y-axis and then translated three units to the right.

11



The graph of a square root function $y = \sqrt{x}$ has been vertically stretched by scale factor 3.

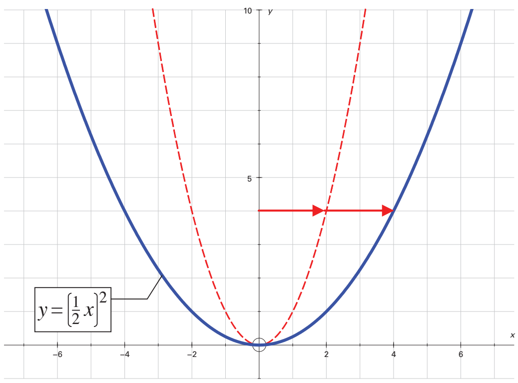
12



The graph of a normal parabola $y = x^2$ has been vertically shrunk by scale factor $\frac{1}{2}$.

Note: Since $y = \frac{1}{2}x^2 = \left(\frac{1}{\sqrt{2}}x\right)^2$, we can interpret this transformation as a horizontal stretch by scale factor $\sqrt{2}$.

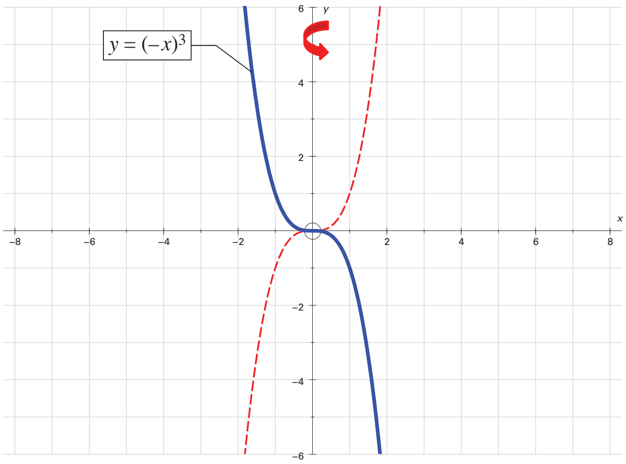
13



The graph of a normal parabola $y = x^2$ has been horizontally stretched by scale factor 2.

Note: Since $y = \left(\frac{1}{2}x\right)^2 = \frac{1}{4}x^2$, we can interpret this transformation as a vertical stretch by scale factor $\frac{1}{4}$.

14



The graph of a cubic function $y = x^3$ has been reflected in the y-axis.

Note: By using the power properties, we can write $y = (-x)^3 = -x^3$, so we can interpret this transformation as a reflection in the x-axis.

15 The normal quadratic function has been reflected in the x-axis and then translated up five units.
$y = x^2 \rightarrow y = -x^2 \rightarrow y = -x^2 + 5$

16 The square root function has been reflected in the y-axis.
$y = \sqrt{x} \rightarrow y = \sqrt{-x}$

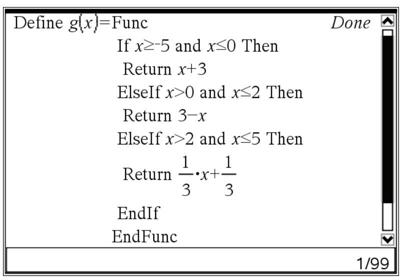
17 The absolute value function has been reflected in the x-axis and then translated one unit to the left.
$y = |x| \rightarrow y = -|x| \rightarrow y = -|x+1|$

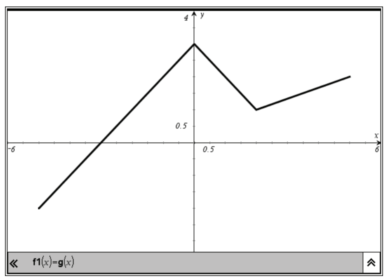
Note: The two transformations applied in reverse order produce the same result.

18 The inverse function has been translated two units to the right and then translated down three units.
$y = \frac{1}{x} \rightarrow y = \frac{1}{x-2} \rightarrow y = \frac{1}{x-2} - 3$
The same function can be written in a different form:
$y = \frac{1}{x-2} - 3 = \frac{1-3x+6}{x-2} = \frac{7-3x}{x-2}$
Again note that we can swap the order of the transformations.

19 Even though this question should be solved without using a calculator to draw the graphs, we are going to use the TI-Nspire calculator. However, for each question part, we are also going to describe the transformation so that we can sketch it without using a calculator or a computer.

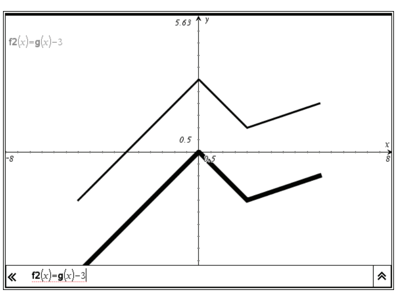
We define a piecewise function (GDC screen on the left) and the function graph is shown below right.



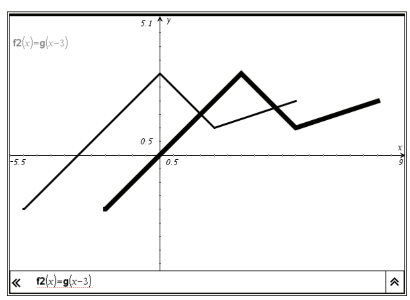


To find the graph of each transformation, we just need to input the function in the order that it is given in the question. On each diagram we will show both the original and the transformed graph. The graph of the transformed function is shown by the thicker line.

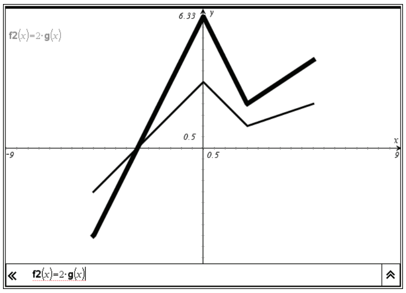
a) We translate the graph down three units.



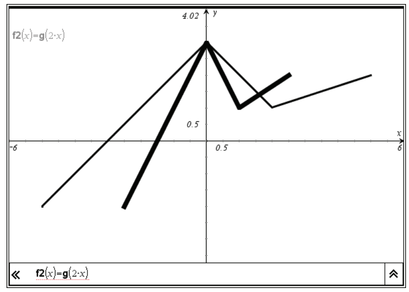
b) We translate the graph three units to the right.



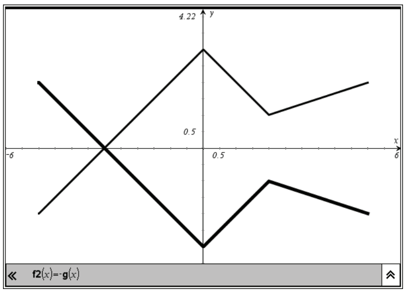
c) We vertically stretch the graph by scale factor 2.



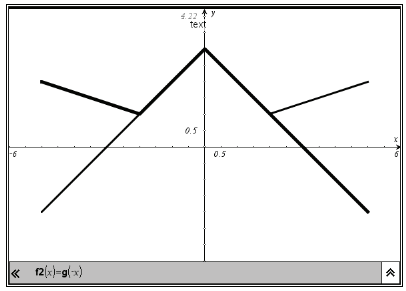
d) We horizontally shrink the graph by scale factor $\frac{1}{2}$.



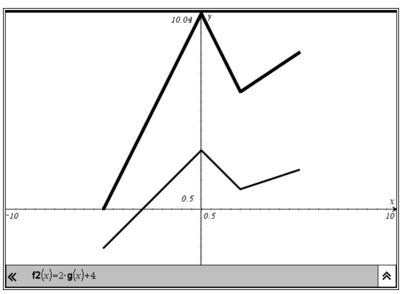
e) We reflect the graph in the x-axis.



f) We reflect the graph in the y-axis.



g) We vertically stretch the graph by scale factor 2 and translate it up four units.



20 We start with $y = x^2$. The first transformation is a translation of three units to the right to obtain $y = (x-3)^2$, followed by a translation of five units up to obtain the final graph, $y = (x-3)^2 + 5$.

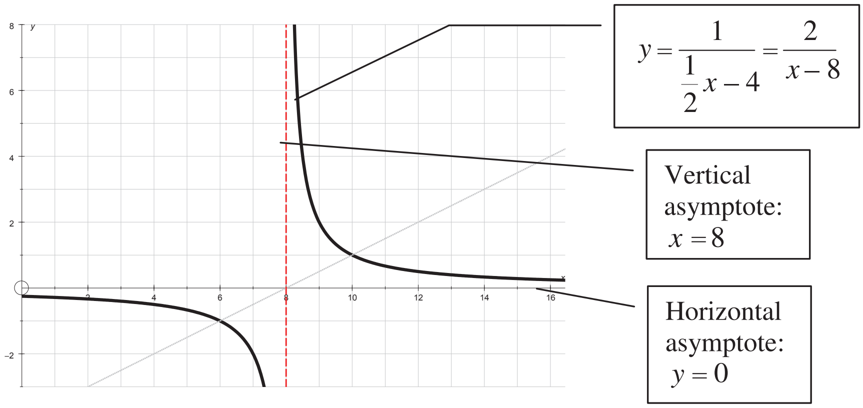
21 We start with $y = x^2$. The first transformation is a reflection in the x-axis to obtain $y = -x^2$, followed by a translation of two units up to obtain the final graph, $y = -x^2 + 2$.

22 We start with $y = x^2$. The first transformation is a translation of four units to the left to obtain $y = (x+4)^2$, followed by a vertical shrink by scale factor $\frac{1}{2}$ to obtain the final graph, $y = \frac{1}{2}(x+4)^2$.

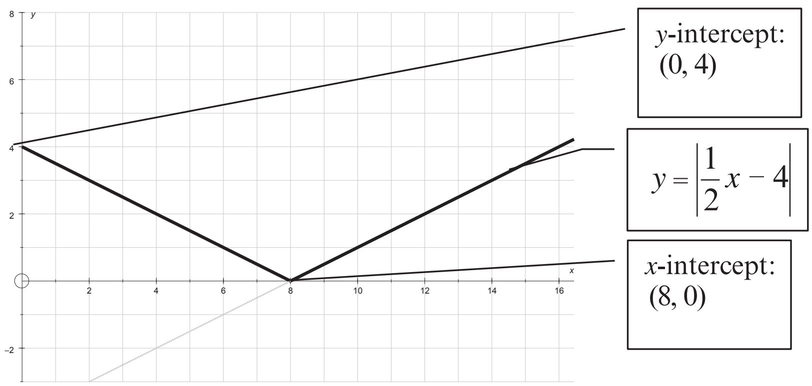
23 We start with $y = x^2$. The first transformation is a horizontal shrink by scale factor $\frac{1}{3}$ to obtain $y = (3x)^2$, followed by a translation of one unit to the right to obtain $y = [3(x-1)]^2$, and, finally, a translation of six units down to obtain the final graph, $y = [3(x-1)]^2 - 6$.

24 On the following graphs, the original function is in grey, whilst the transformed functions and the asymptotes (if any) are in black and red respectively.

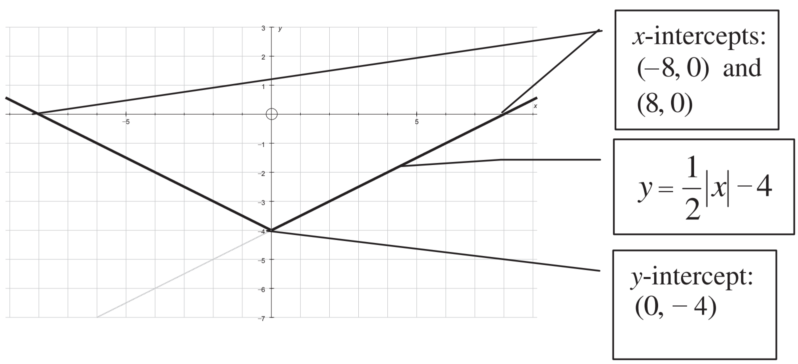
a)


$y = \dfrac{1}{\frac{1}{2}x - 4} = \dfrac{2}{x - 8}$
Vertical asymptote: $x = 8$
Horizontal asymptote: $y = 0$

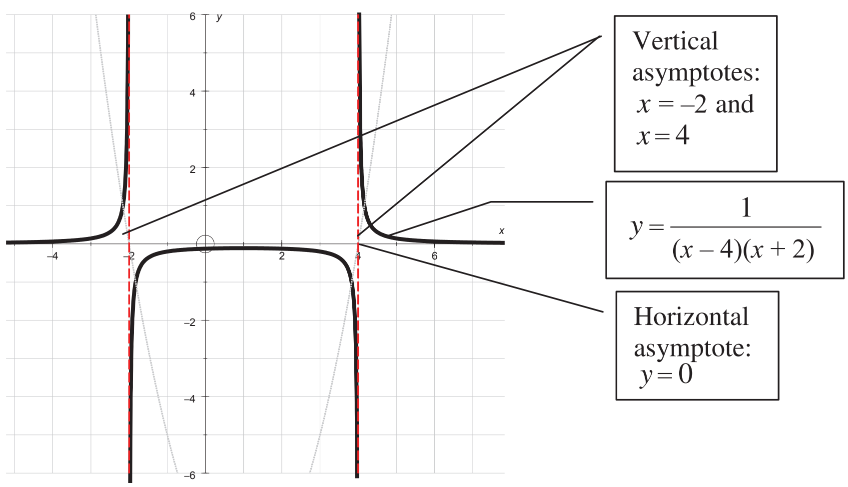
b)


y-intercept: $(0, 4)$
$y = \left|\frac{1}{2}x - 4\right|$
x-intercept: $(8, 0)$

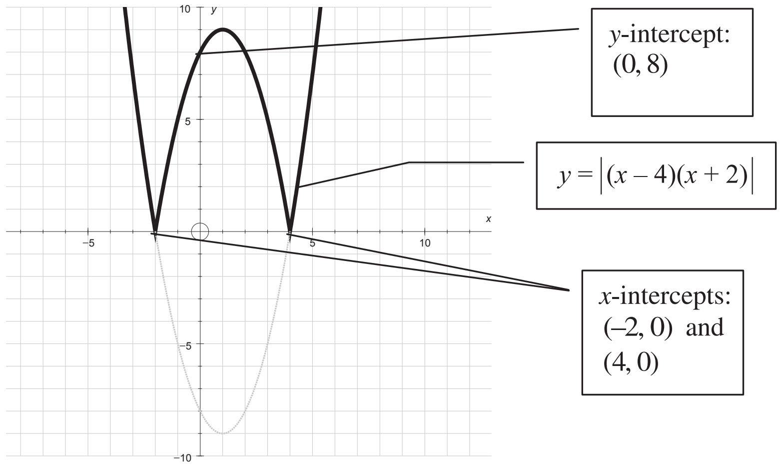
c)


x-intercepts: $(-8, 0)$ and $(8, 0)$
$y = \frac{1}{2}|x| - 4$
y-intercept: $(0, -4)$

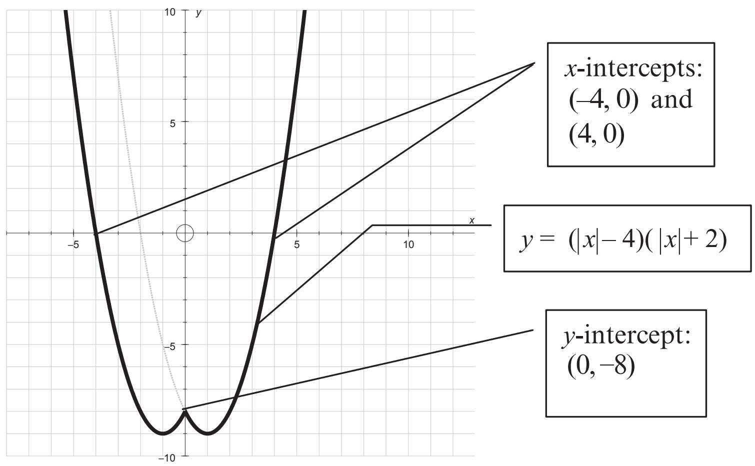
25 a)


Vertical asymptotes: $x = -2$ and $x = 4$
$y = \dfrac{1}{(x-4)(x+2)}$
Horizontal asymptote: $y = 0$

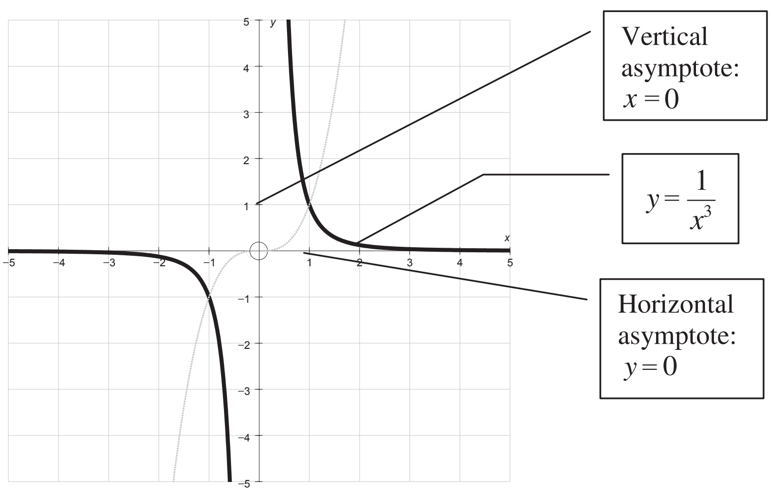
b)


y-intercept: (0, 8)
y = |(x − 4)(x + 2)|
x-intercepts: (−2, 0) and (4, 0)

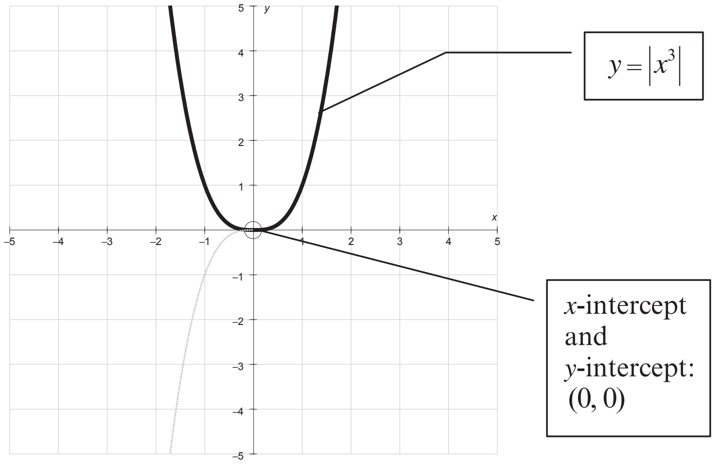
c)


x-intercepts: (−4, 0) and (4, 0)
y = (|x| − 4)(|x| + 2)
y-intercept: (0, −8)

26 a)

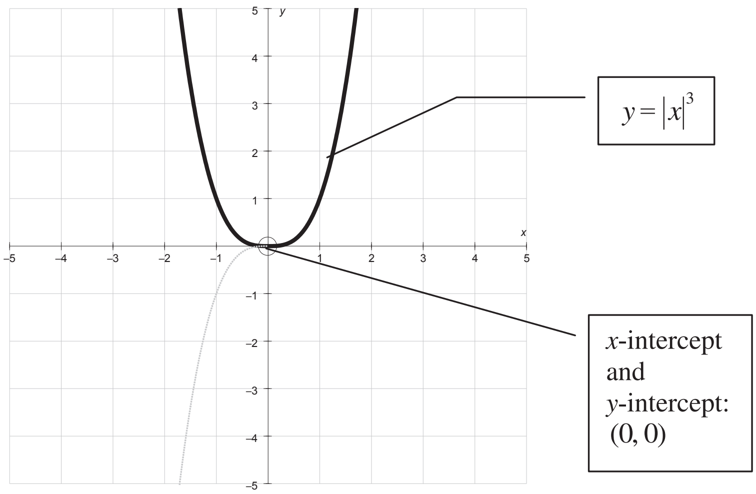

Vertical asymptote: x = 0
y = 1/x³
Horizontal asymptote: y = 0

b)



y
x
$y = |x^3|$
x-intercept and y-intercept: (0, 0)

c)



y
x
$y = |x|^3$
x-intercept and y-intercept: (0, 0)

Chapter 3

Exercise 3.1

In questions 1–4, we will use synthetic substitution to evaluate $P(x)$ for the given values of x.

1

2	1	2	−3	−4	−20
		2	8	10	12
	1	4	5	6	**−8** = ***P*(2)**

−3	1	2	−3	−4	−20
		−3	3	0	12
	1	−1	0	−4	**−8** = ***P*(−3)**

2

−1	2	−1	3	0	−15	−9
		−2	3	−6	6	9
	2	−3	6	−6	−9	**0** = ***P*(−1)**

2	2	−1	3	0	−15	−9
		4	6	18	36	42
	2	3	9	18	21	**33** = ***P*(2)**

3

−2	1	5	3	−6	−9	11
		−2	−6	6	0	18
	1	3	−3	0	−9	**29** = ***P*(2)**

4	1	5	3	−6	−9	11
		4	36	156	600	2364
	1	9	39	150	591	**2375** = ***P*(4)**

4

c	1	$-c-3$	$3c+5$	$-5c$
		c	$-3c$	$5c$
	1	−3	5	**0** = $\boldsymbol{P(c)}$

2	1	$-c-3$	$3c+5$	$-5c$
		2	$-2c-2$	$2c+6$
	1	$-c-1$	$c+3$	$\boldsymbol{-3c+6=P(c)}$

5

-2	k	2	-10	3
		$-2k$	$4k-4$	$-8k+28$
	k	$-2k+2$	$4k-14$	$\mathbf{-8k+31=P(-2)}$

$P(-2) = 15 \Rightarrow -8k + 31 = 15 \Rightarrow k = 2$

6

$-\frac{1}{3}$	3	-2	-10	$3k$	3
		-1	1	3	$-k-1$
	3	-3	-9	$3k+3$	$\mathbf{-k+2=P\left(-\frac{1}{3}\right)}$

$P\left(-\frac{1}{3}\right) = 0 \Rightarrow -k + 2 = 0 \Rightarrow k = 2$

7 a)

-3	2	3	-5	-4
		-6	9	-12
	2	-3	4	$\mathbf{-16=f(-3)}$

-2	2	3	-5	-4
		-4	2	6
	2	-1	-3	$\mathbf{2=f(-2)}$

-1	2	3	-5	-4
		-2	-1	6
	2	1	-6	$\mathbf{2=f(-1)}$

$f(0) = -4$

1	2	3	-5	-4
		2	5	0
	2	5	0	$\mathbf{-4=f(1)}$

2	2	3	-5	-4
		4	14	18
	2	7	9	$\mathbf{14=f(2)}$

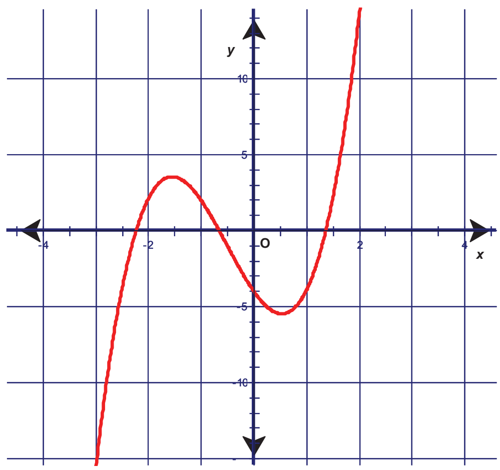
3	2	3	-5	-4
		6	27	66
	2	9	22	$\mathbf{62=f(3)}$

b)

x	−3	−2	−1	0	1	2	3
y	−16	2	2	−4	−4	14	62

The graph must cross the x-axis three times.

c)



8 a)

−3	1	0	−4	−2	1
		−3	9	−15	51
	1	−3	5	−17	**52 = *f*(−3)**

−2	1	0	−4	−2	1
		−2	4	0	4
	1	−2	0	−2	**5 = *f*(−2)**

−1	1	0	−4	−2	1
		−1	1	3	−1
	1	−1	−3	1	**0 = *f*(−1)**

$f(0) = 1$

1	1	0	−4	−2	1
		1	1	−3	−5
	1	1	−3	−5	**−4 = *f*(1)**

2	1	0	−4	−2	1
		2	4	0	−4
	1	2	0	−2	**−3 = *f*(2)**

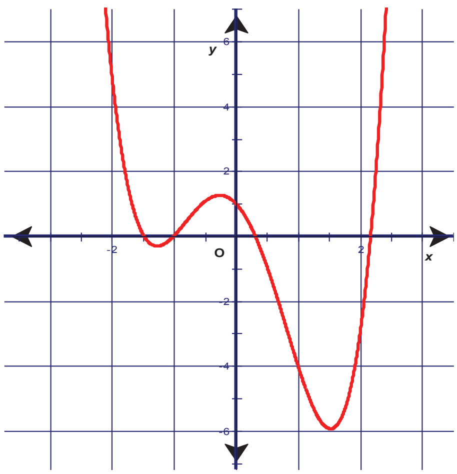
3	1	0	−4	−2	1
		3	9	15	39
	1	3	5	13	**40 = *f*(3)**

b)

x	−3	−2	−1	0	1	2	3
y	52	5	0	1	−4	−3	40

The graph must cross the x-axis four times.

c)



9

2	1	a	−5	$7a$
		2	$2a+4$	$4a-2$
	1	$a+2$	$2a-1$	$\mathbf{11a-2=f(2)}$

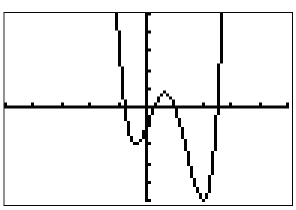
$$f(2)=10 \Rightarrow 11a-2=10 \Rightarrow a=\frac{12}{11}$$

10

$\sqrt{3}$	b	−5	$2b$	10
		$b\sqrt{3}$	$3b-5\sqrt{3}$	$5b\sqrt{3}-15$
	b	$b\sqrt{3}-5$	$5b-5\sqrt{3}$	$\mathbf{5b\sqrt{3}-5=f\left(\sqrt{3}\right)}$

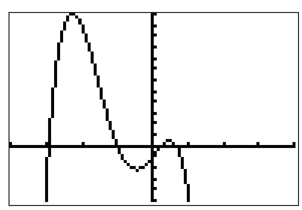
$$f(\sqrt{3})=-20 \Rightarrow 5b\sqrt{3}-5=-20 \Rightarrow b=\frac{-15}{5\sqrt{3}}=-\frac{3\sqrt{3}}{3}=-\sqrt{3}$$

11 **a)** **i)**



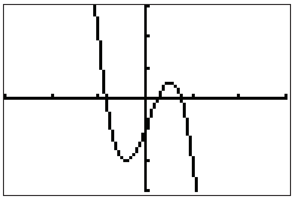
For $P(x)=2x^4-6x^3+x^2+4x-1$, the end behaviour of the function is $(\nwarrow,\nearrow)$.

ii)



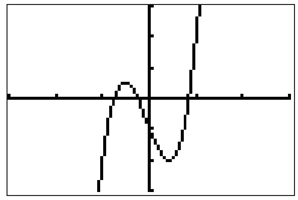
For $P(x)=-2x^4-6x^3+x^2+4x-1$, the end behaviour of the function is $(\swarrow,\searrow)$.

iii)



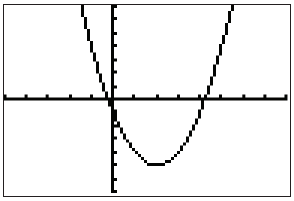
For $P(x) = -6x^3 + x^2 + 4x - 1$, the end behaviour of the function is $(\nwarrow, \searrow)$.

iv)



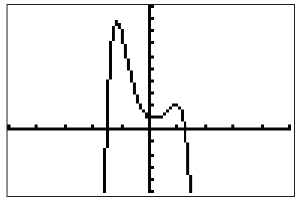
For $P(x) = 6x^3 + x^2 - 4x - 1$, the end behaviour of the function is $(\swarrow, \nearrow)$.

v)



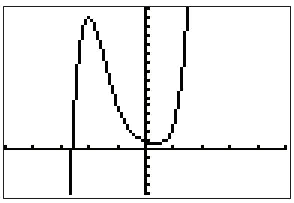
For $P(x) = x^2 - 4x - 1$, the end behaviour of the function is $(\nwarrow, \nearrow)$.

vi)



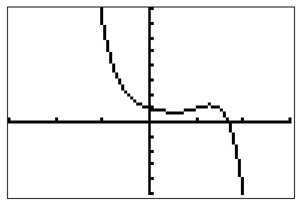
For $P(x) = -2x^6 + x^5 + 2x^4 - 3x^3 + 4x^2 - x + 1$, the end behaviour of the function is $(\swarrow, \searrow)$.

vii)



For $P(x) = x^5 + 2x^4 - x^3 + x^2 - x + 1$, the end behaviour of the function is $(\swarrow, \nearrow)$.

viii)



For $P(x) = -x^5 + 2x^4 - x^3 + x^2 - x + 1$, the end behaviour of the function is $(\nwarrow, \searrow)$.

b) If the leading term has a positive coefficient and even exponent, then the end behaviour is $(\nwarrow, \nearrow)$.

If the leading term has a negative coefficient and even exponent, then the end behaviour is $(\swarrow, \searrow)$.

If the leading term has a positive coefficient and odd exponent, then the end behaviour is $(\swarrow, \nearrow)$.

If the leading term has a negative coefficient and odd exponent, then the end behaviour is $(\nwarrow, \searrow)$.

Exercise 3.2

1 **a)** $f(x) = x^2 - 10x + 32 = (x^2 - 10x + 25) + 7 = (x - 5)^2 + 7$

The axis of symmetry is $x = 5$; the vertex is (5, 7).

b) There are two transformations that need to be applied to $y = x^2$ to obtain $y = x^2 - 10x + 32$: a horizontal translation of five units to the right and a vertical translation of seven units up.

c) Minimum: (5, 7)

2 **a)** $f(x) = x^2 + 6x + 8 = (x^2 + 6x + 9) - 1 = (x+3)^2 - 1$

The axis of symmetry is $x = -3$; the vertex is $(-3, -1)$.

b) There are two transformations that need to be applied to $y = x^2$ to obtain $y = x^2 + 6x + 8$: a horizontal translation of three units to the left and a vertical translation of one unit down.

c) Minimum: $(-3, -1)$

3 **a)** $f(x) = -2x^2 - 4x + 10 = -2(x^2 + 2x + 1) + 12 = -2(x+1)^2 + 12$

The axis of symmetry is $x = -1$; the vertex is $(-1, 12)$.

b) There are four transformations that need to be applied to $y = x^2$ to obtain $y = -2x^2 - 4x + 10$: a horizontal translation of one unit to the left, then a reflection in the x-axis with a vertical stretch by scale factor 2, and, finally, a vertical translation of twelve units up.

c) Maximum: $(-1, 12)$

4 **a)** $f(x) = 4x^2 - 4x + 9 = 4\left(x^2 - 2\cdot\frac{1}{2}x + \frac{1}{4}\right) + 8 = 4\left(x - \frac{1}{2}\right)^2 + 8$

The axis of symmetry is $x = \frac{1}{2}$; the vertex is $\left(\frac{1}{2}, 8\right)$.

b) There are three transformations that need to be applied to $y = x^2$ to obtain $y = 4x^2 - 4x + 9$: a horizontal translation of half a unit to the right, then a vertical stretch by scale factor 4, and, finally, a vertical translation of eight units up.

c) Minimum: $\left(\frac{1}{2}, 8\right)$

5 **a)** $f(x) = \frac{1}{2}x^2 + 7x + 26 = \frac{1}{2}(x^2 + 14x + 49) + \frac{3}{2} = \frac{1}{2}(x+7)^2 + \frac{3}{2}$

The axis of symmetry is $x = -7$; the vertex is $\left(-7, \frac{3}{2}\right)$.

b) There are three transformations that need to be applied to $y = x^2$ to obtain $y = \frac{1}{2}x^2 + 7x + 26$: a horizontal translation of seven units to the left, then a vertical shrink by scale factor $\frac{1}{2}$, and, finally, a vertical translation of one-and-a-half units up.

c) Minimum: $\left(-7, \frac{3}{2}\right)$

6 $x^2 + 2x - 8 = 0 \Rightarrow x^2 + 4x - 2x - 8 = 0 \Rightarrow x(x+4) - 2(x+4) = 0 \Rightarrow$
$(x+4)(x-2) = 0 \Rightarrow x_1 = -4, x_2 = 2$

7 $x^2 = 3x + 10 \Rightarrow x^2 - 3x - 10 = 0 \Rightarrow x^2 + 2x - 5x - 10 = 0 \Rightarrow$
$x(x+2) - 5(x+2) = 0 \Rightarrow (x+2)(x-5) = 0 \Rightarrow x_1 = -2, x_2 = 5$

8 $6x^2 - 9x = 0 \Rightarrow 3x(2x-3) = 0 \Rightarrow x_1 = 0, x_2 = \frac{3}{2}$

9 $6 + 5x = x^2 \Rightarrow x^2 - 5x - 6 = 0 \Rightarrow x^2 - 6x + x - 6 = 0 \Rightarrow$
$x(x-6) + (x-6) = 0 \Rightarrow (x+1)(x-6) = 0 \Rightarrow x_1 = -1, x_2 = 6$

10 $x^2 + 9 = 6x \Rightarrow x^2 - 6x + 9 = 0 \Rightarrow (x-3)^2 = 0 \Rightarrow x_1 = x_2 = 3$

11 $3x^2 + 11x - 4 = 0 \Rightarrow 3x^2 - x + 12x - 4 = 0 \Rightarrow x(3x-1) + 4(3x-1) = 0 \Rightarrow$

$(x+4)(3x-1) = 0 \Rightarrow x_1 = -4, x_2 = \frac{1}{3}$

12 $3x^2 + 18 = 15x \Rightarrow 3x^2 - 15x + 18 = 0 \Rightarrow x^2 - 5x + 6 = 0 \Rightarrow x^2 - 2x - 3x - 6 = 0 \Rightarrow$

$x(x-2) - 3(x-2) = 0 \Rightarrow (x-2)(x-3) = 0 \Rightarrow x_1 = 2, x_2 = 3$

13 $9x - 2 = 4x^2 \Rightarrow 4x^2 - 9x + 2 = 0 \Rightarrow 4x^2 - x - 8x + 2 = 0 \Rightarrow$

$x(4x-1) - 2(4x-1) = 0 \Rightarrow (x-2)(4x-1) = 0 \Rightarrow x_1 = 2, x_2 = \frac{1}{4}$

14 $x^2 + 4x - 3 = 0 \Rightarrow x^2 + 4x + 4 - 4 - 3 = 0 \Rightarrow (x+2)^2 = 7 \Rightarrow$

$x + 2 = \pm\sqrt{7} \Rightarrow x_{1,2} = -2 \pm \sqrt{7}$

15 $x^2 - 4x - 5 = 0 \Rightarrow x^2 - 4x + 4 - 4 - 5 = 0 \Rightarrow (x-2)^2 = 9 \Rightarrow$

$x - 2 = \pm 3 \Rightarrow x_1 = 2 + 3 = 5, x_2 = 2 - 3 = -1$

16 $x^2 - 2x + 3 = 0 \Rightarrow x^2 - 2x + 1 + 2 = 0 \Rightarrow (x-1)^2 = -2 \Rightarrow$

$x - 1 = \pm i\sqrt{2} \Rightarrow x_1 = 1 + i\sqrt{2}, x_2 = 1 - i\sqrt{2}$

(There is no real solution.)

17 $2x^2 + 16x + 6 = 0 \Rightarrow x^2 + 8x + 3 = 0 \Rightarrow x^2 + 8x + 16 - 16 + 3 = 0 \Rightarrow (x+4)^2 = 13 \Rightarrow$

$x + 4 = \pm\sqrt{13} \Rightarrow x_1 = -4 + \sqrt{13}, x_2 = -4 - \sqrt{13}$

18 $x^2 + 2x - 8 = 0 \Rightarrow x^2 + 2x + 1 - 1 - 8 = 0 \Rightarrow (x+1)^2 = 9 \Rightarrow$

$x + 1 = \pm 3 \Rightarrow x_1 = -1 + 3 = 2, x_2 = -1 - 3 = -4$

19 $-2x^2 + 4x + 9 = 0 \Rightarrow x^2 - 2x - \frac{9}{2} = 0 \Rightarrow x^2 - 2x + 1 - 1 - \frac{9}{2} = 0 \Rightarrow (x-1)^2 = \frac{11}{2} \Rightarrow$

$x - 1 = \pm\sqrt{\frac{11}{2}} \Rightarrow x = 1 \pm \frac{\sqrt{22}}{2} \Rightarrow x_1 = \frac{2+\sqrt{22}}{2}, x_2 = \frac{2-\sqrt{22}}{2}$

20 For $f(x) = x^2 - 4x - 1$ we have:

a) $x^2 - 4x - 1 = 0 \Rightarrow x_{1,2} = \frac{4 \pm \sqrt{(-4)^2 - 4\cdot 1\cdot(-1)}}{2\cdot 1} = \frac{4 \pm \sqrt{20}}{2} = \frac{4 \pm 2\sqrt{5}}{2} = 2 \pm \sqrt{5}$

b) The axis of symmetry is $x = \frac{x_1 + x_2}{2} = \frac{(2+\sqrt{5}) + (2-\sqrt{5})}{2} = 2.$

c) As $a = 1 > 0$, the parabola opens up; thus, the minimum value is $f(2) = 2^2 - 4\cdot 2 - 1 = -5$.

21 For $x^2 + 3x + 2 = 0$, the discriminant is $\Delta = 3^2 - 4\cdot 1\cdot 2 = 1 > 0$, so there are two distinct real solutions.

22 For $2x^2 - 3x + 2 = 0$, the discriminant is $\Delta = (-3)^2 - 4\cdot 2\cdot 2 = -7$, so there are no real solutions.

23 For $x^2 - 1 = 0$, the discriminant is $\Delta = 0^2 - 4\cdot 1\cdot(-1) = 4 > 0$, so there are two distinct real solutions.

24 For $2x^2 - \frac{9}{4}x + 1 = 0$, the discriminant is $\Delta = \left(-\frac{9}{4}\right)^2 - 4\cdot 2\cdot 1 = \frac{81}{16} - 8 = -\frac{47}{16} < 0$, so there are no real solutions.

25 The equation $2x^2 + px + 1 = 0$ has one real solution when the discriminant is equal to zero. So:
$\Delta = p^2 - 4 \cdot 2 \cdot 1 = 0 \Rightarrow p^2 - 8 = 0 \Rightarrow p_{1,2} = \pm\sqrt{8} = \pm 2\sqrt{2}$.

26 The equation $x^2 + 4x + k = 0$ has two distinct real solutions when the discriminant is positive. So:
$\Delta = 4^2 - 4 \cdot 1 \cdot k > 0 \Rightarrow 16 - 4k > 0 \Rightarrow k < 4$.

27 The equation $x^2 - 4kx + 4 = 0$ has two distinct real solutions when the discriminant is positive. So:
$\Delta = (4k)^2 - 4 \cdot 1 \cdot 4 > 0 \Rightarrow 16k^2 - 16 > 0 \Rightarrow k^2 - 1 > 0 \Rightarrow (k-1)(k+1) > 0$.

We have two possibilities:

$$\begin{cases} k - 1 < 0 \Rightarrow k < 1 \\ k + 1 < 0 \Rightarrow k < -1 \end{cases} \Rightarrow k < -1 \quad \text{or} \quad \begin{cases} k - 1 > 0 \Rightarrow k > 1 \\ k + 1 > 0 \Rightarrow k > -1 \end{cases} \Rightarrow k > 1$$

28 The graph of the function $g(x) = mx^2 + 6x + m$ does not touch the x-axis when the discriminant is negative. So: $\Delta = 6^2 - 4 \cdot m \cdot m < 0 \Rightarrow 36 - 4m^2 < 0 \Rightarrow 9 - m^2 < 0 \Rightarrow (3-m)(3+m) < 0$.

We have two possibilities:

$$\begin{cases} 3 - m < 0 \Rightarrow 3 < m \\ 3 + m > 0 \Rightarrow m > -3 \end{cases} \Rightarrow m > 3 \quad \text{or} \quad \begin{cases} 3 - m > 0 \Rightarrow m < 3 \\ 3 + m < 0 \Rightarrow m < -3 \end{cases} \Rightarrow m < -3$$

29 For $3x^2 - 12x + k > 0$ to be true for all real values of k, the discriminant must be negative. So:
$\Delta = (-12)^2 - 4 \cdot 3 \cdot k < 0 \Rightarrow 144 - 12k < 0 \Rightarrow k > 12$.

30 The function $f(x) = x - 2 - x^2 = -x^2 + x - 2$ has a negative discriminant, $\Delta = 1^2 - 4 \cdot (-1) \cdot (-2) = -7$, and as its leading coefficient, $a = -1$, is negative, its graph opens downwards and never intersects the x-axis, which means that the expression $x - 2 - x^2$ is negative for all real values of x.

31 Zeros of $x_1 = -1, x_2 = 4 \Rightarrow f(x) = a(x+1)(x-4)$ and
$f(0) = 8 \Rightarrow f(0) = a(0+1)(0-4) = 8 \Rightarrow -4a = 8 \Rightarrow a = -2$

So: $f(x) = -2(x+1)(x-4) = -2(x^2 - 3x - 4) = -2x^2 + 6x + 8$.

32 Zeros of $x_1 = \frac{1}{2}, x_2 = 3 \Rightarrow f(x) = a\left(x - \frac{1}{2}\right)(x-3)$ and

$f(-1) = 4 \Rightarrow f(-1) = a\left(-1 - \frac{1}{2}\right)(-1-3) = 4 \Rightarrow 6a = 4 \Rightarrow a = \frac{2}{3}$

So: $f(x) = \frac{2}{3}\left(x - \frac{1}{2}\right)(x-3) = \frac{2}{3}\left(x^2 - \frac{7}{2} + \frac{3}{2}\right) = \frac{2}{3}x^2 - \frac{7}{3}x + 1$.

33 The equation $2x^2 + (3-k)x + k + 3 = 0$ has two imaginary solutions when the discriminant is negative. So:
$\Delta = (3-k)^2 - 4 \cdot 2 \cdot (k+3) < 0 \Rightarrow 9 - 6k + k^2 - 8k - 24 < 0 \Rightarrow$

$k^2 - 14k - 15 < 0 \Rightarrow (k+1)(k-15) < 0$

We have two possibilities:

$$\begin{cases} k + 1 < 0 \Rightarrow k < -1 \\ k - 15 > 0 \Rightarrow k < 15 \end{cases} \Rightarrow \text{not possible} \quad \text{or} \quad \begin{cases} k + 1 > 0 \Rightarrow k > -1 \\ k - 15 < 0 \Rightarrow k < 15 \end{cases} \Rightarrow k \in \langle -1, 15 \rangle$$

34 The function $f(x) = 5x^2 - mx + 2$ has two distinct real zeros when the discriminant is positive. So:

$\Delta = (-m)^2 - 4 \cdot 5 \cdot 2 > 0 \Rightarrow m^2 - 40 > 0 \Rightarrow$

$\left(m + \sqrt{40}\right)\left(m - \sqrt{40}\right) > 0 \Rightarrow \left(m + 2\sqrt{10}\right)\left(m - 2\sqrt{10}\right) > 0$

We have two possibilities:

$$\begin{cases} m + 2\sqrt{10} < 0 \Rightarrow m < -2\sqrt{10} \\ m - 2\sqrt{10} < 0 \Rightarrow m < 2\sqrt{10} \end{cases} \Rightarrow m < -2\sqrt{10} \text{ or } \begin{cases} m + 2\sqrt{10} > 0 \Rightarrow m > -2\sqrt{10} \\ m - 2\sqrt{10} > 0 \Rightarrow m > 2\sqrt{10} \end{cases} \Rightarrow m > 2\sqrt{10}$$

35 The function $f(x) = ax^2 + bx + c$ passes through:

$(-3, 10) \Rightarrow f(-3) = a(-3)^2 + b(-3) + c = 10$

$\left(\frac{1}{4}, -\frac{9}{16}\right) \Rightarrow f\left(\frac{1}{4}\right) = a\left(\frac{1}{4}\right)^2 + b\left(\frac{1}{4}\right) + c = -\frac{9}{16}$

$(1, 6) \Rightarrow f(1) = a(1)^2 + b(1) + c = 6$

This is a system of linear equations:

$$\begin{cases} a + b + c = 6 \\ 9a - 3b + c = 10 \\ \frac{1}{16}a + \frac{1}{4}b + c = -\frac{9}{16} \end{cases} \Rightarrow \begin{cases} 8a - 4b = 4 \\ -\frac{15}{16}a - \frac{3}{4}b = -\frac{105}{16} \end{cases} \Rightarrow \begin{cases} 2a - b = 1 \\ -5a - 4b = -35 \end{cases}$$

$\Rightarrow -13a = -39 \Rightarrow a = 3, b = 5, c = -2 \Rightarrow f(x) = 3x^2 + 5x - 2$

36 $f(x) = -2(x-1)^2 + 10 \Rightarrow f(2) = -2(2-1)^2 + 10 = 8$. Since $f(3) = f(-1)$ for the function $f(x) = ax^2 + bx + c$, the axis of symmetry is $x = \frac{3-1}{2} = 1$, and the maximum value is 10. Therefore, the vertex has coordinates $(1, 10) \Rightarrow f(x) = a(x-1)^2 + 10$.

$f(3) = 2 \Rightarrow a(3-1)^2 + 10 = 2 \Rightarrow 4a = -8 \Rightarrow a = -2$.

$f(x) = -2(x-1)^2 + 10 \Rightarrow f(2) = -2(2-1)^2 + 10 = 8$.

37 $4x + 1 < x^2 + 4 \Rightarrow x^2 - 4x + 3 > 0 \Rightarrow (x-3)(x-1) > 0$

The graph of the quadratic function $f(x) = (x-3)(x-1)$ opens upwards and has zeros of 1 and 3; so, the solution of the inequality is $x < 1$ or $x > 3$.

38 The discriminant of the equation $2x^2 + (2-t)x + t^2 + 3 = 0$ is

$\Delta = (2-t)^2 - 4 \cdot 2 \cdot (t^2 + 3) = 4 - 4t + t^2 - 8t^2 - 24 = -7t^2 - 4t - 20$

$= -7\left(t^2 + 2 \cdot \frac{2}{7}t + \frac{4}{49}\right) + \frac{28}{49} - 20 = -7\left(t + \frac{2}{7}\right)^2 - \frac{136}{7} < 0.$

Since the discriminant is negative for every $t \in \mathbb{R}$, the equation has no real solutions.

39 $ax^2 + bx + c = a(x-\alpha)(x-\beta) = a(x^2 - \alpha x - \beta x + \alpha\beta) = ax^2 - a(\alpha+\beta)x + a\alpha\beta$

By comparing the corresponding coefficients, we conclude:

a) $b = -a(\alpha + \beta) \Rightarrow \alpha + \beta = -\frac{b}{a}$

b) $c = a\alpha\beta \Rightarrow \alpha\beta = \frac{c}{a}$

40 $ax^2 - a^2x - x + a = 0 \Rightarrow ax(x-a) - (x-a) \Rightarrow (ax-1)(x-a) \Rightarrow x_1 = \frac{1}{a}, x_2 = a$

Exercise 3.3

1

−3	3	5	−5
		−9	12
	3	−4	**7**

Hence: $3x^2 + 5x - 5 = (x+3)(3x-4) + 7$.

2

2	3	−8	0	9	5
		6	−4	−8	2
	3	−2	−4	1	**7**

Hence: $3x^4 - 8x^3 + 9x + 5 = (x-2)(3x^3 - 2x^2 - 4x + 1) + 7$.

3

4	1	−5	3	−7
		4	−4	−4
	1	−1	−1	**−11**

Hence: $x^3 - 5x^2 + 3x - 7 = (x-4)(x^2 - x - 1) - 11$.

4

$\frac{1}{3}$	9	12	−5	1
		3	5	0
	9	15	0	**1**

Hence: $9x^3 + 12x^2 - 5x + 1 = 3\left(x - \frac{1}{3}\right) \cdot \frac{1}{3} \cdot (9x^2 + 15x) + 1 = (3x-1)(3x^2 + 5x) + 1$.

5 Not suitable for synthetic division, so:

$$\begin{array}{r l}
 & x^3 - x + 1 \\
x^2 + x - 7 \,\big) & x^5 + x^4 - 8x^3 + 0x^2 + x + 2 \\
 & \underline{x^5 + x^4 - 7x^3} \\
 & \qquad -x^3 + 0x^2 + x \\
 & \qquad \underline{-x^3 - x^2 + 7x} \\
 & \qquad\qquad x^2 - 6x + 2 \\
 & \qquad\qquad \underline{x^2 + x - 7} \\
 & \qquad\qquad\quad -7x + 9
\end{array}$$

Hence: $\left(x^5 + x^4 - 8x^3 + 0x^2 + x + 2\right) = \left(x^2 + x - 7\right)\left(x^3 - x + 1\right) + (-7x + 9)$.

6

1	2	−17	22	−7
		2	−15	7
	2	−15	7	**0**

$$2x^3 - 17x^2 + 22x - 7 = (x-1)(2x^2 - 15x + 7)$$

Then: $2x^2 - 15x + 7 = 0 \Rightarrow x_{2,3} = \frac{15 \pm \sqrt{15^2 - 4 \cdot 2 \cdot 7}}{2 \cdot 2} = \frac{15 \pm 13}{4} \Rightarrow x_2 = \frac{1}{2}, x_3 = 7$

Hence: $2x^3 - 17x^2 + 22x - 7 = (x-1)(2x-1)(x-7)$.

7

$-\frac{1}{2}$	6	−5	−12	−4
		−3	4	4
	6	−8	−8	**0**

$$6x^3 - 5x^2 - 12x - 4 = (2x+1)(3x^2 - 4x - 4)$$

Then: $3x^2 - 4x - 4 = 0 \Rightarrow x_{2,3} = \frac{4 \pm \sqrt{(-4)^2 - 4 \cdot 3 \cdot (-4)}}{2 \cdot 3} = \frac{4 \pm 8}{6} \Rightarrow x_2 = -\frac{2}{3}, x_3 = 2$

Hence: $6x^3 - 5x^2 - 12x - 4 = (2x+1)(3x+2)(x-2)$.

8

$-\frac{2}{3}$	3	2	−36	24	32
		−2	0	24	−32
2	3	0	−36	48	**0**
		6	12	−48	
	3	6	−24	**0**	

$$3x^4 + 2x^3 - 36x^2 + 24x + 32 = \left(x + \frac{2}{3}\right)(3x^3 - 36x + 48) = \left(x + \frac{2}{3}\right)(x-2)(3x^2 + 6x - 24)$$

Then: $3x^2 + 6x - 24 = 0 \Rightarrow x_{3,4} = \frac{-6 \pm \sqrt{6^2 - 4 \cdot 3 \cdot (-24)}}{2 \cdot 3} = \frac{-6 \pm 18}{6} \Rightarrow x_3 = -4, x_4 = 2$

Hence: $3x^4 + 2x^3 - 36x^2 + 24x + 32 = \left(x + \frac{2}{3}\right)(3x^3 - 36x + 48) = (3x+2)(x-2)^2(x+4)$.

9

3	1	−5	4
		3	−6
	1	−2	**−2**

$Q(x) = x - 2, R(x) = -2$

10

−2	1	2	2	1
		−2	0	−4
	1	0	2	**−3**

$Q(x) = x^2 + 2,\ R(x) = -3$

11

$$
\begin{array}{r}
3 \\
3x^2 - 7x\,\overline{)\,9x^2 - x + 5} \\
\underline{9x^2 - 21x} \\
20x + 5
\end{array}
$$

$Q(x) = 3,\ R(x) = 20x + 5$

12

1	1	0	3	0	0	−6
		1	1	4	4	4
	1	1	4	4	4	**−2**

$Q(x) = x^4 + x^3 + 4x^2 + 4x + 4,\ R(x) = -2$

13

2	2	−3	4	−7
		4	2	12
	2	1	6	**5**

$P(2) = 5$

14

−1	1	−2	0	3	20	3
		−1	3	−3	0	−20
	1	−3	3	0	20	**−17**

$P(-1) = -17$

15

−7	5	30	−40	36	14
		−35	35	35	−497
	5	−5	−5	71	**−483**

$P(-7) = -483$

16

$\frac{1}{4}$	1	0	−1	1
		$\frac{1}{4}$	$\frac{1}{16}$	$-\frac{15}{64}$
	1	$\frac{1}{4}$	$-\frac{15}{16}$	$\frac{49}{64}$

$P\left(\frac{1}{4}\right) = \frac{49}{64}$

17 $x_1 = -6$

-6	1	2	-19	30
		-6	24	-30
	1	-4	5	**0**

$x^3 + 2x^2 - 19x + 30 = (x+6)(x^2 - 4x + 5)$

Then: $x^2 - 4x + 5 = 0 \Rightarrow x_{2,3} = \frac{4 \pm \sqrt{(-4)^2 - 4 \cdot 1 \cdot 5}}{2} = \frac{4 \pm \sqrt{-4}}{2} = 2 \pm i$

18 $x_1 = x_2 = 2$

2	1	-5	7	0	-4
		2	-6	2	4
2	1	-3	1	2	**0**
		2	-2	-2	
	1	-1	-1	**0**	

Then: $x^2 - x - 1 = 0 \Rightarrow x_{3,4} = \frac{1 \pm \sqrt{(-1)^2 - 4 \cdot 1 \cdot (-1)}}{2} = \frac{1 \pm \sqrt{5}}{2} \Rightarrow x_3 = \frac{1+\sqrt{5}}{2}, x_4 = \frac{1-\sqrt{5}}{2}$

19 $f(-3) = 0 \Rightarrow (-3)^3 - (-3)^2 - k^2(-3) = 0 \Rightarrow 3k^2 - 36 = 0 \Rightarrow k^2 = 12 \Rightarrow$

$k = 2\sqrt{3}$ or $k = -2\sqrt{3}$

20 $f(1) = 0 \Rightarrow 2 \cdot 1^4 - 5 \cdot 1^3 - 14 \cdot 1^2 + a \cdot 1 + b = 0 \Rightarrow a + b = 17$

$f(4) = 0 \Rightarrow 2 \cdot 4^4 - 5 \cdot 4^3 - 14 \cdot 4^2 + a \cdot 4 + b = 0 \Rightarrow 4a + b = 32$

$\Rightarrow a = 5, b = 12$

21 $P(x) = (x+2)(x-1)(x-4) = (x^2 + x - 2)(x-4) = x^3 - 3x^2 - 6x + 8$

22 $P(x) = (x+1)(x+2)(x-3)^2 = (x^2 + 3x + 2)(x^2 - 6x + 9) = x^4 - 3x^3 - 7x^2 + 15x + 18$

23 $P(x) = (x-2)^3 = x^3 - 6x^2 + 12x - 8$

24 $P(x) = (x+1)[x - (1-i)][x - (1+i)] = (x+1)(x-1+i)(x-1-i)$

$= (x+1)\left(x^2 - 2x + 2\right) = x^3 - x^2 + 2$

25 $P(x) = (x-2)(x+4)(x-(-3i))(x-3i)$

$= \left(x^2 + 2x - 8\right)\left(x^2 + 9\right) = x^4 + 2x^3 + x^2 + 18x - 72$

26 $P(x) = [x - (3+i)][x + (3+i)][x - (1-2i)][x - (1+2i)]$

$= (x-3-i)(x-3+i)(x-1+2i)(x-1-2i) = \left(x^2 - 6x + 10\right)\left(x^2 - 2x + 5\right)$

$= x^4 - 8x^3 + 27x^2 - 50x + 50$

27 $x_1 = 2 - 3i \Rightarrow x_2 = 2 + 3i$

$(x - x_1)(x - x_2) = (x - 2 + 3i)(x - 2 - 3i) = x^2 - 4x + 13$

$$\begin{array}{r|l} & x - 3 \\ x^2 - 4x + 13 & x^3 - 7x^2 + 25x - 39 \\ & \underline{x^3 - 4x^2 + 13x} \\ & \quad -3x^2 + 12x - 39 \\ & \quad \underline{-3x^2 + 12x - 39} \\ & \qquad\qquad 0 \Rightarrow x_3 = 3 \end{array}$$

28 **a)** The remainder theorem gives us:

$$\left.\begin{array}{l} P(2) = 72 \Rightarrow 6\cdot 2^3 + 7\cdot 2^2 + a\cdot 2 + b = 72 \Rightarrow 2a + b = -4 \\ P(-1) = 0 \Rightarrow 6\cdot(-1)^3 + 7\cdot(-1)^2 + a\cdot(-1) + b = 0 \Rightarrow -a + b = -1 \end{array}\right. \Rightarrow a = -1, b = -2$$

b)

$\frac{1}{2}$	6	7	−1	−2
		3	5	2
−1	6	10	4	**0**
		−6	−4	
	6	4	**0**	

So, $P(x) = (x+1)\,2\left(x - \frac{1}{2}\right)\frac{1}{2}(6x+4) = (x+1)(2x-1)(3x+2)$.

29 The remainder theorem gives us:

$$\left.\begin{array}{l} P(-1) = -1 \Rightarrow (a(-1)+b)^3 = -1 \Rightarrow -a + b = -1 \\ P(2) = 27 \Rightarrow (2a+b)^3 = 27 \Rightarrow 2a + b = 3 \end{array}\right. \Rightarrow a = \frac{4}{3}, b = \frac{1}{3}$$

30 $x^2 - 2x - 3 = 0 \Rightarrow x_{1,2} = \dfrac{2 \pm \sqrt{(-2)^2 - 4\cdot 1\cdot(-3)}}{2} = \dfrac{2 \pm 4}{4} \Rightarrow x_1 = -1, x_2 = 3$

−1	4	−6	−15	−8	−3
		−4	10	5	3
3	4	−10	−5	−3	**0**
		12	6	3	
	4	2	1	**0**	

$$4x^2 + 2x + 1 = 0 \Rightarrow x_{3,4} = \frac{-2 \pm \sqrt{2^2 - 4\cdot 4\cdot 1}}{2\cdot 4} = \frac{-2 \pm \sqrt{-12}}{8} = \frac{-2 \pm 2\sqrt{3}i}{8}$$

$$\Rightarrow x_3 = -\frac{1}{4} + \frac{\sqrt{3}}{4}i,\; x_4 = -\frac{1}{4} - \frac{\sqrt{3}}{4}i$$

31 The remainder theorem gives us:

$P(2) = 0 \Rightarrow 2^3 + a\cdot 2^2 + b\cdot 2 + c = 0$

$P(-2) = 0 \Rightarrow (-2)^3 + a\cdot(-2)^2 + b\cdot(-2) + c = 0$

$P(3) = 10 \Rightarrow 3^3 + a\cdot 3^2 + b\cdot 3 + c = 10$

We have a system of three linear equations:

$$\begin{cases} 4a + 2b + c = -8 \\ 4a - 2b + c = 8 \\ 9a + 3b + c = -17 \end{cases} \Rightarrow \begin{cases} 4b = -16 \\ 5a + b = -9 \end{cases} \Rightarrow b = -4, a = -1, c = 4$$

32 $x_1 = 3, x_2 = 1 + 4i \Rightarrow x_3 = 1 - 4i$

$$P(x) = (x-3)(x-(1+4i))(x-(1-4i)) = (x-3)\left((x-1)^2 - 16i^2\right)$$
$$= (x-3)\left(x^2 - 2x + 17\right) = x^3 - 5x^2 + 23x - 51$$

So, for $P(x) = x^3 + px^2 + qx + r$, we conclude: $p = -5, q = 23, r = -51$.

33 The remainder theorem gives us:

$$P(-2) = 5\cdot(-2)^3 - 3\cdot(-2)^2 + a(-2) + 7 = -45 - 2a = R$$

$$Q(-2) = 4\cdot(-2)^3 + a(-2)^2 + 7\cdot(-2) - 4 = -50 + 4a = 2R$$

We have a system of two linear equations:

$$\begin{cases} 2a + R = -45 \\ 4a - 2R = 50 \end{cases} \Rightarrow 8a = -40 \Rightarrow a = -5,\ R = -35$$

34 If the polynomial $x^3 + mx^2 + nx - 8$ is divisible by $(x + 1 + i)$, it means that one of its zeros is $x_1 = -1 - i$ and, thus, $x_2 = -1 + i$ is the second zero. Let $x_3 = a$ be the third zero. Then:

$$\begin{aligned} x^3 + mx^2 + nx - 8 &= (x-a)(x+1+i)(x+1-i) = (x-a)\left(x^2 + 2x + 2\right) \\ &= x^3 + (2-a)x^2 + (2-2a)x - 2a \end{aligned}$$

By comparing the corresponding coefficients, we obtain the following system of equations:

$$\begin{cases} 2 - a = m \\ 2 - 2a = n \\ -2a = -8 \Rightarrow a = 4 \end{cases} \Rightarrow m = -2,\ n = -6$$

35 If the roots of the equation are consecutive terms in a geometric sequence, they are $x_1 = a$, $x_2 = ar$, and $x_3 = ar^2$. So, the equation can be written as:

$$\begin{aligned} x^3 - 9x^2 + bx - 216 &= (x-a)(x-ar)\left(x-ar^2\right) = \left(x^2 - ax - arx + a^2r\right)\left(x - ar^2\right) \\ &= x^3 - ax^2 - arx^2 + a^2rx - ar^2x^2 + a^2r^2x + a^2r^3x - a^3r^3 \\ &= x^3 - \left(a + ar + ar^2\right)x^2 + \left(a^2r + a^2r^2 + a^2r^3\right)x - a^3r^3 \end{aligned}$$

By comparing the corresponding coefficients, we obtain the following system of equations:

$$\begin{cases} a + ar + ar^2 = 9 \\ a^2r + a^2r^2 + a^2r^3 = b \\ -a^3r^3 = -216 \Rightarrow ar = 6 \end{cases}$$

By rearranging the equations, the system becomes:

$$\begin{cases} a\left(1 + r + r^2\right) = 9 \\ a^2r\left(1 + r + r^2\right) = b \Rightarrow (ar)\,a\left(1 + r + r^2\right) = b \Rightarrow 6\cdot 9 = b \Rightarrow b = 54 \\ ar = 6 \end{cases}$$

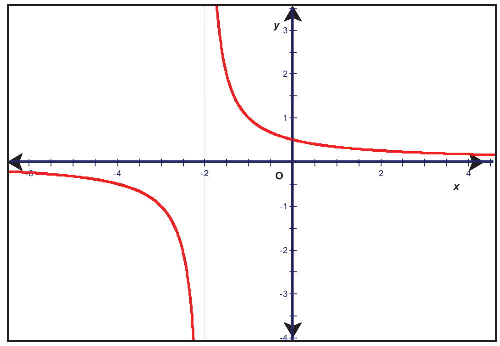
36 **a)** When a polynomial is divided by $(ax - b)$, the quotient is $Q(x)$ and the remainder R is a constant. Then:

$$P(x) = (ax - b)\,Q(x) + R \Rightarrow P\left(\frac{b}{a}\right) = \underbrace{\left(a\cdot\frac{b}{a} - b\right)}_{0} Q\left(\frac{b}{a}\right) + R \Rightarrow R = P\left(\frac{b}{a}\right)$$

b) $R = P\left(-\dfrac{2}{3}\right) = 9\cdot\left(-\dfrac{2}{3}\right)^3 - \left(-\dfrac{2}{3}\right) + 5 = -\dfrac{8}{3} + \dfrac{2}{3} + 5 = 3$

Exercise 3.4

1 $f(x)=\dfrac{1}{x+2}$



No x-intercept since the numerator $\neq 0$.

Vertical asymptote for $x+2=0 \Rightarrow x=-2$.

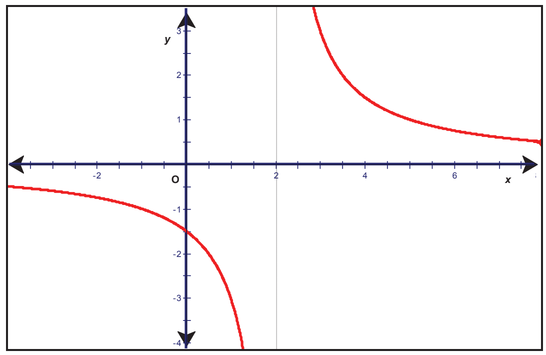
For:

$x<-2,\ f(x)<0 \Rightarrow f(x)\to -\infty$ as $x\to -2^{-}$

$x>-2,\ f(x)>0 \Rightarrow f(x)\to +\infty$ as $x\to -2^{+}$

As $x\to \pm\infty \Rightarrow f(x)\to 0 \Rightarrow y=0$ is a horizontal asymptote.

2 $g(x)=\dfrac{3}{x-2}$



No x-intercept since the numerator $\neq 0$.

Vertical asymptote for $x-2=0 \Rightarrow x=2$.

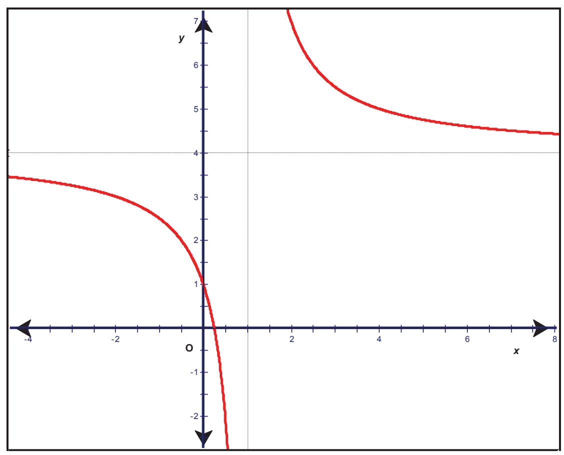
For:

$x<2,\ g(x)<0 \Rightarrow g(x)\to -\infty$ as $x\to 2^{-}$

$x>2,\ g(x)>0 \Rightarrow g(x)\to +\infty$ as $x\to 2^{+}$

As $x\to \pm\infty \Rightarrow g(x)\to 0 \Rightarrow y=0$ is a horizontal asymptote.

3 $h(x)=\dfrac{1-4x}{1-x}$



x-intercept since for $1-4x=0 \Rightarrow x=\dfrac{1}{4}$.

Vertical asymptote for $1-x=0 \Rightarrow x=1$.

When $x\to 1^{-}$: $h(0.9)=\dfrac{1-4\cdot 0.9}{1-0.9}<0 \Rightarrow h(x)\to -\infty$.

When $x\to 1^{+}$: $h(1.1)=\dfrac{1-4\cdot 1.1}{1-1.1}>0 \Rightarrow h(x)\to +\infty$.

As $x\to \pm\infty \Rightarrow h(x)=\dfrac{\frac{1}{x}-4}{\frac{1}{x}-1}=\dfrac{0-4}{0-1}=4 \Rightarrow y=4$ is a horizontal asymptote.

4 $R(x)=\dfrac{x}{x^2-9}=\dfrac{x}{(x+3)(x-3)}$

x-intercept for $x=0$.

Vertical asymptotes for $x+3=0 \Rightarrow x=-3$ and $x-3=0 \Rightarrow x=3$.

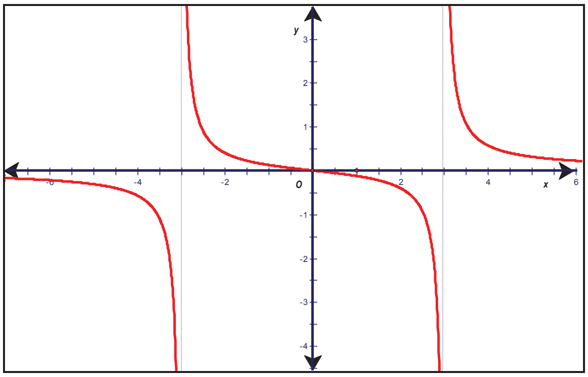
When $x\to -3^{-}$: $R(-3.1)=\dfrac{-3.1}{(-3.1)^2-9}<0 \Rightarrow R(x)\to -\infty$.

When $x\to -3^{+}$: $R(-2.9)=\dfrac{-2.9}{(-2.9)^2-9}>0 \Rightarrow R(x)\to +\infty$.

When $x\to 3^{-}$: $R(2.9)=\dfrac{2.9}{(2.9)^2-9}<0 \Rightarrow R(x)\to -\infty$.

When $x\to 3^{+}$: $R(3.1)=\dfrac{3.1}{(3.1)^2-9}>0 \Rightarrow R(x)\to +\infty$.

As $x \to \pm\infty \Rightarrow R(x) = \dfrac{\dfrac{x}{x}}{\dfrac{x^2}{x} - \dfrac{9}{x}} = \dfrac{1}{x - 0} \to 0 \Rightarrow y = 0$ is a horizontal asymptote.



5 $p(x) = \dfrac{2}{x^2 + 2x - 3} = \dfrac{2}{(x+3)(x-1)}$

No x-intercept since the numerator $\neq 0$.

Vertical asymptotes for $x + 3 = 0 \Rightarrow x = -3$ and $x - 1 = 0 \Rightarrow x = 1$.

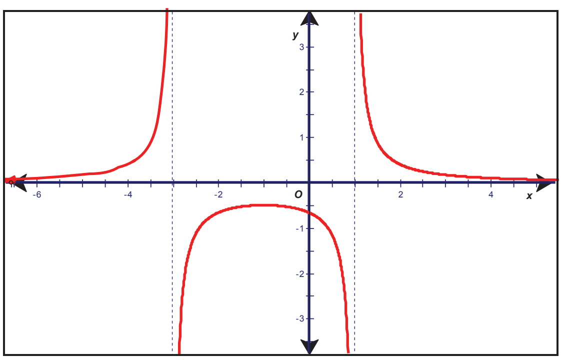
When $x \to -3^- : p(-3.1) = \dfrac{2}{(-3.1)^2 + 2(-3.1) - 3} > 0 \Rightarrow p(x) \to +\infty$.

When $x \to -3^+ : p(-2.9) = \dfrac{2}{(-2.9)^2 + 2(-2.9) - 3} < 0 \Rightarrow p(x) \to -\infty$.

When $x \to 1^- : p(0.9) = \dfrac{2}{(0.9)^2 + 2(0.9) - 3} < 0 \Rightarrow p(x) \to -\infty$.

When $x \to 1^+ : p(1.1) = \dfrac{2}{(1.1)^2 + 2(1.1) - 3} > 0 \Rightarrow p(x) \to +\infty$.

As $x \to \pm\infty \Rightarrow p(x) = \dfrac{\dfrac{2}{x^2}}{\dfrac{x^2}{x^2} + \dfrac{2}{x} - \dfrac{3}{x^2}} = \dfrac{0}{1 + 0 - 0} \to 0 \Rightarrow y = 0$ is a horizontal asymptote.



6 $M(x) = \dfrac{x^2+1}{x}$

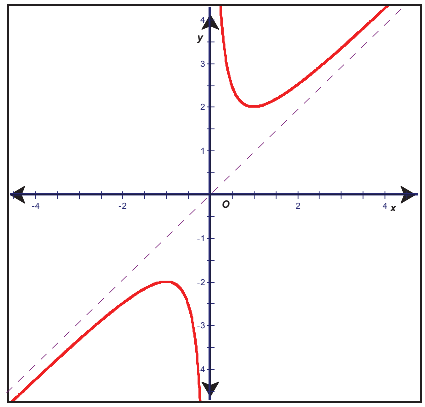
No x-intercept since the numerator $\neq 0$.

Vertical asymptote for $x = 0$.

When $x \to 0^-$: $M(-0.1) = \dfrac{(-0.1)^2+1}{-0.1} < 0 \Rightarrow M(x) \to -\infty$.

When $x \to 0^+$: $M(0.1) = \dfrac{(0.1)^2+1}{0.1} > 0 \Rightarrow M(x) \to +\infty$.

As $x \to \pm\infty \Rightarrow M(x) = x + \dfrac{1}{x} \to x \Rightarrow y = x$ is an oblique asymptote. There is no horizontal asymptote.



7 $f(x) = \dfrac{x}{x^2+4x+4} = \dfrac{x}{(x+2)^2}$

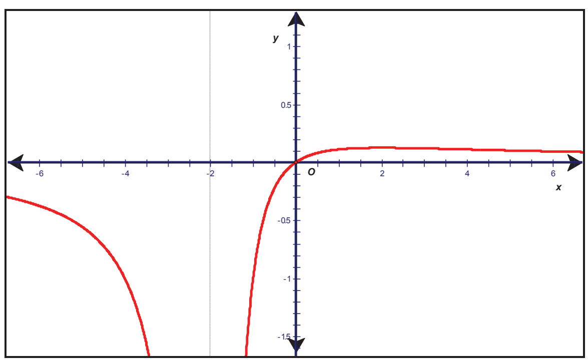
x-intercept for $x = 0$.

Vertical asymptote for $x + 2 = 0 \Rightarrow x = -2$.

When $x \to -2^-$: $f(-2.1) = \dfrac{-2.1}{(-2.1+2)^2} < 0 \Rightarrow f(x) \to -\infty$.

When $x \to -2^+$: $f(-1.9) = \dfrac{-1.9}{(-1.9+2)^2} < 0 \Rightarrow f(x) \to -\infty$.

As $x \to \pm\infty \Rightarrow f(x) = \dfrac{\frac{x}{x}}{\frac{x^2}{x} + 4\frac{x}{x} + \frac{4}{x}} = \dfrac{1}{x+4+0} \to 0 \Rightarrow y = 0$ is a horizontal asymptote.



8 $h(x) = \dfrac{x^2+2x}{x-1} = \dfrac{x(x+2)}{x-1}$

x-intercepts for $x = 0$ and $x + 2 = 0 \Rightarrow x = -2$.

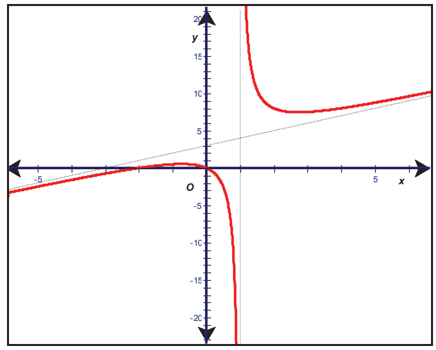
Vertical asymptote for $x - 1 = 0 \Rightarrow x = 1$.

When $x \to 1^-$: $h(0.9) = \dfrac{0.9(0.9+2)}{0.9-1} < 0 \Rightarrow h(x) \to -\infty$.

When $x \to 1^+$: $h(1.1) = \dfrac{1.1(1.1+2)}{1.1-1} > 0 \Rightarrow h(x) \to +\infty$.

Since the degree of the numerator is higher than the degree of the denominator, the function has an oblique asymptote. We divide:

1	1	2	0
		1	3
	1	3	3



The function can be written as: $h(x) = x + 3 + \dfrac{3}{x-1}$.

When $x \to \pm\infty, \dfrac{3}{x-1} \to 0$.

So, the oblique asymptote is $y = x + 3$.

9 $g(x) = \dfrac{2x+8}{x^2 - x - 12} = \dfrac{2x+8}{(x-4)(x+3)}$

x-intercept for $2x + 8 = 0 \Rightarrow x = -4$.

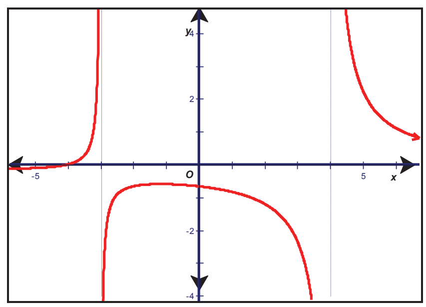
Vertical asymptotes for $x + 3 = 0 \Rightarrow x = -3$ and $x - 4 = 0 \Rightarrow x = 4$.

When $x \to -3^-$: $g(-3.1) = \dfrac{2(-3.1)+8}{(-3.1)^2 - (-3.1) - 12} > 0 \Rightarrow g(x) \to +\infty$.

When $x \to -3^+$: $g(-2.9) = \dfrac{2(-2.9)+8}{(-2.9)^2 - (-2.9) - 12} < 0 \Rightarrow g(x) \to -\infty$.

When $x \to 4^-$: $g(3.9) = \dfrac{2 \cdot 3.9 + 8}{(3.9)^2 - 3.9 - 12} < 0 \Rightarrow g(x) \to -\infty$.

When $x \to 4^+$: $g(4.1) = \dfrac{2 \cdot 4.1 + 8}{(4.1)^2 - 4.1 - 12} > 0 \Rightarrow g(x) \to +\infty$.



As $x \to \pm\infty \Rightarrow g(x) = \dfrac{\dfrac{2}{x} + \dfrac{8}{x^2}}{\dfrac{x^2}{x^2} - \dfrac{1}{x} - \dfrac{12}{x^2}} = \dfrac{0+0}{1-0-0} \to 0 \Rightarrow$

$y = 0$ is a horizontal asymptote.

10 $C(x) = \dfrac{x-2}{x^2 - 4x} = \dfrac{x-2}{x(x-4)}$

x-intercept for $x - 2 = 0 \Rightarrow x = 2$.

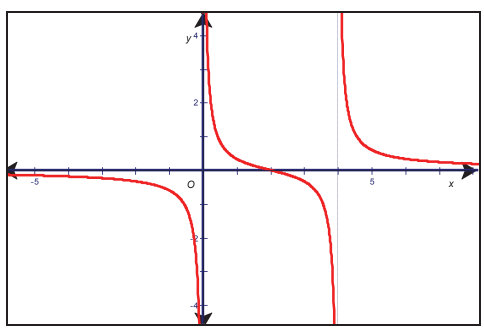
Vertical asymptotes for $x = 0$ and $x - 4 = 0 \Rightarrow x = 4$.

When $x \to 0^-$: $C(-0.1) = \dfrac{-0.1-2}{(-0.1)^2 - 4(-0.1)} < 0 \Rightarrow C(x) \to -\infty$.

When $x \to 0^+$: $C(0.1) = \dfrac{0.1-2}{(0.1)^2 - 4 \cdot 0.1} > 0 \Rightarrow C(x) \to +\infty$.

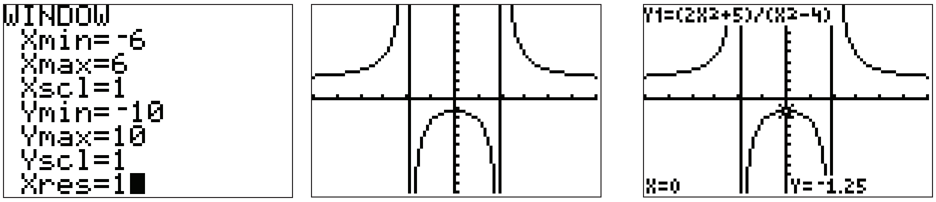
When $x \to 4^-$: $g(3.9) = \dfrac{3.9-2}{(3.9)^2 - 4 \cdot 3.9} < 0 \Rightarrow C(x) \to -\infty$.

When $x \to 4^+$: $C(4.1) = \dfrac{4.1-2}{(4.1)^2 - 4 \cdot 4.1} > 0 \Rightarrow C(x) \to +\infty$.



As $x \to \pm\infty \Rightarrow C(x) = \dfrac{\dfrac{1}{x} - \dfrac{2}{x^2}}{\dfrac{x^2}{x^2} - \dfrac{4}{x}} = \dfrac{0-0}{1-0} \to 0 \Rightarrow y = 0$ is a horizontal asymptote.

11 $f(x)=\dfrac{2x^2+5}{x^2-4}=\dfrac{2x^2+5}{(x-2)(x+2)}$



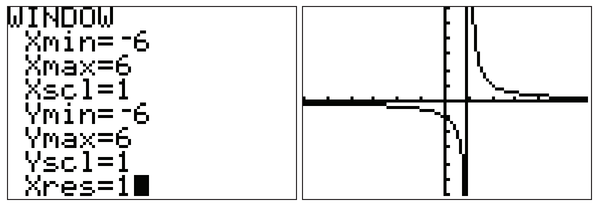
Vertical asymptotes: $x=-2$ and $x=2$.

Horizontal asymptote: $x\to\pm\infty \Rightarrow f(x)=\dfrac{2+\dfrac{5}{x^2}}{1-\dfrac{4}{x^2}}=\dfrac{2+0}{1-0}\Rightarrow y=2.$

Domain: $\{x : x\in\mathbb{R}, x\neq\pm2\}$.

Range: $\left\{y : y\leqslant -\dfrac{5}{4} \text{ or } y>2\right\}$.

12 $g(x)=\dfrac{x+4}{x^2+3x-4}=\dfrac{x+4}{(x+4)(x-1)}=\dfrac{1}{x-1}$



Vertical asymptotes: $x=1$ and $x=-4$.

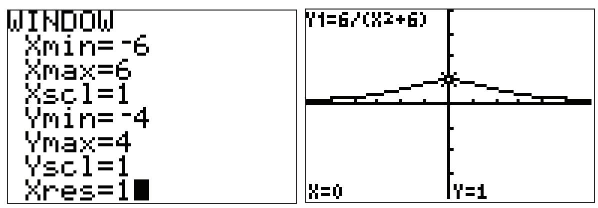
Horizontal asymptote:

$x\to\pm\infty \Rightarrow g(x)=\dfrac{\dfrac{1}{x}}{1-\dfrac{1}{x}}=\dfrac{0}{1-0}\Rightarrow y=0.$

Domain: $\{x : x\in\mathbb{R}, x\neq 1,-4\}$.

Range: $\{y : y\in\mathbb{R}, y\neq 0\}$.

13 $h(x)=\dfrac{6}{x^2+6}$



No vertical asymptotes since $x^2+6\neq 0$ for $x\in\mathbb{R}$.

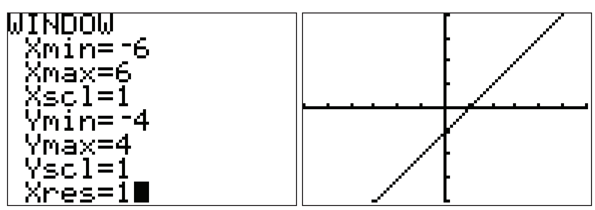
Horizontal asymptote:

$x\to\pm\infty \Rightarrow h(x)=\dfrac{\dfrac{6}{x^2}}{1+\dfrac{6}{x^2}}=\dfrac{0}{1+0}\Rightarrow y=0.$

Domain: $\{x : x\in\mathbb{R}\}$.

Range: $\{y : 0<y\leqslant 1\}$.

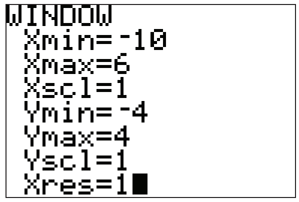
14 $r(x)=\dfrac{x^2-2x+1}{x-1}=\dfrac{(x-1)^2}{x-1}=x-1$

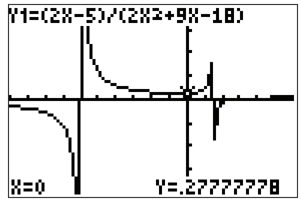


Domain: $\{x : x\in\mathbb{R}, x\neq 1\}$.

Range: $\{y : y\in\mathbb{R}, y\neq 0\}$.

15 $f(x) = \dfrac{2x - 5}{2x^2 + 9x - 18} = \dfrac{2x - 5}{2\left(x - \frac{3}{2}\right)(x + 6)}$

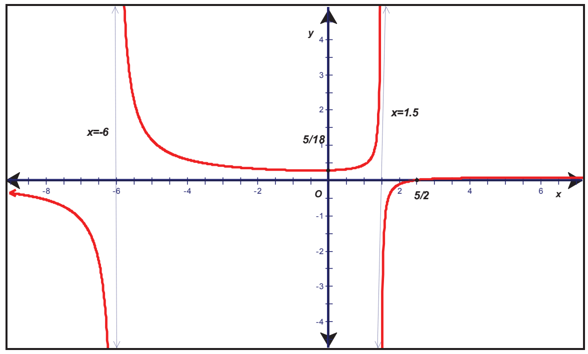




Vertical asymptotes: $x = \dfrac{3}{2}$ and $x = -6$.

Horizontal asymptote:

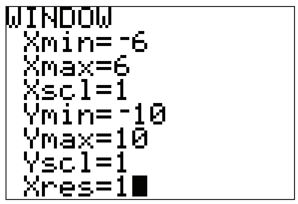
$$x \to \pm\infty \Rightarrow f(x) = \dfrac{\frac{2}{x} - \frac{5}{x^2}}{2 + \frac{9}{x} - \frac{18}{x^2}} = \dfrac{0 - 0}{2 + 0 - 0} \Rightarrow y = 0.$$

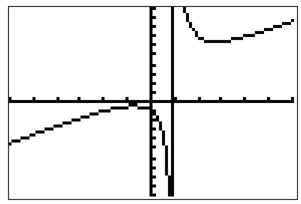


x-intercept: $2x - 5 = 0 \Rightarrow x = \dfrac{5}{2}$.

y-intercept: $f(0) = \dfrac{0 - 5}{0 + 0 - 18} = \dfrac{5}{18}$.

16 $g(x) = \dfrac{x^2 + x + 1}{x - 1}$



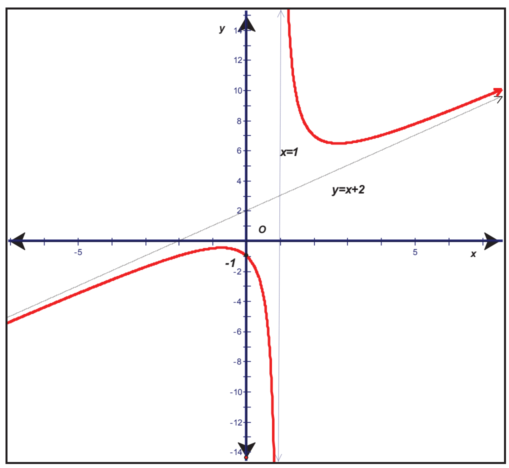


Vertical asymptote: $x = 1$.

No horizontal asymptotes.

Oblique asymptote; we divide:

1	1	1	1
		1	2
	1	2	3



The function can be written as: $g(x) = x + 2 + \dfrac{3}{x - 1}$.

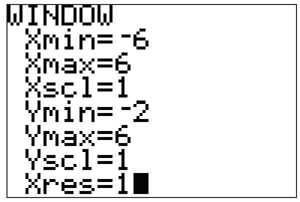
When $x \to \pm\infty$, $\dfrac{3}{x - 1} \to 0$.

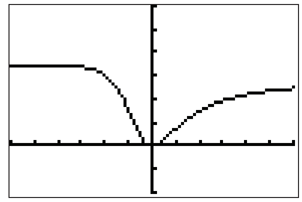
So, the oblique asymptote is $y = x + 2$.

No x-intercept since $x^2 + x + 1 \neq 0$ for $x \in \mathbb{R}$.

y-intercept: $h(0) = \dfrac{0 + 0 + 1}{0 - 1} = -1$.

17 $h(x) = \dfrac{3x^2}{x^2 + x + 2}$





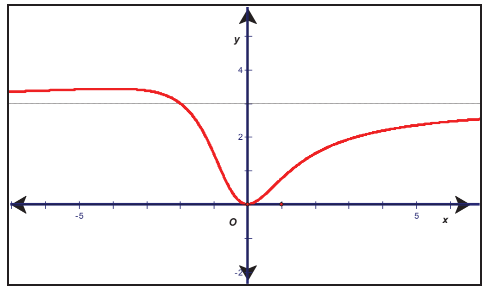
No vertical asymptote since $x^2 + x + 2 \neq 0$ for $x \in \mathbb{R}$.

Horizontal asymptote:

$$x \to \pm\infty \Rightarrow h(x) = \frac{3}{1 + \frac{1}{x} + \frac{2}{x^2}} = \frac{3}{1 + 0 + 0} \Rightarrow y = 3.$$

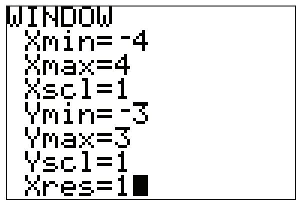
x-intercept for $x = 0$.

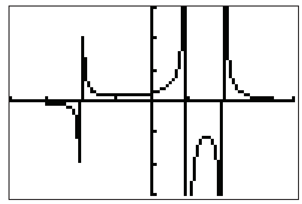
y-intercept: $h(0) = \dfrac{0}{0 + 0 + 2} = 0$.



18 $g(x) = \dfrac{1}{x^3 - x^2 - 4x + 4} = \dfrac{1}{x^2(x-1) - 4(x-1)}$

$= \dfrac{1}{(x-1)(x^2 - 4)} = \dfrac{1}{(x-1)(x-2)(x+2)}$





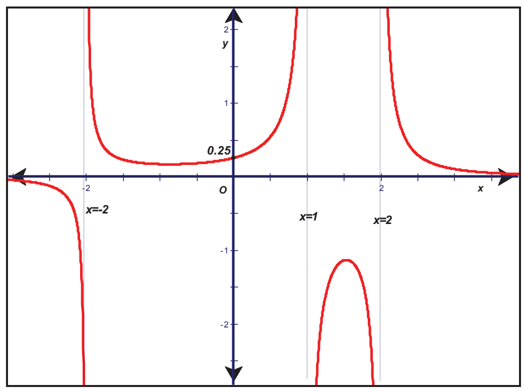
Vertical asymptotes: $x = -2, x = 1$ and $x = 2$.

Horizontal asymptote:

$$x \to \pm\infty \Rightarrow g(x) = \frac{\frac{1}{x^3}}{1 - \frac{1}{x} - \frac{4}{x^2} + \frac{4}{x^3}} = \frac{0}{1 - 0 - 0 + 0} \Rightarrow y = 0.$$

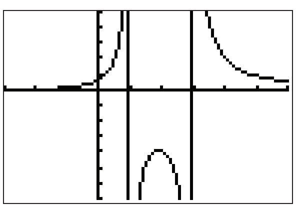
No x-intercept.

y-intercept: $g(0) = \dfrac{1}{0 - 0 - 0 + 4} = \dfrac{1}{4}$.

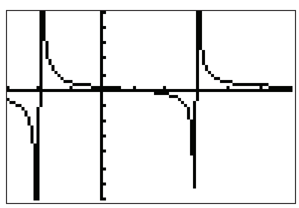


19 $y = \dfrac{x - a}{(x - b)(x - c)}$

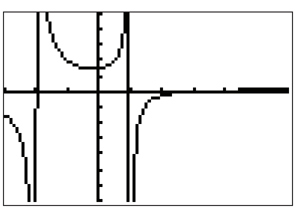
a) $a < b < c : a = -2, b = 1, c = 3$



b) $b < a < c : b = -2, a = 1, c = 3$

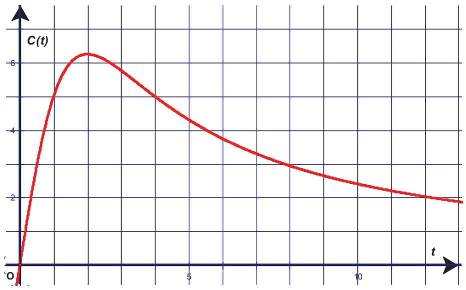


c) $b < c < a : b = -2, c = 1, a = 3$



20 $C(t) = \dfrac{25t}{t^2 + 4}$

a)

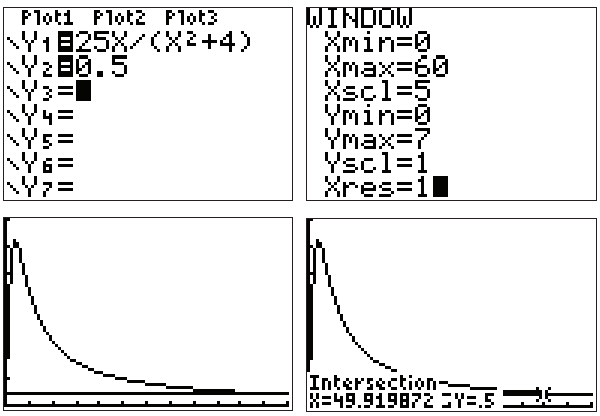


b) $C(2) = \dfrac{25 \cdot 2}{2^2 + 4} = \dfrac{50}{8} = 6.25$

At $t = 2$ minutes the concentration is at its highest: 6.25 mg/l.

c) The concentration continues to decrease and approaches zero as time increases.

d) We are looking for the intersection of $C(t)$ and $c = 0.5$.



The concentration drops below 0.5 mg/l after 49.9 minutes (49 minutes 54 seconds).

Exercise 3.5

1 $\sqrt{x+6} + 2x = 9$

$\sqrt{x+6} = 9 - 2x$

$\left(\sqrt{x+6}\right)^2 = (9 - 2x)^2$

$x + 6 = 81 - 36x + 4x^2$

$4x^2 - 37x + 75 = 0$

$x_{1,2} = \dfrac{37 \pm \sqrt{(-37)^2 - 4 \cdot 4 \cdot 75}}{2 \cdot 4} = \dfrac{37 \pm 13}{8}$

$x_1 = \dfrac{24}{8} = 3, x_2 = \dfrac{50}{8} = \dfrac{25}{4}$

Check:

$x = 3: \sqrt{3+6} + 2 \cdot 3 = 9 \Rightarrow 3 + 6 = 9$

$x = \dfrac{25}{4}: \sqrt{\dfrac{25}{4} + 6} + 2 \cdot \dfrac{25}{4} = 9 \Rightarrow$

$\sqrt{\dfrac{49}{4}} + \dfrac{25}{2} = 9 \Rightarrow \dfrac{7}{2} + \dfrac{25}{2} = 9 \Rightarrow 17 \neq 9$

Therefore, $x = 3$ is the only solution.

2 $\sqrt{x+7}+5=x$

$\sqrt{x+7}=x-5$

$\left(\sqrt{x+7}\right)^2=(x-5)^2$

$x+7=x^2-10x+25$

$x^2-11x+18=0$

$x_{1,2}=\dfrac{11\pm\sqrt{(-11)^2-4\cdot 11}}{2}=\dfrac{11\pm 7}{2}$

$x_1=2, x_2=9$

Check:

$x=2:\sqrt{2+7}+5=2\Rightarrow 3+5=2\Rightarrow 8\neq 2$

$x=9:\sqrt{9+7}+5=9\Rightarrow 4+5=9\Rightarrow 9=9$

Therefore, $x=9$ is the only solution.

3 $\sqrt{7x+14}-2=x$

$\sqrt{7x+14}=x+2$

$\left(\sqrt{7x+14}\right)^2=(x+2)^2$

$7x+14=x^2+4x+4$

$x^2-3x-10=0$

$x_{1,2}=\dfrac{3\pm\sqrt{(-3)^2-4\cdot(-10)}}{2}=\dfrac{3\pm 7}{2}$

$x_1=-2, x_2=5$

Check:

$x=-2:\sqrt{7(-2)+14}-2=-2\Rightarrow -2=-2$

$x=5:\sqrt{7\cdot 5+14}-2=5\Rightarrow 7-2=5\Rightarrow 5=5$

Therefore, $x=-2$ and $x=5$ are both solutions.

4 $\sqrt{2x+3}-\sqrt{x-2}=2$

$\sqrt{2x+3}=2+\sqrt{x-2}$

$\left(\sqrt{2x+3}\right)^2=\left(2+\sqrt{x-2}\right)^2$

$2x+3=4+4\sqrt{x-2}+x-2$

$x+1=4\sqrt{x-2}$

$(x+1)^2=\left(4\sqrt{x-2}\right)^2$

$x^2+2x+1=16x-32$

$x^2-14x+33=0$

$x_{1,2}=\dfrac{14\pm\sqrt{(-14)^2-4\cdot 33}}{2}=\dfrac{14\pm 8}{2}$

$x_1=3, x_2=11$

Check:

$x=3:\sqrt{2\cdot 3+3}-\sqrt{3-2}=2\Rightarrow 3-1=2\Rightarrow 2=2$

$x=11:\sqrt{2\cdot 11+3}-\sqrt{11-2}=2\Rightarrow 5-3=2$

Therefore, $x=3$ and $x=11$ are both solutions.

5 $\dfrac{5}{x+4}-\dfrac{4}{x}=\dfrac{21}{5x+20}$

$\dfrac{5}{x+4}-\dfrac{4}{x}=\dfrac{21}{5(x+4)}\Big/\cdot 5x(x+4)$

$25x-20(x+4)=21x$

$-16x=80\Rightarrow x=-\dfrac{80}{16}=-5$

6 $\dfrac{x+1}{2x+3}=\dfrac{5x-1}{7x+3}$

$(x+1)(7x+3)=(5x-1)(2x+3)$

$7x^2+10x+3=10x^2+13x-3$

$3x^2+3x-6=0$

$x^2+x-2=0$

$(x+2)(x-1)=0$

$x_1=-2, x_2=1$

7 $\dfrac{1}{x}-\dfrac{1}{x+1}=\dfrac{1}{x+4}\Big/\cdot x(x+1)(x+4)$

$(x+1)(x+4)-x(x+4)=x(x+1)$

$x^2+5x+4-x^2-4x=x^2+x$

$x^2-4=0$

$(x+2)(x-2)=0$

$x_1=-2, x_2=2$

8 $\dfrac{2x}{1-x^2}+\dfrac{1}{x+1}=2$

$\dfrac{2x}{(1-x)(1+x)}+\dfrac{1}{x+1}=2/\cdot(1-x)(1+x)$

$2x+1-x=2(1-x)(1+x)$

$x+1=2-2x^2$

$2x^2+x-1=0$

$x_{1,2}=\dfrac{-1\pm\sqrt{1^2-4\cdot 2\cdot(-1)}}{2\cdot 2}=\dfrac{-1\pm 3}{4}$

$x_1=-1, x_2=\dfrac{1}{2}$

$x=-1$ cannot be a solution because that would mean division by zero in the original equation.

Therefore, $x=\dfrac{1}{2}$ is the only solution.

9 $x^4 - 2x^2 - 15 = 0$

We substitute $t = x^2$:

$t^2 - 2t - 15 = 0$

$t_{1,2} = \frac{2 \pm \sqrt{(-2)^2 - 4(-15)}}{2} = \frac{2 \pm 8}{2}$

$t_1 = -3, t_2 = 5 \Rightarrow$

$x^2 = 5 \Rightarrow x_1 = \sqrt{5}, x_2 = -\sqrt{5}$

$x^2 = -3 \Rightarrow$ not real solution

10 $2x^{\frac{2}{3}} - x^{\frac{1}{3}} - 15 = 0$

We substitute $t = x^{\frac{1}{3}}$:

$2t^2 - t - 15 = 0$

$t_{1,2} = \frac{1 \pm \sqrt{(-1)^2 - 4 \cdot 2 \cdot (-15)}}{2 \cdot 2} = \frac{1 \pm 11}{4}$

$t_1 = -\frac{10}{4} = -\frac{5}{2}, t_2 = 3$

$x^{\frac{1}{3}} = -\frac{5}{2} \Rightarrow x = \left(-\frac{5}{2}\right)^3 \Rightarrow x_1 = -\frac{125}{8}$

$x^{\frac{1}{3}} = 3 \Rightarrow x = 3^3 \Rightarrow x_2 = 27$

11 $x^6 - 35x^3 + 216 = 0$

We substitute $t = x^3$:

$t^2 - 35t + 216 = 0$

$t_{1,2} = \frac{35 \pm \sqrt{(-35)^2 - 4 \cdot 216}}{2} = \frac{35 \pm 19}{2}$

$t_1 = 8, t_2 = 27$

$x^3 = 8 \Rightarrow x_1 = 2$

$x^3 = 27 \Rightarrow x_2 = 3$

12 $5x^{-2} - x^{-1} - 2 = 0$

We substitute $t = x^{-1}$:

$5t^2 - t - 2 = 0$

$t_{1,2} = \frac{1 \pm \sqrt{(-1)^2 - 4 \cdot 5 \cdot (-2)}}{2 \cdot 5} = \frac{1 \pm \sqrt{41}}{10}$

$x^{-1} = \frac{1 \pm \sqrt{41}}{10} \Rightarrow x = \frac{10}{1 \pm \sqrt{41}} \cdot \frac{1 \mp \sqrt{41}}{1 \mp \sqrt{41}}$

$= \frac{10\left(1 \mp \sqrt{41}\right)}{-40} = \frac{1 \mp \sqrt{41}}{-4}$

$x_1 = \frac{-1 + \sqrt{41}}{4}, x_2 = \frac{-1 - \sqrt{41}}{4}$

13 $|3x + 4| = 8$

$3x + 4 = 8$ or $3x + 4 = -8$

$3x = 4$ $\quad$ $3x = -12$

$x = \frac{4}{3}$ $\quad$ $x = -4$

The solutions are: $x = \frac{4}{3}$ or $x = -4$.

14 $|x + 6| = |3x - 24|$

There are four possibilities:

(i) $x + 6 = 3x - 24$

(ii) $x + 6 = -(3x + 24)$

(iii) $-(x + 6) = 3x - 24$

(iv) $-(x + 6) = -(3x - 24)$

Equations (i) and (iv), and (ii) and (iii), are equivalent, so we only need to solve (i) and (ii).

$x + 6 = 3x - 24$ or $x + 6 = -3x + 24$

$2x = 30$ $\quad$ $4x = 18$

$x = 15$ $\quad$ $x = \frac{9}{2}$

The solutions are: $x = \frac{9}{2}$ or $x = 15$.

15 $|5x + 1| = 2x$

$5x + 1 = 2x$ or $5x + 1 = -2x$

$3x = -1$ $\quad$ $7x = -1$

$x = -\frac{1}{3}$ $\quad$ $x = -\frac{1}{7}$

We substitute these values into the original equation:

$x = -\frac{1}{3} : \left|-\frac{5}{3} + 1\right| = 2 \cdot \left(-\frac{1}{3}\right) \Rightarrow \frac{2}{3} \neq -\frac{2}{3}$

$x = -\frac{1}{7} : \left|-\frac{5}{7} + 1\right| = 2 \cdot \left(-\frac{1}{7}\right) \Rightarrow \frac{2}{7} \neq -\frac{2}{7}$

Neither value satisfies the original equation, so there is no solution.

16 $|x-1|+|x|=3$

There are four possibilities:

(i) $x-1+x=3 \Rightarrow 2x=4 \Rightarrow x=2$

(ii) $x-1-x=3 \Rightarrow -1=3$, which is not possible

(iii) $-(x-1)+x=3 \Rightarrow 1=3$, which is not possible

(iv) $-(x-1)-x=3 \Rightarrow -2x=2 \Rightarrow x=-1$

The solutions are: $x=2$ or $x=-1$.

17 $\left|\frac{x+1}{x-1}\right|=3$

$\frac{x+1}{x-1}=3$ or $\frac{x+1}{x-1}=-3$

$x+1=3(x-1)$ $\quad$ $x+1=-3(x-1)$

$x+1=3x-3$ $\quad$ $x+1=-3x+3$

$2x=4$ $\quad$ $4x=2$

$x=2$ $\quad$ $x=\frac{1}{2}$

The solutions are: $x=2$ or $x=\frac{1}{2}$.

18 $\sqrt{x}-\frac{6}{\sqrt{x}}=1$

$\frac{\sqrt{x}\sqrt{x}-6}{\sqrt{x}}=1$

$\frac{x-6}{\sqrt{x}}=1$

$x-6=\sqrt{x}$

$(x-6)^2=\left(\sqrt{x}\right)^2$

$x^2-12x+36=x$

$x^2-13x+36=0$

$x_{1,2}=\frac{13\pm\sqrt{(-13)^2-4\cdot 36}}{2}=\frac{13\pm 5}{2}$

$x_1=4, x_2=9$

Check:

$x=4: \sqrt{4}-\frac{6}{\sqrt{4}}=1 \Rightarrow 2-3=1 \Rightarrow -1\neq 1$

$x=9: \sqrt{9}-\frac{6}{\sqrt{9}}=1 \Rightarrow 3-2=1 \Rightarrow 1=1$

Therefore, $x=9$ is the only solution.

19 $\sqrt{4-x}-\sqrt{6+x}=\sqrt{14+2x}$

$\left(\sqrt{4-x}-\sqrt{6+x}\right)^2=\left(\sqrt{14+2x}\right)^2$

$4-x-2\sqrt{(4-x)(6+x)}+6+x=14+2x$

$-4-2x=2\sqrt{(4-x)(6+x)}$

$(-2-x)^2=\left(\sqrt{(4-x)(6+x)}\right)^2$

$4+4x+x^2=(4-x)(6+x)$

$4+4x+x^2=24-2x-x^2$

$2x^2+6x-20=0$

$x^2+3x-10=0$

$x_{1,2}=\frac{-3\pm\sqrt{3^2-4(-10)}}{2}=\frac{-3\pm 7}{2}$

$x_1=-5, x_2=2$

Check:

$x=-5: \sqrt{4-(-5)}-\sqrt{6+(-5)}=\sqrt{14+2(-5)}$

$\Rightarrow \sqrt{9}-\sqrt{1}=\sqrt{4} \Rightarrow 3-1=2 \Rightarrow 2=2$

$x=2: \sqrt{4-2}-\sqrt{6+2}=\sqrt{14+2\cdot 2}$

$\Rightarrow \sqrt{2}-\sqrt{8}=\sqrt{18}$

Therefore, $x=-5$ is the only solution.

20 $\frac{6}{x^2+1}=\frac{1}{x^2}+\frac{10}{x^2+4} \Big/ \cdot x^2\left(x^2+1\right)\left(x^2+4\right)$

$6x^2\left(x^2+4\right)=\left(x^2+1\right)\left(x^2+4\right)+10x^2\left(x^2+1\right)$

$6x^4+24x^2=x^4+5x^2+4+10x^4+10x^2$

$4x^4-9x^2+4=0$

We substitute $t=x^2$:

$5t^2-9t+4=0$

$t_{1,2}=\frac{9\pm\sqrt{(-9)^2-4\cdot 5\cdot 4}}{2\cdot 5}=\frac{9\pm 1}{10}$

$t_1=\frac{4}{5}, t_2=1$

$x^2=\frac{4}{5} \Rightarrow x_{1,2}=\pm\sqrt{\frac{4}{5}}=\pm\frac{2}{\sqrt{5}}=\pm\frac{2\sqrt{5}}{5}$

$x^2=1 \Rightarrow x_{3,4}=\pm 1$

21 $x - \sqrt{x+10} = 0$

$x = \sqrt{x+10}$

$x^2 = \left(\sqrt{x+10}\right)^2$

$x^2 = x + 10$

$x^2 - x - 10 = 0$

$x_{1,2} = \dfrac{1 \pm \sqrt{(-1)^2 - 4(-10)}}{2} = \dfrac{1 \pm \sqrt{41}}{2}$

Check:

Since $\dfrac{1-\sqrt{41}}{2} < 0$, the left side of the equation is negative for $x = \dfrac{1-\sqrt{41}}{2}$; so, the only solution is $x = \dfrac{1+\sqrt{41}}{2}$.

22 $6x - 37\sqrt{x} + 56 = 0$

We substitute $t = \sqrt{x}$:

$6t^2 - 37t + 56 = 0$

$t_{1,2} = \dfrac{37 \pm \sqrt{(-37)^2 - 4 \cdot 6 \cdot 56}}{2 \cdot 6} = \dfrac{37 \pm 5}{12}$

$t_1 = \dfrac{42}{12} = \dfrac{7}{2}, t_2 = \dfrac{32}{12} = \dfrac{8}{3}$

$\sqrt{x} = \dfrac{7}{2} \Rightarrow x_1 = \dfrac{49}{4}$

$\sqrt{x} = \dfrac{8}{3} \Rightarrow x_2 = \dfrac{64}{9}$

23 $3x^2 - 4 < 4x$

$3x^2 - 4x - 4 < 0$

We solve the quadratic equation to factorize the expression.

$3x^2 - 4x - 4 = 0 \Rightarrow$

$x_{1,2} = \dfrac{4 \pm \sqrt{(-4)^2 - 4 \cdot 3 \cdot (-4)}}{2 \cdot 3} = \dfrac{4 \pm 8}{6} \Rightarrow$

$x_1 = -\dfrac{2}{3}, x_2 = 2$

The inequality can be factorized as:

$3\left(x + \dfrac{2}{3}\right)(x-2) < 0$

We analyze the signs of the two factors in a 'sign chart':

x		$-\frac{2}{3}$		2	
$x + \frac{2}{3}$	−	**0**	+		+
$x - 2$	−		−	**0**	+
$\left(x + \frac{2}{3}\right)(x-2)$	+	**0**	−	**0**	+

The solution is: $-\dfrac{2}{3} < x < 2$.

24 $\dfrac{2x-1}{x+2} \geqslant 1$

$\dfrac{2x-1}{x+2} - 1 \geqslant 0$

$\dfrac{2x-1-(x+2)}{x+2} \geqslant 0$

$\dfrac{x-3}{x+2} \geqslant 0$

We analyze the signs of the numerator and denominator in a 'sign chart':

x		-2		3	
$x + 2$	−	**0**	+		+
$x - 3$	−		−	**0**	+
$\frac{x-3}{x+2}$	+	**X**	−	**0**	+

$x \neq -2$ because this would result in a denominator of zero.

The solution is: $x < -2$ or $x \geqslant 3$.

25 $2x^2 + 8x \leqslant 120$

$2x^2 + 8x - 120 \leqslant 0$

$x^2 + 4x - 60 \leqslant 0$

We solve the quadratic equation to factorize the expression.

$x^2 + 4x - 60 = 0 \Rightarrow$

$$x_{1,2} = \frac{-4 \pm \sqrt{4^2 - 4 \cdot (-60)}}{2} = \frac{-4 \pm 16}{2} \Rightarrow$$

$x_1 = -10, x_2 = 6$

The inequality can be factorized as:
$(x+10)(x-6) \leqslant 0$

We analyze the signs of the two factors in a 'sign chart':

		-10		6	x
$x+10$	$-$	**0**	$+$		$+$
$x-6$	$-$		$-$	**0**	$+$
$(x+10)(x-6)$	$+$	**0**	$-$	**0**	$+$

The solution is: $-10 \leqslant x \leqslant 6$.

26 $|1-4x| > 7 \Rightarrow$

$1-4x < -7$ or $1-4x > 7$

$-4x < -8$ $\quad$ $-4x > 6$

$x > 2$ $\quad$ $x < -\frac{3}{2}$

The solution is: $x < -\frac{3}{2}$ or $x > 2$.

Solution Paper 1 type

27 $|x-3| > |x-14|$

Since the expression on both sides must be positive, we can square both sides and remove the absolute value signs.

$(x-3)^2 > (x-14)^2$

$\not{x}^{\not{2}} - 6x + 9 > \not{x}^{\not{2}} - 28x + 196$

$22x > 187$

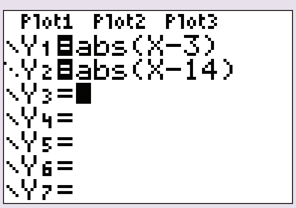
$2x > 17$

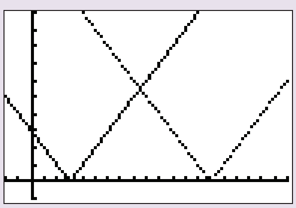
$x > \frac{17}{2}$

The solution is: $x > \frac{17}{2}$.

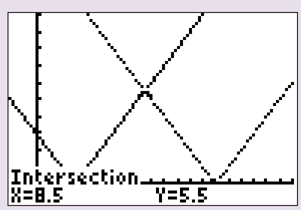
Solution Paper 2 type

27





The solution set is all real numbers for which the graph of the function y_1 (full line) is above the graph of y_2 (dashed). We use the intersect command on the GDC:



The solution is: $x > \frac{17}{2}$.

Solution Paper 1 type

28 $\left|\frac{x^2-4}{x}\right| \leqslant 3$

$-3 \leqslant \frac{x^2-4}{x} \leqslant 3$

We have two inequalities:

$$\frac{x^2-4}{x} \geqslant -3 \qquad \frac{x^2-4}{x} \leqslant 3$$

$$\frac{x^2-4}{x} + 3 \geqslant 0 \qquad \frac{x^2-4}{x} - 3 \leqslant 0$$

$$\frac{x^2+3x-4}{x} \geqslant 0 \qquad \frac{x^2-3x-4}{x} \leqslant 0$$

$$\frac{(x+4)(x-1)}{x} \geqslant 0 \qquad \frac{(x+1)(x-4)}{x} \leqslant 0$$

'Sign chart' for the first inequality:

		-4		0		1		x
$x+4$	−	0	+	⋮	+	|	+	
$x-1$	−	|	−	⋮	−	0	+	
x	−	|	−	0	+	|	+	
$\frac{(x+4)(x-1)}{x}$	−	0	+	⋮	−	0	+	

The solution of the first inequality is: $-4 \leqslant x < 0$ or $x \geqslant 1$.

'Sign chart' for the second inequality:

		-1		0		4		x
$x+1$	−	0	+	⋮	+	|	+	
$x-4$	−	|	−	⋮	−	0	+	
x	−	|	−	0	+	|	+	
$\frac{(x+1)(x-4)}{x}$	−	0	+	⋮	−	0	+	

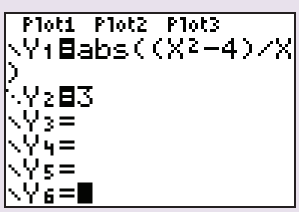
The solution of the second inequality is: $x \leqslant -1$ or $0 < x \leqslant 4$.

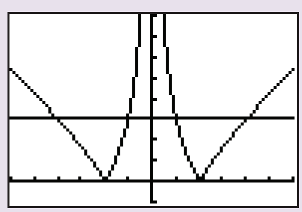
The complete solution is the intersection of the two previous solutions, i.e. $-4 \leqslant x \leqslant -1$ or $1 \leqslant x \leqslant 4$.

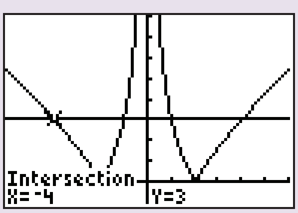
Solution Paper 2 type

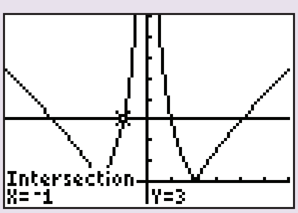
28 The solution set is all real numbers for which the graph of the function y_1 is below the line $y_2 = 3$. We use the intersect command on the GDC repeatedly.

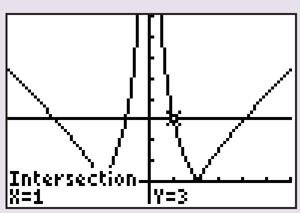
The solution is: $-4 \leqslant x \leqslant -1$ or $1 \leqslant x \leqslant 4$.

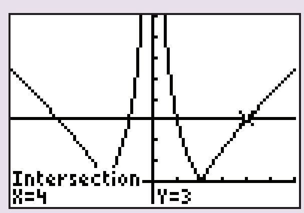












Solution Paper 1 type

29 $\dfrac{x}{x-2} > \dfrac{1}{x+1}$

$$\frac{x}{x-2} - \frac{1}{x+1} > 0$$

$$\frac{x(x+1)-(x-2)}{(x-2)(x+1)} > 0$$

$$\frac{x^2+2}{(x-2)(x+1)} > 0$$

Since the numerator is the sum of the square and 2, it is always positive; so, the inequality can be reduced to $(x+1)(x-2) > 0$.

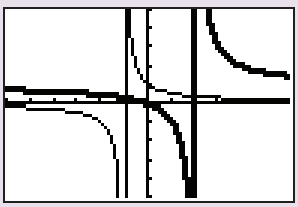
We analyze the signs of the two factors in a 'sign chart':

x		-1		2	
$x+1$	−	**0**	+		+
$x-2$	−		−	**0**	+
$(x+1)(x-2)$	+	**0**	−	**0**	+

The solution is: $x < -1$ or $x > 2$.

Solution Paper 2 type

29

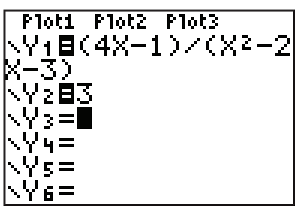


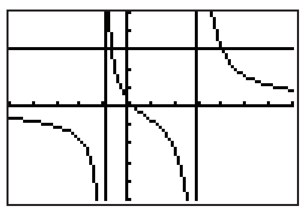
The solution set is all real numbers for which the graph of the function y_1 (thick line) is above the graph of y_2 (thin).

We see that the solution is: $x < -1$ or $x > 2$.

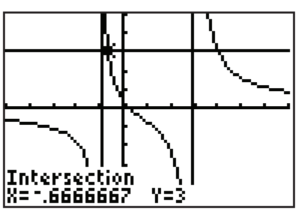
30 $\dfrac{4x-1}{x^2-2x-3} < 3$

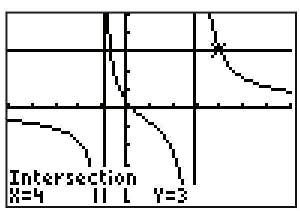
$$\frac{4x-1}{(x+1)(x-3)} < 3$$





The solution set is all real numbers for which the graph of the function y_1 is below the line $y_2 = 3$. We use the intersect command on the GDC repeatedly:





Taking into consideration that the function is not defined for the zeros of the denominator (−1 and 3), we see that the solution is: $x < -1$ or $-\dfrac{2}{3} < x < 3$ or $x > 4$.

31 The type of solution for the quadratic equation $px^2 - 3x + 1 = 0$ depends on its discriminant: $\Delta = (-3)^2 - 4p = 9 - 4p$.

a) The equation has one real solution if $\Delta = 0 \Rightarrow 9 - 4p = 0 \Rightarrow p = \frac{9}{4}$.

b) The equation has two real solutions if $\Delta > 0 \Rightarrow 9 - 4p > 0 \Rightarrow p < \frac{9}{4}$.

c) The equation has no real solutions if $\Delta < 0 \Rightarrow 9 - 4p < 0 \Rightarrow p > \frac{9}{4}$.

32 For $f(x) = x^2 + x(k-1) + k^2$, $f(x) > 0$ for all real values of x if the discriminant is negative so that the function has no zeros, i.e.

$$\Delta = (k-1)^2 - 4k^2 = k^2 - 2k + 1 - 4k^2 = -3k^2 - 2k + 1 < 0$$

$$-3k^2 - 2k + 1 = 0 \Rightarrow k_{1,2} = \frac{2 \pm \sqrt{(-2)^2 - 4(-3)}}{2(-3)} = \frac{2 \pm 4}{-3} \Rightarrow k_1 = -2, k_2 = \frac{2}{3}$$

The inequality can be factorized as: $-3(k+2)\left(k - \frac{2}{3}\right) < 0 \Rightarrow (k+2)\left(k - \frac{2}{3}\right) > 0$

We analyze the signs of the two factors in a 'sign chart':

x		-2		$\frac{2}{3}$	
$k+2$	−	**0**	+		+
$k - \frac{2}{3}$	−		−	**0**	+
$(k+2)\left(k - \frac{2}{3}\right)$	+	**0**	−	**0**	+

The solution is: $k < -2$ or $k > \frac{2}{3}$.

33 For $m > n > 0, m, n \in \mathbb{R}$, we have:

a) $m > n > 0 \Rightarrow mn > n^2 \Rightarrow mn - 2n + 1 > n^2 - 2n + 1 = (n-1)^2 > 0 \Rightarrow$

$mn - 2n + 1 > 0 \Rightarrow mn - 2n + 1 > 0 \Rightarrow mn + 1 > 2n \Rightarrow m + \frac{1}{n} > 2$

b) $(m-n)^2 > 0 \Rightarrow m^2 - 2mn + n^2 > 0 \Rightarrow m^2 + 2mn + n^2 > 4mn \Rightarrow$

$(m+n)^2 > 4mn \Rightarrow (m+n)(m+n) > 4mn \Rightarrow (m+n)\left(\frac{1}{n} + \frac{1}{m}\right)mn > 4mn \Rightarrow$

$(m+n)\left(\frac{1}{m} + \frac{1}{n}\right) > 4$

34 For $\left(x^2 + x\right)^2 = 5x^2 + 5x - 6$, we introduce the substitution $t = x^2 + x$. Then:

$t^2 = 5t - 6$

$t^2 - 5t + 6 = 0$

$(t-2)(t-3) = 0$

$t_1 = 2, t_2 = 3$

$x^2 + x = 2$	$x^2 + x = 3$
$x^2 + x - 2 = 0$	$x^2 + x - 3 = 0$
$(x+2)(x-1) = 0$	$x_{3,4} = \dfrac{-1 \pm \sqrt{1^2 - 4(-3)}}{2}$
$x_1 = -2, x_2 = 1$	$x_{3,4} = \dfrac{-1 \pm \sqrt{13}}{2}$

35 Since a, b and c are positive and unequal, we have:

$(a-b)^2 + (b-c)^2 + (c-a)^2 > 0 \Rightarrow a^2 - 2ab + b^2 - 2bc + c^2 + c^2 - 2ac + a^2 > 0 \Rightarrow$

$2a^2 + 2b^2 + 2c^2 - 2ab - 2ac - 2bc > 0 \Rightarrow 3a^2 + 3b^2 + 3c^2 - 2ab - 2ac - 2bc > a^2 + b^2 + c^2 \Rightarrow$

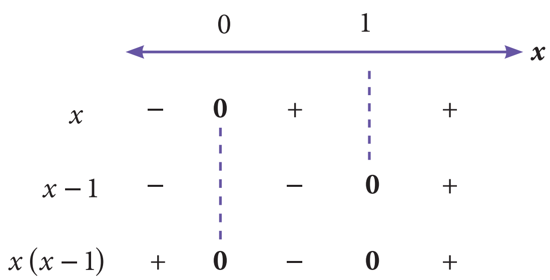
$3a^2 + 3b^2 + 3c^2 > 2ab + 2ac + 2bc + a^2 + b^2 + c^2 \Rightarrow 3\left(a^2 + b^2 + c^2\right) > (a+b+c)^2$

36 **a)** $\left|\dfrac{2x-3}{x}\right| < 1$

$-1 < \dfrac{2x-3}{x} < 1$

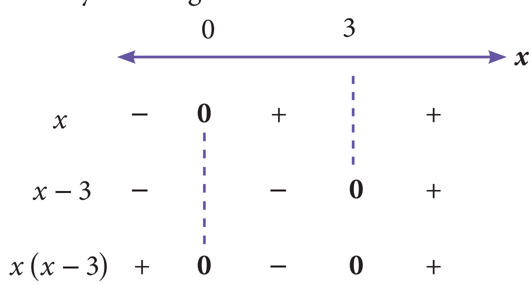
$-1 < \dfrac{2x-3}{x}$	and	$\dfrac{2x-3}{x} < 1$
$\dfrac{2x-3}{x} + 1 > 0$		$\dfrac{2x-3}{x} - 1 < 0$
$\dfrac{x-1}{x} > 0$		$\dfrac{x-3}{x} < 0$

We analyze the signs of the numerator and denominator for the first inequality in a 'sign chart':



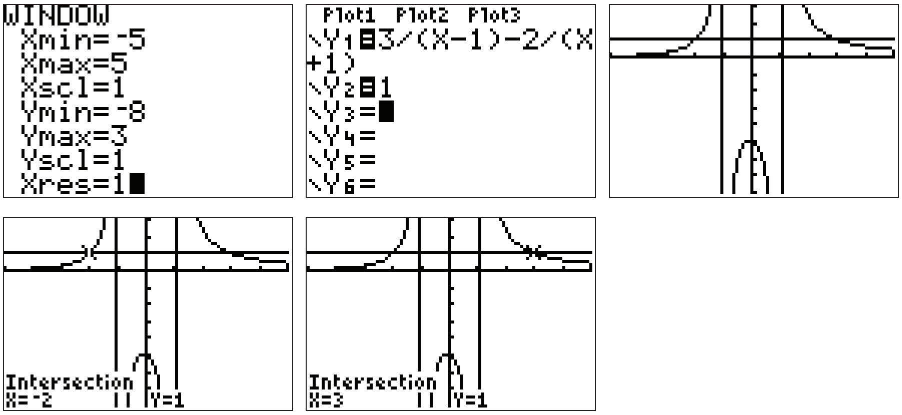
The first solution is: $x < 0$ or $x > 1$.

Next, we analyze the signs of the numerator and denominator for the second inequality:



The second solution is: $0 < x < 3$.

The final solution is the intersection of the two solutions, i.e. $1 < x < 3$.

b) 

We see that $\dfrac{3}{x-1} - \dfrac{2}{x+1}$ is below the line $y = 2$ for: $x < -2, -1 < x < 1, x > 3$.

37 If a and b have the same sign, then $|a+b| = |a| + |b|$.

If a and b are of opposite sign, then $|a+b| < |a| + |b| \Rightarrow$

$|a+b| \leqslant |a| + |b|$.

Chapter 4

Exercise 4.1

In questions 1–6, we have to substitute n (or k) = 1, 2, …, 5 into the given formula.

1 $s(n) = 2n - 3 \Rightarrow s(1) = -1, s(2) = 1, s(3) = 3, s(4) = 5, s(5) = 7$

2 $g(k) = 2^k - 3 \Rightarrow g(1) = -1, g(2) = 1, g(3) = 5, g(4) = 13, g(5) = 29$

3 $f(k) = 3 \times 2^{-n} \Rightarrow f(1) = \frac{3}{2}, f(2) = \frac{3}{4}, f(3) = \frac{3}{8}, f(4) = \frac{3}{16}, f(5) = \frac{3}{32}$

4 $\begin{cases} a_1 = 5 \\ a_n = a_{n-1} + 3 \end{cases} \Rightarrow a_1 = 5, a_2 = 8, a_3 = 11, a_4 = 14, a_5 = 17$

5 $a_n = (-1)^n(2^n) + 3 \Rightarrow a_1 = 1, a_2 = 7, a_3 = -5, a_4 = 19, a_5 = -29$

6 $\begin{cases} b_1 = 3 \\ b_n = b_{n-1} + 2n \end{cases} \Rightarrow b_1 = 3, b_2 = 7, b_3 = 13, b_4 = 21, b_5 = 31$

In questions 7–14, we have to substitute n = 1, 2, …, 5 into the given formula.

7 $-1, 1, 3, 5, 7;\ a_{50} = 2 \cdot 50 - 3 = 97$

8 $2, 6, 18, 54, 162;\ b_{50} = 2 \cdot 3^{49} = 4.786 \times 10^{23}$

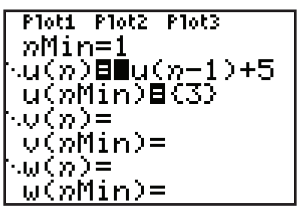
9 $\frac{2}{3}, -\frac{2}{3}, \frac{6}{11}, -\frac{4}{9}, \frac{10}{27};\ u_{50} = (-1)^{49}\frac{2 \cdot 50}{50^2 + 2} = -\frac{100}{2502} = -\frac{50}{1251}$

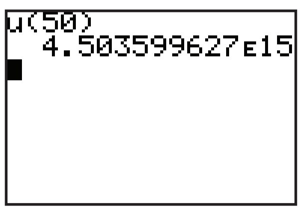
10 $1, 2, 9, 64, 625;\ a_{50} = 50^{49} = 1.776 \cdot 10^{83}$

In questions 11–14, we start with the first term and substitute it in the given formula to find the second term, and so on. To find the 50th term, we will use a GDC in sequential mode.

11 $3, 11, 27, 59, 123$

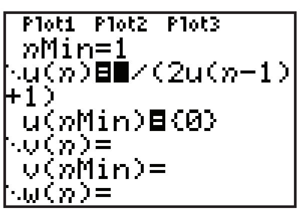
$a_{50} = 4.50 \times 10^{15}$

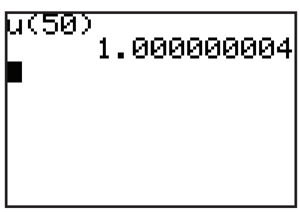




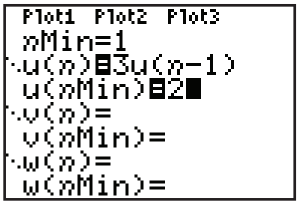
12 $0, 3, \frac{3}{7}, \frac{3}{2 \cdot \frac{3}{7} + 1} = \frac{21}{13}, \frac{39}{55}$

$u_{50} = 1.00$

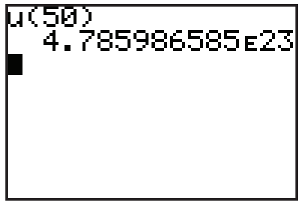




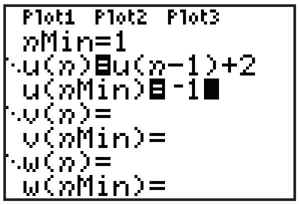
13 2, 6, 18, 54, 162



$b_{50} = 4.786 \times 10^{23}$



14 −1, 1, 3, 5, 7



$a_{50} = 97$

15 $u_n = \frac{1}{4}u_{n-1}, u_1 = \frac{1}{3}$

16 $u_n = \frac{4a^2}{3}u_{n-1}, u_1 = \frac{1}{2}a$

17 $u_n = u_{n-1} + a - k, u_1 = a - 5k$

18 $u_n = n^2 + 3$

19 $u_n = 3n - 1$

20 $u_n = \frac{2n-1}{n^2}$

21 $u_n = \frac{2n-1}{n+3}$

22 For the Fibonacci sequence, 1, 1, 2, 3, 5, 8, 13, 21, 34, 55, 89, ..., we have:

a) $a_1 = \frac{F_2}{F_1} = \frac{1}{1} = 1, a_2 = \frac{F_3}{F_2} = \frac{2}{1} = 2, a_3 = \frac{F_4}{F_3} = \frac{3}{2}, a_4 = \frac{F_5}{F_4} = \frac{5}{3}, a_5 = \frac{F_6}{F_5} = \frac{8}{5},$

$a_6 = \frac{F_7}{F_6} = \frac{13}{8}, a_7 = \frac{F_8}{F_7} = \frac{21}{13}, a_8 = \frac{F_9}{F_8} = \frac{34}{21}, a_9 = \frac{F_{10}}{F_9} = \frac{55}{34}, a_{10} = \frac{F_{11}}{F_{10}} = \frac{89}{55}$

b) $a_n = \frac{F_{n+1}}{F_n} = \frac{F_{n-1} + F_n}{F_n} = \frac{F_n}{F_n} + \frac{F_{n-1}}{F_n} = 1 + \frac{1}{\frac{F_n}{F_{n-1}}} = 1 + \frac{1}{a_{n-1}}$

23 For $F_n = \frac{1}{\sqrt{5}}\left(\frac{\left(1+\sqrt{5}\right)^n - \left(1-\sqrt{5}\right)^n}{2^n}\right)$ we have:

a) $F_1 = \frac{1}{\sqrt{5}}\left(\frac{\left(1+\sqrt{5}\right) - \left(1-\sqrt{5}\right)}{2}\right) = 1$

$F_2 = \frac{1}{\sqrt{5}}\left(\frac{\left(1+2\sqrt{5}+5\right) - \left(1-2\sqrt{5}+5\right)}{4}\right) = 1$

$F_3 = \frac{1}{\sqrt{5}}\left(\frac{\left(1+3\sqrt{5}+15+5\sqrt{5}\right) - \left(1-3\sqrt{5}+15-5\sqrt{5}\right)}{8}\right) = 2$

$F_4 = \frac{1}{\sqrt{5}}\left(\frac{\left(1+4\sqrt{5}+30+20\sqrt{5}+25\right) - \left(1-4\sqrt{5}+30-20\sqrt{5}+25\right)}{16}\right) = 3$

$F_5 = \frac{1}{\sqrt{5}}\left(\frac{\left(1+5\sqrt{5}+50+50\sqrt{5}+125+25\sqrt{5}\right) - \left(1-5\sqrt{5}+50-50\sqrt{5}+125-25\sqrt{5}\right)}{32}\right) = 5$

$F_6 = \frac{1}{\sqrt{5}}\left(\frac{\left(1+6\sqrt{5}+75+100\sqrt{5}+375+150\sqrt{5}+125\right) - \left(1-6\sqrt{5}+75-100\sqrt{5}+375-150\sqrt{5}+125\right)}{64}\right) = 8$

$F_7 = 13, F_8 = 21, F_9 = 34, F_{10} = 55$

This sequence is equal to the Fibonacci sequence.

b) $3 \pm \sqrt{5} = \frac{6 \pm 2\sqrt{5}}{2} = \frac{\left(1 \pm 2\sqrt{5} + 5\right)}{2} = \frac{\left(1 \pm \sqrt{5}\right)^2}{2}$

c)
$$F_{n-1} + F_n = \frac{1}{\sqrt{5}}\left(\frac{\left(1+\sqrt{5}\right)^{n-1} - \left(1-\sqrt{5}\right)^{n-1}}{2^{n-1}}\right) + \frac{1}{\sqrt{5}}\left(\frac{\left(1+\sqrt{5}\right)^{n} - \left(1-\sqrt{5}\right)^{n}}{2^{n}}\right)$$
$$= \frac{1}{\sqrt{5}}\left[\frac{2\left(1+\sqrt{5}\right)^{n-1} + \left(1+\sqrt{5}\right)^{n}}{2^n} - \frac{2\left(1-\sqrt{5}\right)^{n-1} + \left(1-\sqrt{5}\right)^{n}}{2^n}\right]$$
$$= \frac{1}{\sqrt{5}}\left[\frac{\left(1+\sqrt{5}\right)^{n-1}\left(2+1+\sqrt{5}\right)}{2^n} - \frac{\left(1-\sqrt{5}\right)^{n-1}\left(2+1-\sqrt{5}\right)}{2^n}\right]$$
$$= \frac{1}{\sqrt{5}}\left[\frac{\left(1+\sqrt{5}\right)^{n-1}\left(3+\sqrt{5}\right)}{2^n} - \frac{\left(1-\sqrt{5}\right)^{n-1}\left(3-\sqrt{5}\right)}{2^n}\right]$$
$$= \frac{1}{\sqrt{5}}\left[\frac{\left(1+\sqrt{5}\right)^{n-1}}{2^n} \cdot \frac{\left(1+\sqrt{5}\right)^{2}}{2} - \frac{\left(1-\sqrt{5}\right)^{n-1}}{2^n} \cdot \frac{\left(1-\sqrt{5}\right)^{2}}{2}\right]$$
$$= \frac{1}{\sqrt{5}}\left[\frac{\left(1+\sqrt{5}\right)^{n+1}}{2^{n+1}} - \frac{\left(1-\sqrt{5}\right)^{n+1}}{2^{n+1}}\right] = F_{n+1}$$

Exercise 4.2

1 $a_1 = 3$

$\underline{a_6 = 7}$

$a_6 = a_1 + 5d \Rightarrow 7 = 3 + 5d \Rightarrow d = \frac{4}{5} = 0.8$

The sequence is: $3, \frac{19}{5}, \frac{23}{5}, \frac{27}{5}, \frac{31}{5}, 7$.

2 a) Arithmetic: $a_{n+1} - a_n = [2(n+1) - 3] - (2n - 3) = 2 \Rightarrow d = 2 \Rightarrow a_{50} = a_1 + 49d = -1 + 49 \cdot 2 = 97$

b) Arithmetic: $b_{n+1} - b_n = (n + 1 + 2) - (n + 2) = 1 \Rightarrow d = 1 \Rightarrow b_{50} = b_1 + 49d = 3 + 49 \cdot 1 = 52$

c) Arithmetic: $c_1 = -1, c_2 = 1, c_3 = 3 \Rightarrow d = 2 \Rightarrow c_{50} = c_1 + 49d = -1 + 49 \cdot 2 = 97$

d) u_1 is not defined, so the sequence cannot be determined.

e) $e_2 - e_1 = 5 - 2 = 3, e_3 - e_2 = 7 - 5 = 2$

There is no common difference, so the sequence is not arithmetic.

f) Arithmetic: $f_2 - f_1 = f_3 - f_2 = f_4 - f_3 = -7 \Rightarrow d = -7 \Rightarrow f_{50} = f_1 + 49d = 2 + 49 \cdot (-7) = -341$

3 a) $a_1 = -2, d = 4$: $a_8 = a_1 + 7d = -2 + 7 \cdot 4 = 26$

b) $a_n = -2 + (n - 1) \cdot 4 = 4n - 6$

c) $a_n = a_{n-1} + 4, a_1 = -2$

4 **a)** $a_1 = 29, d = -4{:}\ a_8 = a_1 + 7d = 29 + 7 \cdot (-4) = 1$

b) $a_n = 29 + (n-1) \cdot (-4) = -4n + 33$

c) $a_n = a_{n-1} - 4,\ a_1 = 29$

5 **a)** $a_1 = -6, d = 9{:}\ a_8 = a_1 + 7d = -6 + 7 \cdot 9 = 57$

b) $a_n = -6 + (n-1) \cdot 9 = 9n - 15$

c) $a_n = a_{n-1} + 9,\ a_1 = -6$

6 **a)** $a_1 = 10.07, d = -0.12{:}\ a_8 = a_1 + 7d = 10.07 + 7 \cdot (-0.12) = 9.23$

b) $a_n = 10.07 + (n-1) \cdot (-0.12) = -0.12n + 10.19$

c) $a_n = a_{n-1} - 0.12,\ a_1 = 10.07$

7 **a)** $a_1 = 100, d = -3{:}\ a_8 = a_1 + 7d = 100 + 7 \cdot (-3) = 79$

b) $a_n = 100 + (n-1) \cdot (-3) = -3n + 103$

c) $a_n = a_{n-1} - 3,\ a_1 = 100$

8 **a)** $a_1 = 2, d = -\frac{5}{4}{:}\ a_8 = a_1 + 7d = 2 + 7 \cdot \left(-\frac{5}{4}\right) = -\frac{27}{4}$

b) $a_n = 2 + (n-1) \cdot \left(-\frac{5}{4}\right) = -\frac{5}{4}n + \frac{13}{4}$

c) $a_n = a_{n-1} - \frac{5}{4},\ a_1 = 2$

9 $a_1 = 13$

$\underline{a_7 = -23}$

$a_7 = a_1 + 6d \Rightarrow -23 = 13 + 6d \Rightarrow d = -6$

The sequence is: 13, 7, 1, –5, –11, –17, –23.

10 $a_1 = 299$

$\underline{a_5 = 300}$

$a_5 = a_1 + 4d \Rightarrow 300 = 299 + 4d \Rightarrow d = \frac{1}{4} = 0.25$

The sequence is: 299, 299.25, 299.5, 299.75, 300.

11 $a_5 = 6$

$\underline{a_{14} = 42}$

$a_1 + 4d = 6$

$\underline{a_1 + 13d = 42}$

$d = 4, a_1 = -10 \quad \Rightarrow a_n = -10 + (n-1) \cdot 4 = 4n - 14$

12 $a_3 = -40$

$\underline{a_9 = -18}$

$a_1 + 2d = -40$

$\underline{a_1 + 8d = -18}$

$d = \frac{11}{3}, a_1 = -\frac{142}{3} \quad \Rightarrow a_n = -\frac{142}{3} + (n-1) \cdot \frac{11}{3} = \frac{11}{3}n - 51$

13 $a_1 = 3$

$d = 6$

$\underline{a_n = 525}$

$a_n = a_1 + (n-1)d \Rightarrow 525 = 3 + (n-1)\cdot 6 \Rightarrow n = 88$

14 $a_1 = 9$

$d = -6$

$\underline{a_n = -201}$

$a_n = a_1 + (n-1)d \Rightarrow -201 = 9 + (n-1)\cdot(-6) \Rightarrow n = 36$

15 $a_1 = 3\frac{1}{8}$

$d = 4\frac{1}{4} - 3\frac{1}{8} = \frac{9}{8}$

$\underline{a_n = 14\frac{3}{8}}$

$a_n = a_1 + (n-1)d \Rightarrow \frac{115}{8} = \frac{25}{8} + (n-1)\cdot\frac{9}{8} \Rightarrow n = 11$

16 $a_1 = \frac{1}{3}$

$d = \frac{1}{2} - \frac{1}{3} = \frac{1}{6}$

$\underline{a_n = 2\frac{5}{6}}$

$a_n = a_1 + (n-1)d \Rightarrow \frac{17}{6} = \frac{1}{3} + (n-1)\cdot\frac{1}{6} \Rightarrow n = 16$

17 $a_1 = 1 - k$

$d = (1+k) - (1-k) = 2k$

$\underline{a_n = 1 + 19k}$

$a_n = a_1 + (n-1)d \Rightarrow \cancel{1} + 19k = \cancel{1} - k + (n-1)\cdot 2k \Rightarrow 20\cancel{k} = (n-1)\cdot 2\cancel{k} \Rightarrow n = 11$

18 $a_1 = 15$

$\underline{a_7 = -21}$

$a_7 = a_1 + 6d \Rightarrow -21 = 15 + 6d \Rightarrow d = -\frac{36}{6} = -6$

The sequence is: 15, 9, 3, –3, –9, –15, –21.

19 $a_1 = 99$

$\underline{a_5 = 100}$

$a_5 = a_1 + 4d \Rightarrow 100 = 99 + 4d \Rightarrow d = \frac{1}{4}$

The sequence is: 99, 99.25, 99.5, 99.75, 100.

20 $\begin{cases} a_3 = 11 \\ a_{12} = 47 \end{cases} \Rightarrow \begin{cases} a_1 + 2d = 11 \\ a_1 + 11d = 47 \end{cases} \Rightarrow d = 4, a_1 = 3$

The sequence is defined by: $a_1 = 3, a_n = 3 + (n-1) \cdot 4 = 4n - 1$ for $n > 1$.

21 $\begin{cases} a_7 = -48 \\ a_{13} = -10 \end{cases} \Rightarrow \begin{cases} a_1 + 6d = -48 \\ a_1 + 12d = -10 \end{cases} \Rightarrow d = \frac{19}{3}, a_1 = -86$

The sequence is defined by: $a_1 = -86, a_n = -86 + (n-1) \cdot \frac{19}{3} = \frac{19n - 277}{3}$ for $n > 1$.

22 $a_{30} = 147$

$d = 4$

$a_{30} = a_1 + 29d \Rightarrow 147 = a_1 + 29 \cdot 4 \Rightarrow a_1 = 31$

$a_n = a_1 + (n-1)d = 31 + (n-1) \cdot 4 = 4n + 27$

23 $a_1 = -7$

$d = 3$

$a_n = 9803$

$a_n = a_1 + (n-1)d \Rightarrow 9803 = -7 + (n-1) \cdot 3 \Rightarrow n = 3271$

Yes, 9803 is the 3271th term of the sequence.

24 $a_1 = 9689$

$a_{100} = 8996$

$a_n = a_1 + (n-1)d \Rightarrow 8996 = 9689 + 99d \Rightarrow d = -7$

$a_{110} = a_1 + 109d = 9689 + 109 \cdot (-7) = 8926$

$a_n = 1 \Rightarrow 9689 + (n-1) \cdot (-7) = 1 \Rightarrow n = 1385$

Yes, 1 is the 1385th term of the sequence.

25 $a_1 = 2$

$a_{30} = 147$

$a_n = a_1 + (n-1)d \Rightarrow 147 = 2 + 29d \Rightarrow d = 5$

$a_n = 995 \Rightarrow 2 + (n-1) \cdot 5 = 995 \Rightarrow n = \frac{998}{5}$

As a fractional result is not possible for n, we conclude that 995 is not a term of this sequence.

Exercise 4.3

1 $3, 3^{a+1}, 3^{2a+1}, 3^{3a+1}, \ldots$

The sequence is geometric.

$$r = \frac{3^{a+1}}{3} = \frac{3^{2a+1}}{3^{a+1}} = \frac{3^{3a+1}}{3^{2a+1}} = 3^a$$

$u_{10} = u_1 r^9 = 3 \cdot (3^a)^9 = 3^{9a+1}$

2 $0, 3, 6, 9, \ldots$

The sequence is arithmetic.

$d = a_n - a_{n-1} = (3n - 3) - [3(n-1) - 3] = 3$

$a_{10} = a_1 + 9d = 0 + 9 \cdot 3 = 27$

3 $8, 16, 32, 64, \ldots$

The sequence is geometric.

$r = b_n : b_{n-1} = 2^{n+2} : 2^{n+1} = 2$

$b_{10} = b_1 r^9 = 8 \cdot 2^9 = 4096$

4 $-1, -4, -10, -22, \ldots$

$c_{10} = -1534$

The sequence is neither arithmetic nor geometric.

5 $4, 12, 36, 108, \ldots$

The sequence is geometric.

$r = u_n : u_{n-1} = 3$

$u_{10} = u_1 r^9 = 4 \cdot 3^9 = 78\,732$

6 $2, 5, 12.5, 31.25, 78.125, \ldots$

The sequence is geometric.

$r = 5 : 2 = 12.5 : 5 = 31.25 : 12.5$

$= 78.125 : 31.25 = 2.5$

$u_{10} = u_1 r^9 = 2 \cdot 2.5^9 = \frac{1\,953\,125}{256} \approx 7629.39$

7 $2, -5, 12.5, -31.25, 78.125, \ldots$

The sequence is geometric.

$r = (-5) : 2 = 12.5 : (-5) = (-31.25) : 12.5$

$= 78.125 : (-31.25) = -2.5$

$u_{10} = u_1 r^9 = 2 \cdot (-2.5)^9 = -\frac{1\,953\,125}{256} \approx -7629.39$

8 $2, 2.75, 3.5, 4.25, 5, \ldots$

The sequence is arithmetic.

$d = 2.75 - 2 = 3.5 - 2.75 = 4.25 - 3.5$

$= 5 - 4.25 = 0.75$

$u_{10} = u_1 + 9d = 2 + 9 \cdot 0.75 = 8.75$

9 $18, -12, 8, -\frac{16}{3}, \frac{32}{9}, \ldots$

The sequence is geometric.

$r = (-12) : 18 = 8 : (-12) = \left(-\frac{16}{3}\right) : 8$

$= \frac{32}{9} : \left(-\frac{16}{3}\right) = -\frac{2}{3}$

$u_{10} = u_1 r^9 = 18 \cdot \left(-\frac{2}{3}\right)^9 = -\frac{1024}{2187} \approx -0.468$

10 $52, 55, 58, 61, \ldots$

The sequence is arithmetic.

$d = 55 - 52 = 58 - 55 = 61 - 58 = 3$

$u_{10} = u_1 + 9d = 52 + 9 \cdot 3 = 79$

11 $-1, 3, -9, 27, -81, \ldots$

The sequence is geometric.

$r = 3 : (-1) = (-9) : 3 = 27 : (-9)$

$= (-81) : 27 = -3$

$u_{10} = u_1 r^9 = (-1) \cdot (-3)^9 = 19\,683$

12 $0.1, 0.2, 0.4, 0.8, 1.6, 3.2, \ldots$

The sequence is geometric.

$r = 0.2 : 0.1 = 0.4 : 0.2 = 0.8 : 0.4 = 1.6 : 0.8$

$= 3.2 : 1.6 = 2$

$u_{10} = u_1 r^9 = 0.1 \cdot 2^9 = 51.2$

13 $3, 6, 12, 18, 21, 27, \ldots$

The sequence is neither arithmetic nor geometric.

14 $6, 14, 20, 28, 34, \ldots$

The sequence is neither arithmetic nor geometric.

15 $2.4, 3.7, 5, 6.3, 7.6$

The sequence is arithmetic.

$d = 3.7 - 2.4 = 5 - 3.7 = 6.3 - 5 = 7.6 - 6.3 = 1.3$

$u_{10} = u_1 + 9d = 2.4 + 9 \cdot 1.3 = 14.1$

16 a) Arithmetic:

$d = 2 - (-3) = 7 - 2 = 12 - 7 = 5$

$a_8 = a_1 + 7d = -3 + 7 \cdot 5 = 32$

b) $a_n = -3 + (n-1) \cdot 5 = 5n - 8$

c) $a_1 = -3, a_n = a_{n-1} + 5$ for $n > 1$

17 a) Arithmetic:

$d = 15 - 19 = 11 - 15 = 7 - 11 = -4$

$a_8 = a_1 + 7d = 19 + 7 \cdot (-4) = -9$

b) $a_n = 19 + (n-1) \cdot (-4) = 23 - 4n$

c) $a_1 = 19, a_n = a_{n-1} - 4$ for $n > 1$

18 a) Arithmetic:

$d = 3 - (-8) = 14 - 3 = 25 - 14 = 11$

$a_8 = a_1 + 7d = -8 + 7 \cdot 11 = 69$

b) $a_n = -8 + (n-1) \cdot 11 = 11n - 19$

c) $a_1 = -8, a_n = a_{n-1} + 11$ for $n > 1$

19 **a)** Arithmetic:

$d = 9.95 - 10.05 = 9.85 - 9.95$

$= 9.75 - 9.85 = -0.1$

$a_8 = a_1 + 7d = 10.05 + 7 \cdot (-0.1) = 9.35$

b) $a_n = 10.05 + (n-1) \cdot (-0.1) = 10.15 - 0.1n$

c) $a_1 = 10.05,\ a_n = a_{n-1} - 0.1$ for $n > 1$

20 **a)** Arithmetic:

$d = 99 - 100 = 98 - 99 = 97 - 98 = -1$

$a_8 = a_1 + 7d = 100 + 7 \cdot (-1) = 93$

b) $a_n = 100 + (n-1) \cdot (-1) = 101 - n$

c) $a_1 = 100,\ a_n = a_{n-1} - 1$ for $n > 1$

21 **a)** Arithmetic:

$d = \frac{1}{2} - 2 = -1 - \frac{1}{2} = -\frac{5}{2} - (-1) = -\frac{3}{2}$

$a_8 = a_1 + 7d = 2 + 7 \cdot \left(-\frac{3}{2}\right) = -\frac{17}{2}$

b) $a_n = 2 + (n-1) \cdot \left(-\frac{3}{2}\right) = \frac{7-3n}{2}$

c) $a_1 = 2,\ a_n = a_{n-1} - \frac{3}{2}$ for $n > 1$

22 **a)** Geometric:

$r = 6 : 3 = 12 : 6 = 24 : 12 = 2$

$a_8 = a_1 \cdot r^7 = 3 \cdot 2^7 = 384$

b) $a_n = 3 \cdot 2^{n-1}$

c) $a_1 = 3,\ a_n = 2a_{n-1}$ for $n > 1$

23 **a)** Geometric:

$r = 12 : 4 = 36 : 12 = 108 : 36 = 3$

$a_8 = a_1 \cdot r^7 = 4 \cdot 3^7 = 8748$

b) $a_n = 4 \cdot 3^{n-1}$

c) $a_1 = 4,\ a_n = 3a_{n-1}$ for $n > 1$

24 **a)** Geometric:

$r = -5 : 5 = 5 : (-5) = -1$

$a_8 = a_1 \cdot r^7 = 5 \cdot (-1)^7 = -5$

b) $a_n = 5 \cdot (-1)^{n-1}$

c) $a_1 = 5,\ a_n = -a_{n-1}$ for $n > 1$

25 **a)** Geometric:

$r = (-6) : 3 = 12 : (-6) = (-24) : 12 = -2$

$a_8 = a_1 \cdot r^7 = 3 \cdot (-2)^7 = -384$

b) $a_n = 3 \cdot (-2)^{n-1}$

c) $a_1 = 3,\ a_n = -2a_{n-1}$ for $n > 1$

26 The sequence is neither arithmetic nor geometric.

27 **a)** Geometric:

$r = 3 : (-2) = \left(-\frac{9}{2}\right) : 3 = \frac{27}{4} : \left(-\frac{9}{2}\right) = -\frac{3}{2}$

$a_8 = a_1 \cdot r^7 = (-2) \cdot \left(-\frac{3}{2}\right)^7 = \frac{2187}{64} \approx 34.17$

b) $a_n = -2 \cdot \left(-\frac{3}{2}\right)^{n-1} = \frac{3^{n-1}}{(-2)^{n-2}}$

c) $a_1 = -2,\ a_n = \left(-\frac{3}{2}\right) a_{n-1}$ for $n > 1$

28 **a)** Geometric:

$r = 25 : 35 = \frac{125}{7} : 25 = \frac{625}{49} : \frac{125}{7} = \frac{5}{7}$

$a_8 = a_1 \cdot r^7 = 35 \cdot \left(\frac{5}{7}\right)^7 \approx 3.32$

b) $a_n = 35 \cdot \left(\frac{5}{7}\right)^{n-1} = \frac{5^n}{7^{n-2}}$

c) $a_1 = 35,\ a_n = \frac{5}{7} a_{n-1}$ for $n > 1$

29 **a)** Geometric:

$r = (-3) : (-6) = \left(-\frac{3}{2}\right) : (-3) = \left(-\frac{3}{4}\right) : \left(-\frac{3}{2}\right) = \frac{1}{2}$

$a_8 = a_1 \cdot r^7 = (-6) \cdot \left(\frac{1}{2}\right)^7 = -\frac{3}{64} \approx 0.0469$

b) $a_n = -6 \cdot \left(\frac{1}{2}\right)^{n-1} = -\frac{3}{2^{n-2}}$

c) $a_1 = -6,\ a_n = \frac{1}{2} a_{n-1}$ for $n > 1$

30 **a)** Geometric:

$r = 19 : 9.5 = 38 : 19 = 76 : 38 = 2$

$a_8 = a_1 \cdot r^7 = 9.5 \cdot 2^7 = 1216$

b) $a_n = 9.5 \cdot (2)^{n-1}$

c) $a_1 = 9.5,\ a_n = 2a_{n-1}$ for $n > 1$

31 **a)** Geometric:
$r = 95 : 100 = 90.25 : 95 = 0.95$
$a_8 = a_1 \cdot r^7 = 100 \cdot 0.95^7 \approx 69.83$
b) $a_n = 100 \cdot (0.95)^{n-1}$
c) $a_1 = 100, a_n = 0.95a_{n-1}$ for $n > 1$

32 **a)** Geometric:
$$r = \frac{3}{4} : 2 = \frac{9}{32} : \frac{3}{4} = \frac{27}{256} : \frac{9}{32} = \frac{3}{8}$$
$$a_8 = a_1 \cdot r^7 = 2 \cdot \left(\frac{3}{8}\right)^7 \approx 0.002\,09$$
b) $a_n = 2 \cdot \left(\frac{3}{8}\right)^{n-1}$
c) $a_1 = 2, a_n = \frac{3}{8} a_{n-1}$ for $n > 1$

33 $a_1 = 3$
$\underline{a_6 = a_1 r^5 = 96}$
$3r^5 = 96 \Rightarrow r^5 = 32 \Rightarrow r = 2$
(3), 6, 12, 24, 48, (96)

34 $a_1 = 7$
$\underline{a_5 = a_1 r^4 = 4375}$
$7r^4 = 4375 \Rightarrow r^4 = 625 \Rightarrow r^4 = 5^4 \Rightarrow r = 5$
(7), 35, 175, 875, (4375)

35 $a_1 = 16$
$\underline{a_3 = a_1 r^2 = 81}$
$$16r^2 = 81 \Rightarrow r^2 = \frac{81}{16} \Rightarrow r_{1,2} = \pm\frac{9}{4}$$
There are two solutions for the common ratio r, but, as we require the geometric **mean**, the only correct one is the positive one. Therefore, the sequence is: (16), 36, (81).

36 $a_1 = 7$
$\underline{a_6 = a_1 r^5 = 1701}$
$7r^5 = 1701 \Rightarrow r^5 = 243 \Rightarrow r = 3$
(7), 21, 63, 189, 567, (1701)

37 $a_1 = 9$
$\underline{a_3 = a_1 r^2 = 64}$
$$9r^2 = 64 \Rightarrow r^2 = \frac{64}{9} \Rightarrow r_{1,2} = \pm\frac{8}{3}$$
There are two solutions for the common ratio r, but, as we require the geometric **mean**, the only correct one is the positive one. Therefore, the sequence is: (9), 24, (64).

38 $a_1 = 24$
$\underline{a_4 = a_1 r^3 = 3}$
$$24r^3 = 3 \Rightarrow r^3 = \frac{3}{24} = \frac{1}{8} \Rightarrow r = \frac{1}{2}$$
$$a_5 = a_1 r^4 = 24 \cdot \left(\frac{1}{2}\right)^4 = \frac{24}{16} = \frac{3}{2}$$
$$a_n = 24\left(\frac{1}{2}\right)^{n-1} = \frac{3}{2^{n-4}}$$

39 $a_1 = 24$
$\underline{a_3 = a_1 r^2 = 6}$
$$24r^2 = 6 \Rightarrow r^2 = \frac{6}{24} = \frac{1}{4} \Rightarrow r = \pm\frac{1}{2}$$
We have two solutions:
$$\text{For } r = \frac{1}{2} \Rightarrow a_4 = a_1 r^3 = 24 \cdot \left(\frac{1}{2}\right)^3 = \frac{24}{8} = 3$$
$$a_n = 24\left(\frac{1}{2}\right)^{n-1} = \frac{3}{2^{n-4}}$$
$$\text{For } r = -\frac{1}{2} \Rightarrow a_4 = a_1 r^3 = 24 \cdot \left(-\frac{1}{2}\right)^3 = -\frac{24}{8} = -3$$
$$a_n = 24\left(-\frac{1}{2}\right)^{n-1}$$

40 $r = \frac{2}{7}$
$\underline{a_4 = a_1 r^3 = \frac{14}{3}}$
$$a_1 = a_4 : r^3 = \frac{14}{3} : \left(\frac{2}{7}\right)^3 = \frac{2401}{12}$$
$$a_3 = a_1 r^2 = \frac{2401}{12} \cdot \left(\frac{2}{7}\right)^2 = \frac{49}{3}$$

41 6, 18, 54, ...
$a_1 = 6, r = 3$
$a_n = 118\,098$
$a_n = a_1 r^{n-1} \Rightarrow 118\,098 = 6 \cdot 3^{n-1} \Rightarrow$
$3^{n-1} = 19\,683 \Rightarrow 3^{n-1} = 3^9 \Rightarrow$
$n - 1 = 9 \Rightarrow n = 10$
So, 118 098 is the 10th term of the given sequence.

42 $a_4 = 18$

$a_7 = \frac{729}{8}$

$a_1 r^3 = 18 \quad (1)$

$a_1 r^6 = \frac{729}{8} \quad (2)$

We can divide (2) by (1) to obtain:

$$r^3 = \frac{81}{16} \Rightarrow r = \sqrt[3]{\frac{81}{16}} = \sqrt[3]{\frac{3^4}{2^4}} = \frac{3}{2}\sqrt[3]{\frac{3}{2}}$$

and so:

$$a_1 = \frac{18}{r^3} = \frac{18}{\frac{81}{16}} = \frac{32}{9}$$

Now we can proceed as in the previous question:

$a_n = \frac{59\,049}{128}$

$$a_n = a_1 r^{n-1} \Rightarrow \frac{59\,049}{128} = \frac{32}{9} \cdot \left(\left(\frac{3}{2}\right)^{\frac{4}{3}}\right)^{n-1} \Rightarrow$$

$$\left(\frac{3}{2}\right)^{\frac{4n-4}{3}} = \frac{531\,441}{4096} \Rightarrow \left(\frac{3}{2}\right)^{\frac{4n-4}{3}} = \left(\frac{3}{2}\right)^{12} \Rightarrow$$

$$\frac{4n-4}{3} = 12 \Rightarrow n = 10$$

Since n is a natural number, we can conclude that $\frac{59\,049}{128}$ is the 10th term of the sequence.

43 $a_3 = 18$

$a_6 = \frac{243}{4}$

$a_1 r^2 = 18 \quad (1)$

$a_1 r^5 = \frac{243}{4} \quad (2)$

We can divide (2) by (1) to obtain:

$$r^3 = \frac{27}{8} \Rightarrow r = \frac{3}{2}$$

$$a_1 = \frac{18}{r^2} = \frac{18}{\frac{9}{4}} = 8$$

Now we can proceed as in the previous question:

$a_n = \frac{19\,683}{64}$

$$a_n = a_1 r^{n-1} \Rightarrow \frac{19\,683}{64} = 8 \cdot \left(\frac{3}{2}\right)^{n-1} \Rightarrow$$

$$\left(\frac{3}{2}\right)^{n-1} = \frac{19\,683}{512} \Rightarrow \left(\frac{3}{2}\right)^{n-1} = \left(\frac{3}{2}\right)^{9} \Rightarrow$$

$$n - 1 = 9 \Rightarrow n = 10$$

Since n is a natural number, we can conclude that $\frac{19\,683}{64}$ is the 10th term of the sequence.

44 The interest is paid $n = 2$ times per year. For $t = 10$ years, annual interest rate $r = 0.04$, and principal $P = 1500$, we have:

$$A = P\left(1 + \frac{r}{n}\right)^{nt} = 1500\left(1 + \frac{0.04}{2}\right)^{2 \cdot 10}$$
$$= 1500 \cdot 1.02^{20} \approx 2228.92$$

Jim will have €2228.92 in his account.

45 $P = 500, r = 0.04, n = 4, t = 16$

$$A = P\left(1 + \frac{r}{n}\right)^{nt} = 500\left(1 + \frac{0.04}{4}\right)^{4 \cdot 16} \approx 945.23$$

Jane will have £945.23 on her 16th birthday.

46 $A = 4000, t = 6, n = 4, r = 0.05$

$$A = P\left(1 + \frac{r}{n}\right)^{nt} \Rightarrow P = A : \left(1 + \frac{r}{n}\right)^{nt}$$
$$= 4000 : \left(1 + \frac{0.05}{4}\right)^{4 \cdot 6} \approx 2968.79$$

You should invest €2968.79 now.

47 This situation can be modelled by a geometric sequence whose first term is 7554 and whose common ratio is 1.005. Since we count the population of 2007 among the terms, the number of terms is 6.

$$a_6 = 7554 \cdot 1.005^5 \approx 7744.748$$

The population in 2012 would be 7745 thousand.

48 $r = \frac{3}{7}$

$a_4 = \frac{14}{3}$

$$a_3 = \frac{a_4}{r} = \frac{\frac{14}{3}}{\frac{3}{7}} = \frac{98}{9}$$

49 $a_1 = 7$

$r = 3$

$a_n = 137\,781 \Rightarrow 7 \cdot 3^{n-1} = 137\,781 \Rightarrow$

$3^{n-1} = 19\,683 \Rightarrow 3^{n-1} = 3^9 \Rightarrow n = 10$

137 781 is the 10th term.

50 $P = 2500, n = 2, t = 10, r = 0.04$

$$A = P\left(1+\frac{r}{n}\right)^{nt} = 2500\left(1+\frac{0.04}{2}\right)^{20} \approx 3714.87$$

Tim's account will hold €3714.87.

51 $P = 1000, n = 4, t = 18, r = 0.06$

$$A = P\left(1+\frac{r}{n}\right)^{nt} = 1000\left(1+\frac{0.06}{4}\right)^{4\cdot 18} \approx 2921.16$$

William will have £2921.16 on his 18th birthday.

Exercise 4.4

1 Firstly, we need to determine the number of terms in the series.

$a_1 = 11, d = 6$

$a_n = 365$

$a_n = a_1 + (n-1)d \Rightarrow 365 = 11 + (n-1)\cdot 6 \Rightarrow n = 60$

The sum of the series is: $S_{60} = \frac{60}{2}(11+365) = 11\,280$.

2 The series is geometric. Firstly, we need to determine the number of terms in the series.

$a_1 = 2, r = -\frac{3}{2}$

$a_n = -\frac{177\,147}{1024}$

$$a_n = a_1 r^{n-1} \Rightarrow -\frac{177\,147}{1024} = 2\cdot\left(-\frac{3}{2}\right)^{n-1} \Rightarrow \left(-\frac{3}{2}\right)^{n-1} = -\frac{177\,147}{2048} \Rightarrow \left(-\frac{3}{2}\right)^{n-1} = \left(-\frac{3}{2}\right)^{11} \Rightarrow n = 12$$

The sum of the series is: $S_{12} = \dfrac{2\left(\left(-\frac{3}{2}\right)^{12}-1\right)}{-\frac{3}{2}-1} = -\dfrac{105\,469}{1024} \approx -103$.

3 $\sum_{k=0}^{13}(2-0.3k) = 2 + 1.7 + 1.4 + \ldots + (-1.9)$

The series is arithmetic with 14 terms, $a_1 = 2$ and $d = -0.3$.

The sum of the series is: $S_{14} = \frac{14}{2}(2+(-1.9)) = 0.7 = \frac{7}{10}$.

4 $2 - \frac{4}{5} + \frac{8}{25} - \frac{16}{125} + \ldots$ is an infinite geometric series with $a_1 = 2$ and $r = -\frac{2}{5}$.

The sum is: $S_\infty = \dfrac{2}{1-\left(-\frac{2}{5}\right)} = \dfrac{10}{7}$.

5 $\frac{1}{3} + \frac{\sqrt{3}}{12} + \frac{1}{16} + \frac{\sqrt{3}}{64} + \frac{3}{256} + \ldots$ is an infinite geometric series with $a_1 = \frac{1}{3}$ and $r = \frac{\sqrt{3}}{4}$.

The sum is: $S_\infty = \dfrac{\frac{1}{3}}{1-\frac{\sqrt{3}}{4}} = \dfrac{4}{3(4-\sqrt{3})} \cdot \dfrac{4+\sqrt{3}}{4+\sqrt{3}} = \dfrac{16+4\sqrt{3}}{39}$.

6 In each case we have an infinite geometric series:

a) $0.\overline{52} = 0.525\,252\,52... = 0.52 + 0.0052 + 0.000\,052 + ... = 52 \cdot 10^{-2} + 52 \cdot 10^{-4} + 52 \cdot 10^{-6} + ...$

$a_1 = \dfrac{52}{100}, r = \dfrac{1}{100}$

$0.\overline{52} = S_\infty = \dfrac{\frac{52}{100}}{1-\frac{1}{100}} = \dfrac{52}{99}$

b) $0.4\overline{53} = 0.453\,535\,353... = 0.4 + 0.053 + 0.000\,53 + 0.000\,0053 + ...$

$= 0.4 + 53 \cdot 10^{-3} + 53 \cdot 10^{-5} + 53 \cdot 10^{-7} + ... = 0.4 + S_\infty$

$a_1 = \dfrac{53}{1000}, r = \dfrac{1}{100}$

$0.4\overline{53} = 0.4 + S_\infty = 0.4 + \dfrac{\frac{53}{1000}}{1-\frac{1}{100}} = \dfrac{4}{10} + \dfrac{53}{990} = \dfrac{449}{990}$

c) $3.01\overline{37} = 3.013\,737\,37... = 3.01 + 0.0037 + 0.000\,037 + 0.000\,000\,37 + ...$

$= 3.01 + 37 \cdot 10^{-4} + 37 \cdot 10^{-6} + 37 \cdot 10^{-8} + ... = 3.01 + S_\infty$

$a_1 = \dfrac{37}{10\,000}, r = \dfrac{1}{100}$

$3.01\overline{37} = 3.01 + S_\infty = 3.01 + \dfrac{\frac{37}{10\,000}}{1-\frac{1}{100}} = \dfrac{301}{100} + \dfrac{37}{9900} = \dfrac{29\,836}{9900} = \dfrac{7459}{2475}$

7 Maggie invests £150 (R) at the beginning of every month for six years, so we are calculating future value (FV) for annuity due. For annual rate $r = 0.06$ and $m = 6 \cdot 12 = 72$ periods, we have:

$i = \dfrac{0.06}{12} = 0.005$

$FV = R\left(\dfrac{(1+i)^{m+1}-1}{i} - 1\right) = 150\left(\dfrac{(1+0.005)^{73}-1}{0.005} - 1\right) = 13\,026.135$

There will be £13 026.14 in her account after six years.

8 The series $9 + 13 + 17 + ... + 85$ is arithmetic. Firstly, we need to determine the number of terms in the series.

$a_1 = 9, d = 4$

$a_n = 85$

$a_n = a_1 + (n-1)d \Rightarrow 85 = 9 + (n-1) \cdot 4 \Rightarrow n = 20$

The sum of the series is: $S_{20} = \dfrac{20}{2}(9 + 85) = 940.$

9 The series $8 + 14 + 20 + ... + 278$ is arithmetic. Firstly, we need to determine the number of terms in the series.

$a_1 = 8, d = 6$

$a_n = 278$

$a_n = a_1 + (n-1)d \Rightarrow 278 = 8 + (n-1) \cdot 6 \Rightarrow n = 46$

The sum of the series is: $S_{46} = \frac{46}{2}(8 + 278) = 6578$.

10 The series $155 + 158 + 161 + ... + 527$ is arithmetic. Firstly, we need to determine the number of terms in the series.

$a_1 = 155, d = 3$

$a_n = 527$

$a_n = a_1 + (n-1)d \Rightarrow 527 = 155 + (n-1) \cdot 3 \Rightarrow n = 125$

The sum of the series is: $S_{125} = \frac{125}{2}(155 + 527) = 42\,625$.

11 $a_k = 2 + 3k \Rightarrow a_1 = 2 + 3 \cdot 1 = 5, a_n = 2 + 3n$

$S_n = \frac{n}{2}[5 + 2 + 3n] = \frac{n(3n+7)}{2}$

12 For the arithmetic series $17 + 20 + 23...$, we have:

$a_1 = 17, d = 3$

$S_n = \frac{n}{2}[2a_1 + (n-1)d] = \frac{n}{2}[34 + (n-1) \cdot 3] = \frac{n(3n+31)}{2}$

$S_n > 678 \Rightarrow \frac{n(3n+31)}{2} > 678 \Rightarrow 3n^2 + 31n - 1356 > 0$

The solutions of the quadratic equation $3n^2 + 31n - 1356 = 0$ are 16.71 and -27.05, so the solutions of the inequality are $n > 16.71$ or $n < -27.05$. Since $n \in \mathbb{N}$, we conclude that we need to add 17 terms to exceed 678.

13 For the arithmetic series $-18 - 11 - 4...$, we have:

$a_1 = -18, d = 7$

$S_n = \frac{n}{2}[2a_1 + (n-1)d] = \frac{n}{2}[-36 + (n-1) \cdot 7] = \frac{n(7n-43)}{2}$

$S_n > 2335 \Rightarrow \frac{n(7n-43)}{2} > 2335 \Rightarrow 7n^2 - 43n - 4670 > 0$

The solutions of the quadratic equation $7n^2 - 43n - 4670 = 0$ are 29.08 and -22.94, so the solutions of the inequality are $n > 29.08$ or $n < -22.94$. Since $n \in \mathbb{N}$, we conclude that we need to add 30 terms to exceed 2335.

14 First sequence:

$a, a + 2d, a + 4d, ..., a + 98d$

$S_{50} = \frac{50}{2}(a + a + 98d) = 25(2a + 98d) = T$

Second sequence:

$a + d, a + 3d, a + 5d, ..., a + 99d$

Combined sequence:

$a, a+d, a+2d, a+3d, ..., a+99d$

$$S_{100} = \frac{100}{2}(a+a+99d) = 50(2a+99d) = S$$

Then:

$2T + 200 = S \Rightarrow 2 \cdot 25(2a+98d) + 200 = 50(2a+99d) \Rightarrow$

$100a + 4900d + 200 = 100a + 4950d \Rightarrow 50d = 200 \Rightarrow d = 4$

15 **a)** For the arithmetic sequence 3, 7, 11, ..., 999, we have:

$a_1 = 3, d = 4$

$a_n = 999$

$a_n = a_1 + (n-1)d \Rightarrow 999 = 3 + (n-1) \cdot 4 \Rightarrow n = 250$

$$S_{250} = \frac{250}{2}(3+999) = 125\,250$$

b) The removed terms, 11, 23, 35, ..., 995, form an arithmetic sequence with 83 terms and

$b_1 = 11$ and $d = 12$.

$$S_{83} = \frac{83}{2}[2 \cdot 11 + 82 \cdot 12] = 41\,749$$

The sum of the remaining terms is then $125\,250 - 41\,749 = 83\,501$.

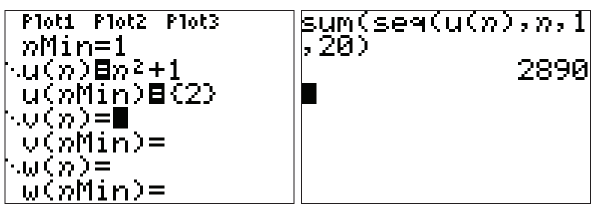
16 We have the following system of equations:

$$\begin{cases} a + (a+d) + (a+2d) + ... + (a+9d) = 235 \\ (a+10d) + (a+11d) + ... + (a+19d) = 735 \end{cases}$$

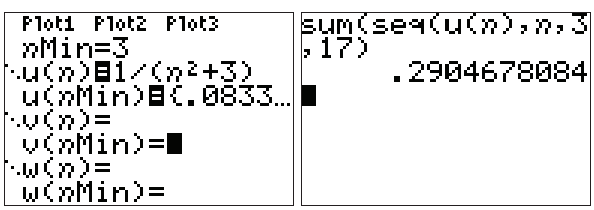
$$\begin{cases} \frac{10}{2}[a + (a+9d)] = 235 \\ \frac{10}{2}[(a+10d) + (a+19d)] = 735 \end{cases}$$

$$\begin{cases} 2a + 9d = 47 \\ 2a + 29d = 147 \end{cases} \Rightarrow d = 5, a = 1$$

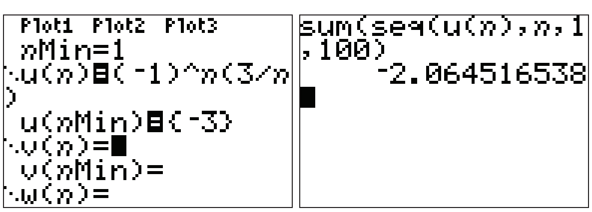
17 For $\sum_{k=1}^{20}(k^2+1)$, using a GDC in sequential mode:



18 For $\sum_{i=3}^{17}\frac{1}{i^2+3}$:



19 For $\sum_{n=1}^{100}(-1)^n\frac{3}{n}$:



20 For the arithmetic series 13 + 19 + ... + 367, we have:

$a_1 = 13, d = 6$

$a_n = 367$

$a_n = a_1 + (n-1)d \Rightarrow 367 = 13 + (n-1)\cdot 6 \Rightarrow n = 60$

$S_{60} = \frac{60}{2}(13 + 367) = 11\,400$

21 For the geometric series $2 - \frac{4}{3} + \frac{8}{9} - \frac{16}{27} + ... - \frac{4096}{177\,147}$, we have:

$a_1 = 2, r = -\frac{2}{3}$

$a_n = -\frac{4096}{177\,147}$

$a_n = a_1 r^{n-1} \Rightarrow 2\cdot\left(-\frac{2}{3}\right)^{n-1} = -\frac{4096}{177\,147} \Rightarrow \left(-\frac{2}{3}\right)^{n-1} = \left(-\frac{2}{3}\right)^{11} \Rightarrow n = 12$

$S_{12} = a_1\frac{1-r^{12}}{1-r} = 2\frac{1-\left(-\frac{2}{3}\right)^{12}}{1-\left(-\frac{2}{3}\right)} \approx 1.191$

22 $\sum_{k=0}^{11}(3 + 0.2k) = 3 + 3.2 + 3.4 + ... + 5.2$ is an arithmetic series with:

$a_1 = 3, d = 0.2, n = 12.$

So: $\sum_{k=0}^{11}(3 + 0.2k) = S_{12} = \frac{12}{2}(3 + 5.2) = 49.2.$

23 For the infinite geometric series $2 - \frac{4}{3} + \frac{8}{9} - \frac{16}{27} + ...$, we have:

$a_1 = 2, r = -\frac{2}{3}$

$S_\infty = \frac{a_1}{1-r} = \frac{2}{1+\frac{2}{3}} = \frac{6}{5}$

24 For the infinite geometric series $\frac{1}{2} + \frac{\sqrt{2}}{2\sqrt{3}} + \frac{1}{3} + \frac{\sqrt{2}}{2\sqrt{3}} + \frac{2}{9}...$, we have:

$a_1 = \frac{1}{2}, r = \frac{\sqrt{2}}{\sqrt{3}}$

$S_\infty = \frac{a_1}{1-r} = \frac{\frac{1}{2}}{1-\frac{\sqrt{2}}{\sqrt{3}}} = \frac{\sqrt{3}}{2\left(\sqrt{3}-\sqrt{2}\right)} = \frac{\sqrt{3}}{2\left(\sqrt{3}-\sqrt{2}\right)}\cdot\frac{\sqrt{3}+\sqrt{2}}{\sqrt{3}+\sqrt{2}} = \frac{3+\sqrt{6}}{2}$

25 For $u_n = \frac{3}{5^n}$, we have:

$$S_1 = \frac{3}{5}$$

$$S_2 = \frac{3}{5} + \frac{3}{25} = \frac{18}{25}$$

$$S_3 = \frac{3}{5} + \frac{3}{25} + \frac{3}{125} = \frac{93}{125}$$

$$S_4 = \frac{3}{5} + \frac{3}{25} + \frac{3}{125} + \frac{3}{625} = \frac{468}{625}$$

$$S_n = 3\frac{1-\left(\frac{1}{5}\right)^n}{1-\frac{1}{5}} = \frac{15}{4}\left(1-\frac{1}{5^n}\right)$$

26 For $v_n = \frac{1}{n^2+3n+2} = \frac{1}{n+1} - \frac{1}{n+2}$, we have:

$$S_1 = \frac{1}{6} = \frac{1}{2\cdot 3}$$

$$S_2 = \frac{1}{6} + \frac{1}{12} = \frac{3}{12} = \frac{1}{4} = \frac{2}{8}$$

$$S_3 = \frac{1}{6} + \frac{1}{12} + \frac{1}{20} = \frac{6}{20} = \frac{3}{10}$$

$$S_4 = \frac{1}{6} + \frac{1}{12} + \frac{1}{20} + \frac{1}{30} = \frac{10}{30} = \frac{1}{3} = \frac{4}{12}$$

$$S_n = \frac{n}{2n+4}$$

27 For $u_n = \sqrt{n+1} - \sqrt{n}$, we have:

$$S_1 = \sqrt{2} - \sqrt{1} = \sqrt{2} - 1$$

$$S_2 = \left(\sqrt{2}-\sqrt{1}\right) + \left(\sqrt{3}-\sqrt{2}\right) = \sqrt{3} - 1$$

$$S_3 = \left(\sqrt{2}-\sqrt{1}\right) + \left(\sqrt{3}-\sqrt{2}\right) + \left(\sqrt{4}-\sqrt{3}\right) = \sqrt{4} - 1 = 1$$

$$S_4 = \left(\sqrt{2}-\sqrt{1}\right) + \left(\sqrt{3}-\sqrt{2}\right) + \left(\sqrt{4}-\sqrt{3}\right) + \left(\sqrt{5}-\sqrt{4}\right) = \sqrt{5} - 1$$

$$S_n = \sqrt{n+1} - 1$$

28 The heights that the ball reaches after each bounce form an infinite geometric sequence: $16\cdot 0.81, 16\cdot 0.81^2, \ldots$

a) After the 10th bounce: $16\cdot 0.81^{10} \approx 1.945$ m

b) $16 + 2\cdot\left(16\cdot 0.81 + 16\cdot 0.81^2 + 16\cdot 0.81^3 + \ldots\right) = 16 + 2\cdot\frac{16}{1-0.81} \approx 184.42$ m

29 The shaded area in the first square is $2\cdot\frac{1}{8}\cdot 16^2 = \frac{1}{4}\cdot 16^2 = 64$.

In each successive square, the shaded area is one-half of the shaded area of the previous square; so, in the second square, the shaded area is $\frac{1}{2}\cdot 64 = 32$, in the third 16, etc.

Total shaded area forms a geometric series with $a_1 = 64, r = \frac{1}{2}$.

a) $S_{10} = 64 \cdot \dfrac{1-\left(\frac{1}{2}\right)^{10}}{1-\frac{1}{2}} = \dfrac{1023}{8} = 127.875$

b) $S_\infty = \dfrac{64}{1-\frac{1}{2}} = 128$

30 **a)** The first shaded area is: $4 \cdot 2 - 2 \cdot 1 = 6$.

The second shaded area is: $1 \cdot \frac{1}{2} - \frac{1}{2} \cdot \frac{1}{4} = \frac{3}{8}$.

The third shaded area is: $\frac{1}{4} \cdot \frac{1}{8} - \frac{1}{8} \cdot \frac{1}{16} = \frac{3}{128}$.

Total shaded area is: $6 + \frac{3}{8} + \frac{3}{128} = \frac{819}{128}$.

b) If the process is repeated indefinitely, the total shaded area forms an infinite geometric sequence with $a_1 = 6, r = \frac{1}{16}$: $S_\infty = \dfrac{6}{1-\frac{1}{16}} = \dfrac{32}{5}$

31 Arithmetic series: $a_1 = 7, d = 5$

$a_n = 342$

$a_n = a_1 + (n-1)d \Rightarrow 342 = 7 + (n-1) \cdot 5 \Rightarrow n = 68$

$S_{68} = \frac{68}{2}(7 + 342) = 11\,866$

32 Arithmetic series: $a_1 = 9486, d = -7$

$a_n = 8912$

$a_n = a_1 + (n-1)d \Rightarrow 8912 = 9486 + (n-1) \cdot (-7) \Rightarrow n = 83$

$S_{83} = \frac{83}{2}(9486 + 8912) = 763\,517$

33 Geometric series: $a_1 = 2, r = 3$

$a_n = 9\,565\,938$

$a_n = a_1 r^{n-1} \Rightarrow 2 \cdot 3^{n-1} = 9\,565\,938 \Rightarrow 3^{n-1} = 4\,782\,969 \Rightarrow 3^{n-1} = 3^{14} \Rightarrow n = 15$

$S_{15} = a_1 \dfrac{r^{15}-1}{r-1} = 2 \cdot \dfrac{3^{15}}{3-1} = 14\,348\,907$

34 Geometric series:

$a_1 = 120, r = \frac{1}{5}$

$a_n = \dfrac{24}{78\,125}$

$a_n = a_1 r^{n-1} \Rightarrow 120 \cdot \left(\frac{1}{5}\right)^{n-1} = \dfrac{24}{78\,125} \Rightarrow \left(\frac{1}{5}\right)^{n-1} = \dfrac{1}{390\,625} \Rightarrow \left(\frac{1}{5}\right)^{n-1} = \left(\frac{1}{5}\right)^{8} \Rightarrow n = 9$

$S_9 = 120 \cdot \dfrac{1-\left(\frac{1}{5}\right)^{9}}{1-\frac{1}{5}} \approx 150$

Exercise 4.5

1 **a)** $^{5}P_{5} = \dfrac{5\,!}{(5-5)\,!} = \dfrac{1\cdot2\cdot3\cdot4\cdot5}{1} = 120$

b) $5\,! = 1\cdot2\cdot3\cdot4\cdot5 = 120$

c) $^{20}P_{1} = \dfrac{20\,!}{(20-1)\,!} = \dfrac{1\cdot2\cdot3\cdot\ldots\cdot19\cdot20}{1\cdot2\cdot3\cdot\ldots\cdot19} = 20$

d) $^{8}P_{3} = \dfrac{8\,!}{(8-3)\,!} = \dfrac{1\cdot2\cdot3\cdot4\cdot5\cdot6\cdot7\cdot8}{1\cdot2\cdot3\cdot4\cdot5} = 6\cdot7\cdot8 = 336$

2 **a)** $\binom{5}{5} = \dfrac{\cancel{5\,!}}{(5-5)\,!\,\cancel{5\,!}} = 1$

b) $\binom{5}{0} = \dfrac{\cancel{5\,!}}{\cancel{(5-0)\,!}\,0\,!} = 1$

c) $\binom{10}{3} = \dfrac{10\,!}{(10-3)\,!\,3\,!} = \dfrac{\cancel{1\cdot2\cdot3\cdot4\cdot5\cdot6\cdot7}\cdot8\cdot9\cdot10}{\cancel{(1\cdot2\cdot3\cdot4\cdot5\cdot6\cdot7)}1\cdot2\cdot3} = 120$

d) $\binom{10}{7} = \dfrac{10\,!}{(10-7)\,!\,7\,!} = \dfrac{\cancel{1\cdot2\cdot3\cdot4\cdot5\cdot6\cdot7}\cdot8\cdot9\cdot10}{(1\cdot2\cdot3)\cancel{1\cdot2\cdot3\cdot4\cdot5\cdot6\cdot7}} = 120$

3 **a)** $\binom{7}{3}+\binom{7}{4} = \dfrac{7\,!}{(7-3)\,!\,3\,!}+\dfrac{7\,!}{(7-4)\,!\,4\,!} = 2\cdot\dfrac{7\,!}{4\,!\,3\,!} = 2\cdot\dfrac{\cancel{1\cdot2\cdot3\cdot4}\cdot5\cdot6\cdot7}{\cancel{(1\cdot2\cdot3\cdot4)}(1\cdot2\cdot3)} = 70$

b) $\binom{8}{4} = \dfrac{8\,!}{(8-4)\,!\,4\,!} = \dfrac{1\cdot2\cdot3\cdot4\cdot5\cdot6\cdot7\cdot8}{(1\cdot2\cdot3\cdot4)^2} = 70$

c) $\binom{10}{6}+\binom{10}{7} = \dfrac{10\,!}{(10-6)\,!\,6\,!}+\dfrac{10\,!}{(10-7)\,!\,7\,!}$

$= \dfrac{\cancel{1\cdot2\cdot3\cdot4\cdot5\cdot6}\cdot7\cdot8\cdot9\cdot10}{(1\cdot2\cdot3\cdot4)\cancel{1\cdot2\cdot3\cdot4\cdot5\cdot6}}+\dfrac{\cancel{1\cdot2\cdot3\cdot4\cdot5\cdot6\cdot7}\cdot8\cdot9\cdot10}{(1\cdot2\cdot3)\cancel{1\cdot2\cdot3\cdot4\cdot5\cdot6\cdot7}} = 210+120 = 330$

d) $\binom{11}{7} = \dfrac{11\,!}{(11-7)\,!\,7\,!} = \dfrac{\cancel{1\cdot2\cdot3\cdot4\cdot5\cdot6\cdot7}\cdot8\cdot9\cdot10\cdot11}{(1\cdot2\cdot3\cdot4)\cancel{1\cdot2\cdot3\cdot4\cdot5\cdot6\cdot7}} = 330$

4 **a)** $\binom{8}{5}-\binom{8}{3} = \dfrac{8\,!}{(8-5)\,!\,5\,!}-\dfrac{8\,!}{(8-3)\,!\,3\,!} = \dfrac{8\,!}{3\,!\,5\,!}-\dfrac{8\,!}{5\,!\,3\,!} = 0$

b) $11\cdot10\,! = 11\cdot1\cdot2\cdot3\cdot4\cdot5\cdot6\cdot7\cdot8\cdot9\cdot10 = 1\cdot2\cdot3\cdot4\cdot5\cdot6\cdot7\cdot8\cdot9\cdot10\cdot11 = 11\,! = 39\,916\,800$

c) $\binom{10}{3}-\binom{10}{7} = \dfrac{10\,!}{(10-3)\,!\,3\,!}-\dfrac{10\,!}{(10-7)\,!\,7\,!} = \dfrac{10\,!}{7\,!\,3\,!}-\dfrac{10\,!}{3\,!\,7\,!} = 0$

d) $\binom{10}{1} = \dfrac{10\,!}{(10-1)\,!\,1\,!} = \dfrac{\cancel{1\cdot2\cdot3\cdot4\cdot5\cdot6\cdot7\cdot8\cdot9}\cdot10}{\cancel{(1\cdot2\cdot3\cdot4\cdot5\cdot6\cdot7\cdot8\cdot9)}\cdot1} = 10$

5 a) $\dfrac{10!}{5!} = \dfrac{1\cdot2\cdot3\cdot4\cdot5\cdot6\cdot7\cdot8\cdot9\cdot10}{1\cdot2\cdot3\cdot4\cdot5} = 30\,240$

$2! = 2$

$\Rightarrow \dfrac{10!}{5!} \neq 2!$

b) $(5!)^2 = (1\cdot2\cdot3\cdot4\cdot5)^2 = 14\,400$

$25! = 1\cdot2\cdot3\cdot4\cdot5\cdot6\cdot7\cdot8\cdot9\cdot10\cdot11\cdot12\cdot13\cdot14\cdot15\cdot16\cdot17\cdot18\cdot19\cdot20\cdot21\cdot22\cdot23\cdot24\cdot25$

$= 1.551\,121 \times 10^{25}$

$\Rightarrow (5!)^2 \neq 25!$

c) $\dbinom{101}{8} = \dfrac{101!}{(101-8)!\,8!} = \dfrac{101!}{93!\,8!}$

$\dbinom{101}{93} = \dfrac{101!}{(101-93)!\,93!} = \dfrac{101!}{8!\,93!}$

$\Rightarrow \dbinom{101}{8} = \dbinom{101}{93}$

6 Fundamental principle of counting $\Rightarrow 3\cdot2\cdot4 = 24$ different systems to choose from.

7 Fundamental principle of counting $\Rightarrow 3\cdot4\cdot2\cdot3 = 72$ different choices.

8 Fundamental principle of counting $\Rightarrow 8\cdot3\cdot13 = 312$ different combinations of choices.

9 Fundamental principle of counting $\Rightarrow \underbrace{4\cdot4\cdot\ldots\cdot4}_{12\text{ times}} = 4^{12} = 16\,777\,216$ ways to answer all the questions.

10 Fundamental principle of counting $\Rightarrow \underbrace{2\cdot2\cdot\ldots\cdot2}_{6\text{ times}}\cdot\underbrace{4\cdot4\cdot\ldots\cdot4}_{6\text{ times}} = 2^6 4^6 = 262\,144$ ways to answer all the questions.

11 Fundamental principle of counting $\Rightarrow$ each letter can be chosen in 26 ways and each digit in 10 ways, so there are $26^3\cdot10^5 = 1\,757\,600\,000$ different passwords.

12 The first and last digit can be chosen in 9 ways (cannot be 0) and the three middle digits in 10 ways, so there are $9\cdot10^3\cdot9 = 81\,000$ such numbers.

13 a) Eight people can be seated in a row in $8! = 40\,320$ different ways (permutations).

b) If every member of each couple likes to sit together, four couples can be seated in a row in $4!$ different ways, and for each member of a couple there are two possible places; so, altogether, there are $4!\cdot 2^4 = 384$ different ways that they can be seated.

14 a) Eight children can be arranged in single file in $8! = 40\,320$ different ways (permutations).

b) If the girls must go first, the five girls can be arranged in $5!$ ways and the three boys in $3!$ ways; so, altogether, there are $5!\,3! = 720$ different orders.

15 In alphabetical order:

AEJN, AENJ, AJEN, AJNE, ANEJ, ANJE, EAJN, EANJ, EJAN, EJNA, ENAJ, ENJA,

JAEN, JANE, JEAN, JENA, JNAE, JNEA, NAEJ, NAJE, NEAJ, NEJA, NJAE, NJEA

16

ACG	AGC	CAG	CGA	GAC	GCA
ACI	AIC	CAI	CIA	IAC	ICA
ACM	AMC	CAM	CMA	MAC	MCA
AGI	AIG	GAI	GIA	IAG	IGA
AGM	AMG	GAM	GMA	MAG	MGA
AIM	AMI	IAM	IMA	MAI	MIA
CGI	CIG	GCI	GIC	ICG	IGC
CGM	CMG	GCM	GMC	MCG	MGC
CIM	CMI	ICM	IMC	MCI	MIC
GIM	GMI	IGM	IMG	MGI	MIG

17 **a)** Three letters can be chosen in 26^3 ways and four digits in 10^4 ways; so, altogether, there are $26^3 \cdot 10^4 = 175\,760\,000$ possible codes.

b) The letters can be chosen in $26^3 - 97$ ways and the four digits in 10^4 ways; so, altogether, there are $(26^3 - 97) \cdot 10^4 = 174\,790\,000$ possible codes.

18 **a)** The president can be chosen in 17 ways, the deputy in 16 ways, and the treasurer in 15 ways; so, altogether, there are $17 \cdot 16 \cdot 15 = 4080$ ways.

b) If the president is male, he can be chosen in 7 ways, and, thus, the deputy can be chosen in 16 ways and the treasurer in 15 ways; so, altogether, there are $7 \cdot 16 \cdot 15 = 1680$ ways.

c) The deputy (male) can be chosen in 7 ways, the treasurer (female) in 10 ways, and the president in 15 ways; so, altogether, there are $7 \cdot 10 \cdot 15 = 1050$ ways.

d) If the president and deputy are both male, then they can be chosen in $7 \cdot 6$ ways; if they are both female, in $10 \cdot 9$ ways. The treasurer can be chosen in 15 ways; so, altogether, there are $(7 \cdot 6 + 10 \cdot 9) \cdot 15 = 1980$ ways.

e) If all three are male, they can be chosen in $7 \cdot 6 \cdot 5$ ways; if all three are female, there are $10 \cdot 9 \cdot 8$ ways. So, there are $4080 - (7 \cdot 6 \cdot 5 + 10 \cdot 9 \cdot 8) = 3150$ cases where all three officers are not the same gender.

19 Since the order is not important, we have combinations.

a) Three officers of the same specialization can be chosen as 3 mathematicians out of 8 (in $\binom{8}{3}$ ways), or 3 computer scientists out of 12 (in $\binom{12}{3}$ ways), or 3 engineers out of 6 (in $\binom{6}{3}$ ways). So, altogether, there are $\binom{8}{3} + \binom{12}{3} + \binom{6}{3} = 56 + 220 + 20 = 296$ ways.

b) There are a total of $\binom{26}{3}$ ways to choose three officers from 26 people. Three officers that are **not** engineers can be chosen as 3 out of 20 mathematicians and computer scientists in $\binom{20}{3}$ ways. So, there are $\binom{26}{3} - \binom{20}{3} = 2600 - 1140 = 1460$ combinations with at least one engineer.

c) Two mathematicians can be chosen in $\binom{8}{2}$ ways, and the third member in 18 ways out of 12 computer scientists and 6 engineers; so, altogether, there are $\binom{8}{2}\cdot 18 = 28\cdot 18 = 504$ ways.

20 Three numbers each in the range 1 to 50.

a) There are $50^3 = 125\,000$ different combinations.

b) There are $50\cdot 49\cdot 48 = 117\,600$ combinations without duplicates.

c) If the first and second number are matching, then the first number can be chosen in 50 ways, the second in one way (must be the same as the first), and the third in 49 ways; so, altogether, there are $50\cdot 1\cdot 49 = 2450$ combinations.

d) If two out of three numbers are matching, it may be the first and the second, or the first and the third, or the second and the third; so, altogether, there are $3\cdot 2450 = 7350$ combinations.

21 Five couples can be permutated in $5!$ ways, but, as they are sitting around a circle, all circular permutations that come in groups of five (ABCDE, BCDEA, CDEAB, …) are equivalent, so there are actually $\frac{5!}{5}$ arrangements. For each couple, there are two different ways of sitting (male right or left of female); so, altogether, there are $\frac{5!}{5}\cdot 2^5 = 768$ different seating arrangements.

22 a) Two elements out of nine can be chosen in $\binom{9}{2} = 36$ ways, so there are 36 two-element subsets.

b) There are $\binom{9}{1}$ one-element subsets, $\binom{9}{3}$ three-element subsets, $\binom{9}{5}$ five-element subsets, $\binom{9}{7}$ seven-element subsets, and $\binom{9}{9}$ nine-element subsets. Altogether, we have $\binom{9}{1}+\binom{9}{3}+\binom{9}{5}+\binom{9}{7}+\binom{9}{9} = 9+84+126+36+1 = 256$ subsets with an odd number of elements.

23 a) Four students out of $9+12 = 21$ members can be chosen in $\binom{21}{4} = 5985$ ways.

b) Two juniors out of 12 can be chosen in $\binom{12}{2}$ ways, and two seniors out of nine can be chosen in $\binom{9}{2}$ ways; so, altogether, there are $\binom{12}{2}\cdot\binom{9}{2} = 66\cdot 36 = 2376$ teams.

c) More juniors than seniors can be realized if there are three or four juniors, so there are $\binom{12}{3}\cdot\binom{9}{1}+\binom{12}{4} = 220\cdot 9+495 = 2475$ such teams.

24 a) Teams with one 'mathlete': $2\cdot\binom{20}{3} = 2\cdot 1140 = 2280$

b) Two juniors (one is Tim) can be chosen in $1\cdot\binom{11}{1}$ ways, and two seniors (no Gwen) can be chosen in $\binom{8}{2}$ ways; two juniors (no Tim) can be chosen in $\binom{11}{2}$ ways, and two seniors (one is Gwen) can be chosen in $1\cdot\binom{8}{1}$ ways. Altogether, there are

$1\cdot\binom{11}{1}\cdot\binom{8}{2}+\binom{11}{2}\cdot 1\cdot\binom{8}{1} = 1\cdot 11\cdot 28+55\cdot 1\cdot 8 = 748$ teams.

c) More juniors than seniors can be realized in the following cases:

3 juniors (one is Tim), 1 senior (no Gwen) $\Rightarrow 1 \cdot \binom{11}{2} \cdot \binom{8}{1}$

3 juniors (no Tim), 1 senior (Gwen) $\Rightarrow \binom{11}{3} \cdot 1$

4 juniors (one is Tim) $\Rightarrow 1 \cdot \binom{11}{3}$

Altogether: $1 \cdot \binom{11}{2} \cdot \binom{8}{1} + \binom{11}{3} \cdot 1 + 1 \cdot \binom{11}{3} = 55 \cdot 8 + 165 + 165 = 770$ teams.

25 a) In a sample of six disks there can be 0, 1, 2, 3 or 4 defective disks. So, we have:

$$\binom{96}{6} \cdot \binom{4}{0} + \binom{96}{5} \cdot \binom{4}{1} + \binom{96}{4} \cdot \binom{4}{2} + \binom{96}{3} \cdot \binom{4}{3} + \binom{96}{2} \cdot \binom{4}{4}$$

$$= 927\,048\,304 + 61\,124\,064 \cdot 4 + 3\,321\,960 \cdot 6 + 142\,880 \cdot 4 + 4560 \cdot 1 = 1\,192\,052\,400$$

b) All four defective disks could be in $\binom{96}{2} \cdot \binom{4}{4} = 4560$ samples, which gives us

$\frac{4560}{1\,192\,052\,400} = 0.000\,003\,83$, i.e. 0.000 383% of the total.

c) At least one defective disk could be in

$$\binom{96}{5} \cdot \binom{4}{1} + \binom{96}{4} \cdot \binom{4}{2} + \binom{96}{3} \cdot \binom{4}{3} + \binom{96}{2} \cdot \binom{4}{4}$$

$= 61\,124\,064 \cdot 4 + 3\,321\,960 \cdot 6 + 142\,880 \cdot 4 + 4560 \cdot 1 = 265\,004\,096$ samples,

which gives us $\frac{265\,004\,096}{1\,192\,052\,400} = 0.2223$, i.e. 22.23% of the total.

26 a) Six people out of $10 + 8 + 4 = 22$ can be chosen in $\binom{22}{6} = 74\,613$ ways.

b) Two members out of each party can be chosen in $\binom{10}{2} \cdot \binom{8}{2} \cdot \binom{4}{2} = 45 \cdot 28 \cdot 6 = 7560$ ways.

27 First we count the number of ways in which we can form a line of six boys and six girls, where boys and girls alternate, starting with a boy. The first boy can be chosen in 9 ways, the first girl in 6 ways, the second boy in 8 ways, the second girl in 5 ways, and so on. So, altogether, we have $\frac{9!6!}{3!}$ combinations. We must multiply the answer by 2, because the line can start with a boy or with a girl; so, we have $\frac{9!6!}{3!} \cdot 2$ combinations. The three boys that remain can be placed anywhere in the line; the first boy in 13 possible places (the beginning of the line, the end of the line, or anywhere in between), the second in 14 possible places (because now there are 13 children in the line), and the last one in 15 possible places. Therefore, there are: $\frac{9!6!}{3!} \cdot 2 \cdot 13 \cdot 14 \cdot 15 = \frac{9!5! \cdot \not{6}}{\not{6}} \cdot 2 \cdot 13 \cdot 14 \cdot 15 \approx 2.3776 \times 10^{11}$ possible ways for nine boys and six girls to stand in a line so that no two girls stand next to each other.

Exercise 4.6

1 In this question we will use Pascal's triangle.

1
1 1
1 2 1
1 3 3 1
1 4 6 4 1
1 5 10 10 5 1
1 6 15 20 15 6 1
1 7 21 35 35 21 7 1

a) $$(x+2y)^5 = 1x^5 + 5x^4(2y) + 10x^3(2y)^2 + 10x^2(2y)^3 + 5x(2y)^4 + 1(2y)^5$$
$$= x^5 + 10x^4y + 40x^3y^2 + 80x^2y^3 + 80xy^4 + 32y^5$$

b) $$(a-b)^4 = 1a^4 + 4a^3(-b) + 6a^2(-b)^2 + 4a(-b)^3 + 1(-b)^4$$
$$= a^4 - 4a^3b + 6a^2b^2 - 4ab^3 + b^4$$

c) $$(x-3)^6 = 1x^6 + 6x^5(-3) + 15x^4(-3)^2 + 20x^3(-3)^3 + 15x^2(-3)^4 + 6x(-3)^5 + 1(-3)^6$$
$$= x^6 - 18x^5 + 135x^4 - 540x^3 + 1215x^2 - 1458x + 729$$

d) $$(2-x^3)^4 = 1\cdot 2^4 + 4\cdot 2^3(-x^3) + 6\cdot 2^2(-x^3)^2 + 4\cdot 2(-x^3)^3 + 1\cdot(-x^3)^4$$
$$= 16 - 32x^3 + 24x^6 - 8x^9 + x^{12}$$

e) $$(x-3b)^7 = 1\cdot x^7 + 7x^6(-3b) + 21x^5(-3b)^2 + 35x^4(-3b)^3 + 35x^3(-3b)^4 + 21x^2(-3b)^5 + 7x(-3b)^6 + (-3b)^7$$
$$= x^7 - 21x^6b + 189x^5b^2 - 945x^4b^3 + 2835x^3b^4 - 5103x^2b^5 + 5103xb^6 - 2187b^7$$

f) $$\left(2n + \frac{1}{n^2}\right)^6 = 1\cdot(2n)^6 + 6(2n)^5\frac{1}{n^2} + 15(2n)^4\left(\frac{1}{n^2}\right)^2 + 20(2n)^3\left(\frac{1}{n^2}\right)^3 + 15(2n)^2\left(\frac{1}{n^2}\right)^4$$
$$+ 6(2n)\left(\frac{1}{n^2}\right)^5 + 1\cdot\left(\frac{1}{n^2}\right)^6 = 64n^6 + 192n^3 + 240 + \frac{160}{n^3} + \frac{60}{n^6} + \frac{12}{n^9} + \frac{1}{n^{12}}$$

g) $$\left(\frac{3}{x} - 2\sqrt{x}\right)^4 = 1\cdot\left(\frac{3}{x}\right)^4 + 4\left(\frac{3}{x}\right)^3(-2\sqrt{x}) + 6\left(\frac{3}{x}\right)^2(-2\sqrt{x})^2 + 4\left(\frac{3}{x}\right)^1(-2\sqrt{x})^3 + 1\cdot(-2\sqrt{x})^4$$
$$= \frac{81}{x^4} - \frac{216\sqrt{x}}{x^3} + \frac{216}{x} - 96\sqrt{x} + 16x^2$$

2 **a)** $$\binom{8}{3} = \frac{8!}{3!5!} = \frac{\cancel{1}\cdot\cancel{2}\cdot\cancel{3}\cdot\cancel{4}\cdot\cancel{5}\cdot\cancel{6}\cdot 7\cdot 8}{(1\cdot\cancel{2}\cdot\cancel{3})(\cancel{1}\cdot\cancel{2}\cdot\cancel{3}\cdot\cancel{4}\cdot\cancel{5})} = 56$$

b) $$\binom{18}{5} - \binom{18}{13} = \frac{18!}{5!13!} - \frac{18!}{13!5!} = 0$$

c) $$\binom{7}{4}\binom{7}{3} = \frac{7!}{4!3!}\cdot\frac{7!}{3!4!} = \left(\frac{7!}{3!4!}\right)^2 = \left(\frac{\cancel{1}\cdot\cancel{2}\cdot\cancel{3}\cdot\cancel{4}\cdot 5\cdot\cancel{6}\cdot 7}{(1\cdot\cancel{2}\cdot\cancel{3})(\cancel{1}\cdot\cancel{2}\cdot\cancel{3}\cdot\cancel{4})}\right)^2 = 35^2 = 1225$$

d) $$\binom{5}{0} + \binom{5}{1} + \binom{5}{2} + \binom{5}{3} + \binom{5}{4} + \binom{5}{5} = (1+1)^5 = 32$$

e) $$\binom{6}{0} - \binom{6}{1} + \binom{6}{2} - \binom{6}{3} + \binom{6}{4} - \binom{6}{5} + \binom{6}{6} = (1-1)^6 = 0$$

3 **a)**

$$(x+2y)^7 = \sum_{i=0}^{7}\binom{7}{i}x^{n-i}(2y)^i$$

$$= x^7 + \binom{7}{1}x^6(2y)^1 + \binom{7}{2}x^5(2y)^2 + \binom{7}{3}x^4(2y)^3 + \binom{7}{4}x^3(2y)^4 + \binom{7}{5}x^2(2y)^5 + \binom{7}{6}x^1(2y)^6 + (2y)^7$$

$$= x^7 + 14x^6y + 84x^5y^2 + 280x^4y^3 + 560x^3y^4 + 672x^2y^5 + 448xy^6 + 128y^7$$

b)

$$(a-b)^6 = \sum_{i=0}^{6}\binom{6}{i}a^{6-i}(-b)^i$$

$$= \binom{6}{0}a^6 + \binom{6}{1}a^{6-1}(-b)^1 + \binom{6}{2}a^{6-2}(-b)^2 + \binom{6}{3}a^{6-3}(-b)^3 + \binom{6}{4}a^{6-4}(-b)^4 + \binom{6}{5}a^{6-5}(-b)^5 + \binom{6}{6}(-b)^6$$

$$= a^6 - 6a^5b + 15a^4b^2 - 20a^3b^3 + 15a^2b^4 - 6ab^5 + b^6$$

c)

$$(x-3)^5 = \sum_{i=0}^{5}\binom{5}{i}x^{5-i}(-3)^i$$

$$= \binom{5}{0}x^{5-i}(-3)^i + \binom{5}{1}x^{5-1}(-3)^1 + \binom{5}{2}x^{5-2}(-3)^2 + \binom{5}{3}x^{5-3}(-3)^3 + \binom{5}{4}x^{5-4}(-3)^4 + \binom{5}{5}x^{5-5}(-3)^5$$

$$= x^5 - 15x^4 + 90x^3 - 270x^2 + 405x - 243$$

d)

$$(2-x^3)^6 = \sum_{i=0}^{6}\binom{6}{i}2^{6-i}(-x^3)^i$$

$$= \binom{6}{0}2^6 + \binom{6}{1}2^{6-1}(-x^3)^1 + \binom{6}{2}2^{6-2}(-x^3)^2 + \binom{6}{3}2^{6-3}(-x^3)^3 + \binom{6}{4}2^{6-4}(-x^3)^4 + \binom{6}{5}2^{6-5}(-x^3)^5 + \binom{6}{6}(-x^3)^6$$

$$= 64 - 192x^3 + 240x^6 - 160x^9 + 60x^{12} - 12x^{15} + x^{18}$$

e)

$$(x-3b)^7 = \sum_{i=0}^{7}\binom{7}{i}x^{7-i}(-3b)^i = \binom{7}{0}x^{7-0}(-3b)^0 + \binom{7}{1}x^{7-1}(-3b)^1 + \binom{7}{2}x^{7-2}(-3b)^2$$

$$+ \binom{7}{3}x^{7-3}(-3b)^3 + \binom{7}{4}x^{7-4}(-3b)^4 + \binom{7}{5}x^{7-5}(-3b)^5 + \binom{7}{6}x^{7-6}(-3b)^6 + \binom{7}{7}x^{7-7}(-3b)^7$$

$$= x^7 - 21x^6b + 189x^5b^2 - 945x^4b^3 + 2835x^3b^4 - 5103x^2b^5 + 5103xb^6 - 2187b^7$$

f)

$$\left(2n+\frac{1}{n^2}\right)^6 = \sum_{i=0}^{6}\binom{6}{i}(2n)^{6-i}\left(\frac{1}{n^2}\right)^i = \binom{6}{0}(2n)^6 + \binom{6}{1}(2n)^{6-1}\left(\frac{1}{n^2}\right)^1 + \binom{6}{2}(2n)^{6-2}\left(\frac{1}{n^2}\right)^2$$

$$+ \binom{6}{3}(2n)^{6-3}\left(\frac{1}{n^2}\right)^3 + \binom{6}{4}(2n)^{6-4}\left(\frac{1}{n^2}\right)^4 + \binom{6}{5}(2n)^{6-5}\left(\frac{1}{n^2}\right)^5 + \binom{6}{6}\left(\frac{1}{n^2}\right)^6$$

$$= 64n^6 + 192n^3 + 240 + \frac{160}{n^3} + \frac{60}{n^6} + \frac{12}{n^9} + \frac{1}{n^{12}}$$

g) $$\left(\frac{3}{x}-2\sqrt{x}\right)^4=\sum_{i=0}^{4}\binom{4}{i}\left(\frac{3}{x}\right)^{4-i}(-2\sqrt{x})^i=\binom{4}{0}\left(\frac{3}{x}\right)^4+\binom{4}{1}\left(\frac{3}{x}\right)^{4-1}(-2\sqrt{x})^1+\binom{4}{2}\left(\frac{3}{x}\right)^{4-2}(-2\sqrt{x})^2$$
$$+\binom{4}{3}\left(\frac{3}{x}\right)^{4-3}(-2\sqrt{x})^3+\binom{4}{4}(-2\sqrt{x})^4$$
$$=\frac{81}{x^4}-\frac{216\sqrt{x}}{x^3}+\frac{216}{x}-96\sqrt{x}+16x^2$$

h) $$\left(1+\sqrt{5}\right)^4+\left(1-\sqrt{5}\right)^4=\sum_{i=0}^{4}\binom{4}{i}1^{4-i}(\sqrt{5})^i+\sum_{i=0}^{4}\binom{4}{i}1^{4-i}(-\sqrt{5})^i=\binom{4}{0}(\sqrt{5})^0+\binom{4}{1}(\sqrt{5})^1+\binom{4}{2}(\sqrt{5})^2$$
$$+\binom{4}{3}(\sqrt{5})^3+\binom{4}{4}(\sqrt{5})^4-\left[\binom{4}{0}(\sqrt{5})^0-\binom{4}{1}(\sqrt{5})^1+\binom{4}{2}(\sqrt{5})^2-\binom{4}{3}(\sqrt{5})^3+\binom{4}{4}(\sqrt{5})^4\right]$$
$$=2\left(\binom{4}{0}(\sqrt{5})^0+\binom{4}{2}(\sqrt{5})^2+\binom{4}{4}(\sqrt{5})^4\right)=2(1+30+25)=112$$

i)
$$\left(\sqrt{3}+1\right)^8-\left(\sqrt{3}-1\right)^8=\sum_{i=0}^{8}\binom{8}{i}(\sqrt{3})^{8-i}1^i-\sum_{i=0}^{8}\binom{8}{i}(\sqrt{3})^{8-i}(-1)^i$$
$$=(\sqrt{3})^8+\binom{8}{1}(\sqrt{3})^7+\binom{8}{2}(\sqrt{3})^6+\binom{8}{3}(\sqrt{3})^5+\binom{8}{4}(\sqrt{3})^4+\binom{8}{5}(\sqrt{3})^3+\binom{8}{6}(\sqrt{3})^2+\binom{8}{7}(\sqrt{3})^1+(\sqrt{3})^0-$$
$$\left[(\sqrt{3})^8-\binom{8}{1}(\sqrt{3})^7+\binom{8}{2}(\sqrt{3})^6-\binom{8}{3}(\sqrt{3})^5+\binom{8}{4}(\sqrt{3})^4-\binom{8}{5}(\sqrt{3})^3+\binom{8}{6}(\sqrt{3})^2-\binom{8}{7}(\sqrt{3})^1+(\sqrt{3})^0\right]$$
$$=2\left[\binom{8}{1}(\sqrt{3})^7+\binom{8}{3}(\sqrt{3})^5+\binom{8}{5}(\sqrt{3})^3+\binom{8}{7}(\sqrt{3})^1\right]=2\left[216\sqrt{3}+504\sqrt{3}+168\sqrt{3}+8\sqrt{3}\right]=1792\sqrt{3}$$

j) For $i^2=-1\Rightarrow i=\sqrt{-1}, i^3=-i$, and $i^4=1$:
$$(1+i)^8=\sum_{k=0}^{8}\binom{8}{k}1^{8-k}i^k=\binom{8}{0}i^0+\binom{8}{1}i^1+\binom{8}{2}i^2+\binom{8}{3}i^3+\binom{8}{4}i^4+\binom{8}{5}i^5+\binom{8}{6}i^6+\binom{8}{7}i^7+\binom{8}{8}i^8$$
$$=1+8i+28(-1)+56(-i)+70+56i+28(-1)+8(-i)+1=16$$

k) For $i^2=-1\Rightarrow i=\sqrt{-1}, i^3=-i$, and $i^4=1$:
$$\left(\sqrt{2}-i\right)^6=\sum_{k=0}^{6}\binom{6}{k}(\sqrt{2})^{6-k}(-i)^k=\binom{6}{0}(\sqrt{2})^6(-i)^0+\binom{6}{1}(\sqrt{2})^{6-1}(-i)^1+\binom{6}{2}(\sqrt{2})^{6-2}(-i)^2$$
$$+\binom{6}{3}(\sqrt{2})^{6-3}(-i)^3+\binom{6}{4}(\sqrt{2})^{6-4}(-i)^4+\binom{6}{5}(\sqrt{2})^{6-5}(-i)^5+\binom{6}{6}(\sqrt{2})^{6-6}(-i)^6$$
$$=8-24\sqrt{2}i-60+40\sqrt{2}i+30-6\sqrt{2}i-1=-23+10\sqrt{2}i$$

4 **a)** and **c)**
$$\left(x-\frac{2}{x}\right)^{45}=\sum_{i=0}^{45}\binom{45}{i}x^{45-i}\left(-\frac{2}{x}\right)^i$$
$$=\binom{45}{0}x^{45}+\binom{45}{1}x^{44}\left(-\frac{2}{x}\right)^1+\binom{45}{2}x^{43}\left(-\frac{2}{x}\right)^2+\ldots+\binom{45}{43}x^2\left(-\frac{2}{x}\right)^{43}+\binom{45}{44}x^1\left(-\frac{2}{x}\right)^{44}+\binom{45}{45}x^0\left(-\frac{2}{x}\right)^{45}$$
$$=x^{45}-90x^{43}+3960x^{41}+\ldots-\frac{990\cdot 2^{43}}{x^{41}}+\frac{45\cdot 2^{44}}{x^{43}}-\frac{2^{45}}{x^{45}}$$

b) For the constant term, i should be:

$$x^{45-i} = \frac{1}{x^i} \Rightarrow x^{45-i} = x^i \Rightarrow 45 - i = i \Rightarrow i = \frac{45}{2}$$

Since i is a natural number, this is not possible, i.e. there is no constant term in the expansion.

d) For the term containing x^3, i should be: $x^{45-i}\frac{1}{x^i} = x^3 \Rightarrow x^{45-2i} = x^3 \Rightarrow 45 - 2i = 3 \Rightarrow i = 21$

For $i = 21$, the 22nd term is: $\binom{45}{21} x^{45-21}\left(-\frac{2}{x}\right)^{21} = -\binom{45}{21} 2^{21} x^3.$

5 $\binom{n}{k} = \frac{n!}{k!(n-k)!} = \frac{n!}{(n-k)!(n-(n-k))!} = \binom{n}{n-k}$

6 $2^n = (1+1)^n = \binom{n}{0}1^n + \binom{n}{1}1^{n-1}\cdot 1 + \ldots + \binom{n}{n}1^n \Rightarrow$

$2^n = 1 + \binom{n}{1} + \ldots + \binom{n}{n} \Rightarrow \binom{n}{1} + \ldots + \binom{n}{n} = 2^n - 1$

7 **a)** $k! = 1\cdot 2\cdot 3\cdot \ldots (k-1)\cdot k = [1\cdot 2\cdot 3\cdot \ldots (k-1)]\,k = (k-1)!\,k = k(k-1)!$

b) $(n-k+1)! = 1\cdot 2\cdot 3\cdot \ldots \cdot (n-k)\cdot (n-k+1) = [1\cdot 2\cdot 3\cdot \ldots \cdot (n-k)](n-k+1) = (n-k)!(n-k+1)$

c) For $n = 10$ and $r = 4$ we have:

$$\binom{n}{r-1} + \binom{n}{r} = \binom{10}{4-1} + \binom{10}{4} = \frac{10!}{3!7!} + \frac{10!}{4!6!} = \frac{10!\cdot 4}{4\cdot 3!7!} + \frac{10!\cdot 7}{4!6!\cdot 7}$$

$$= \frac{10!\cdot 4}{4!7!} + \frac{10!\cdot 7}{4!7!} = \frac{10!(4+7)}{4!7!} = \frac{10!\cdot 11}{4!7!} = \frac{11!}{4!7!} = \binom{11}{4} = \binom{n+1}{r}$$

8 $\binom{6}{0}\left(\frac{1}{3}\right)^6 + \binom{6}{1}\left(\frac{1}{3}\right)^5\left(\frac{2}{3}\right) + \binom{6}{2}\left(\frac{1}{3}\right)^4\left(\frac{2}{3}\right)^2 + \ldots + \binom{6}{6}\left(\frac{2}{3}\right)^6 = \left(\frac{1}{3} + \frac{2}{3}\right)^6 = 1$

9 $\binom{8}{0}\left(\frac{2}{5}\right)^8 + \binom{8}{1}\left(\frac{2}{5}\right)^7\left(\frac{3}{5}\right) + \binom{8}{2}\left(\frac{2}{5}\right)^6\left(\frac{3}{5}\right)^2 + \ldots + \binom{8}{8}\left(\frac{3}{5}\right)^8 = \left(\frac{2}{5} + \frac{3}{5}\right)^8 = 1$

10 $\binom{n}{0}\left(\frac{1}{7}\right)^n + \binom{n}{1}\left(\frac{1}{7}\right)^{n-1}\left(\frac{6}{7}\right) + \binom{n}{2}\left(\frac{1}{7}\right)^{n-2}\left(\frac{6}{7}\right)^2 + \ldots + \binom{n}{n}\left(\frac{6}{7}\right)^n = \left(\frac{1}{7} + \frac{6}{7}\right)^n = 1^n = 1$

11 $\left(x^2 - \frac{1}{x}\right)^6 = \sum_{i=0}^{6}\binom{6}{i}(x^2)^{6-i}\left(-\frac{1}{x}\right)^i$

To determine i we have to solve the exponential equation:

$(x^2)^{6-i}\left(\frac{1}{x}\right)^i = 1 \Rightarrow x^{12-2i-i} = x^0 \Rightarrow 12 - 3i = 0 \Rightarrow i = 4$

So, for $i = 4$, we have the term

$\binom{6}{4}(x^2)^{6-4}\left(-\frac{1}{x}\right)^4 = 15\,(x^4)\left(\frac{1}{x^4}\right) = 15.$

12 $\left(3x - \frac{2}{x}\right)^8 = \sum_{i=0}^{8}\binom{8}{i}(3x)^{8-i}\left(-\frac{2}{x}\right)^i$

To determine i we have to solve the exponential equation:

$(x)^{8-i}\left(\frac{1}{x}\right)^i = 1 \Rightarrow x^{8-i-i} = x^0 \Rightarrow 8 - 2i = 0 \Rightarrow i = 4$

So, for $i = 4$, we have the term

$$\binom{8}{4}(3x)^{8-4}\left(-\frac{2}{x}\right)^4 = 70\left(81x^4\right)\left(\frac{16}{x^4}\right) = 90\,720.$$

13 $$\left(2x - \frac{3}{x^3}\right)^8 = \sum_{i=0}^{8}\binom{8}{i}(2x)^{8-i}\left(-\frac{3}{x^3}\right)^i$$

To determine i we have to solve the exponential equation:

$$(x)^{8-i}\left(\frac{1}{x^3}\right)^i = 1 \Rightarrow x^{8-i-3i} = x^0 \Rightarrow 8 - 4i = 0 \Rightarrow i = 2$$

So, for $i = 2$, we have the term

$$\binom{8}{2}(2x)^{8-2}\left(-\frac{3}{x^3}\right)^2 = 28\left(64x^6\right)\left(\frac{9}{x^6}\right) = 16\,128.$$

14 $$(1+x)^{10} = 1 + \binom{10}{1}x + \binom{10}{2}x^2 + \ldots = 1 + 10x + 45x^2 + \ldots$$

a) $1.01^{10} = (1+0.01)^{10} \approx 1 + 10 \cdot 0.01 + 45 \cdot 0.01^2 = 1.1045$

b) $0.99^{10} = (1-0.01)^{10} \approx 1 - 10 \cdot 0.01 + 45 \cdot 0.01^2 = 0.9045$

15 $$\binom{n}{r-1} + 2\binom{n}{r} + \binom{n}{r+1} = \frac{n!}{(n-r+1)!(r-1)!} + \frac{2n!}{(n-r)!\,r!}$$

$$= \frac{n!}{(n-r-1)!(r-1)!}\left[\frac{1}{(n-r)(n-r+1)} + \frac{2}{(n-r)r} + \frac{1}{r(r+1)}\right]$$

$$= \frac{n!(n^2+3n+2)}{(n-r+1)!(r+1)!} = \frac{n!(n+1)(n+2)}{(n-r+1)!(r+1)!}$$

$$= \frac{(n+2)!}{[(n+2)-(r+1)]!(r+1)!} = \binom{n+2}{r+1}$$

Each element is the sum of the two elements above it, which are the sums of the two terms above them.

$$\begin{array}{ccccccccc} & & & & 1 & & & & \\ & & & 1 & & 1 & & & \\ & & 1 & & 2 & & 1 & & \\ & 1 & & 3 & & 3 & & 1 & \\ 1 & & 4 & & 6 & & 4 & & 1 \end{array}$$

$6 = 3 + 3 = 1 + 2 + 2 + 1 = 1 + 2 \cdot\ 2 + 1$

16 In each case we have an infinite geometric series.

a) $0.\overline{7} = 0.7777\ldots = 0.7 + 0.07 + 0.007 + \ldots = 7 \cdot 10^{-1} + 7 \cdot 10^{-2} + 7 \cdot 10^{-3} + \ldots$

$$a_1 = \frac{7}{10},\ r = \frac{1}{10}$$

$$0.\overline{7} = S_\infty = \frac{\frac{7}{10}}{1 - \frac{1}{10}} = \frac{7}{9}$$

b) $0.3\overline{45} = 0.3 + 0.045\,4545... = 0.3 + 0.045 + 0.000\,45 + ... = 0.3 + 45 \cdot 10^{-3} + 45 \cdot 10^{-5} + 45 \cdot 10^{-7} + ...$

$a_1 = \frac{45}{1000}, r = \frac{1}{100}$

$$0.3\overline{45} = 0.3 + S_\infty = 0.3 + \frac{\frac{45}{1000}}{1 - \frac{1}{100}} = \frac{3}{10} + \frac{45}{990} = \frac{19}{55}$$

c) $3.21\overline{29} = 3.21 + 0.002\,929\,29... = 3.21 + 0.0029 + 0.000\,029 + ... = 3.21 + 29 \cdot 10^{-4} + 29 \cdot 10^{-6} + 29 \cdot 10^{-8} + ...$

$a_1 = \frac{29}{10\,000}, r = \frac{1}{100}$

$$3.21\overline{29} = 3.21 + S_\infty = 3.21 + \frac{\frac{29}{10\,000}}{1 - \frac{1}{100}} = \frac{321}{100} + \frac{29}{9900} = \frac{7952}{2475}$$

17 $(2x-3)^9 = \sum_{i=0}^{9} \binom{9}{i} (2x)^{9-i}(-3)^i$

$i = 3 \Rightarrow \binom{9}{3}(2x)^{9-3}(-3)^3 = 84 \cdot 64 \cdot x^6(-27) = -145\,152x^6$

The coefficient of x^6 is $-145\,152$.

18 $(ax+b)^7 = \sum_{i=0}^{7} \binom{7}{i} (ax)^{7-i}(-3)^i$

$i = 4 \Rightarrow \binom{7}{4}(ax)^{7-4}b^4 = 35a^3x^3b^4$

The coefficient of x^3b^4 is $35a^3$.

19 $\left(\frac{2}{z^2} - z\right)^{15} = \sum_{i=0}^{15} \binom{15}{i} \left(\frac{2}{z^2}\right)^{15-i} (-z)^i$

$i = 10 \Rightarrow \binom{15}{10}\left(\frac{2}{z^2}\right)^{15-10} (-z)^{10} = 3003 \cdot \frac{32}{z^{10}} z^{10} = 96\,096$

20

$(3n-2m)^5$

$= \binom{5}{0}(3n)^5(-2m)^0 + \binom{5}{1}(3n)^4(-2m)^1 + \binom{5}{2}(3n)^3(-2m)^2 + \binom{5}{3}(3n)^2(-2m)^3 + \binom{5}{4}(3n)^1(-2m)^4 + \binom{5}{5}(3n)^0(-2m)^5$

$= 243n^5 - 810n^4m + 1080n^3m^2 - 720n^2m^3 + 240nm^4 - 32m^5$

21 $(4+3r^2)^9 = \sum_{i=0}^{9} \binom{9}{i} 4^{9-i}(3r^2)^i$

$i = 5 \Rightarrow \binom{9}{5} 4^4 (3r^2)^5 = 126 \cdot 256 \cdot 243r^{10} = 7\,838\,208r^{10}$

The coefficient of r^{10} is 7 838 208.

Exercise 4.7

1 For $a_1 = 2, d = 2 \Rightarrow S_n = \frac{n}{2}[2 \cdot 2 + (n-1) \cdot 2] = n(n+1)$

Proof:

Let $S(n)$ be the statement: $2 + 4 + 6 + ... + 2n = n(n+1)$ for $n \geqslant 1$.

Basis step:

$S(1)$:

LHS $= 2 \cdot 1 = 2$

RHS $= 1 \cdot (1+1) = 2$

Inductive step:

Assume $S(k)$ is true, i.e. assume that $2 + 4 + 6 + ... + 2k = k(k+1)$ (*).

Then $S(k + 1)$: $\underbrace{2 + 4 + 6 + ... + 2k}_{*} + 2(k+1) = k(k+1) + 2(k+1) = (k+1)(k+2)$

This shows that $S(k + 1)$ is true whenever $S(k)$ is true.

Therefore: $2 + 4 + 6 + ... + 2n = n(n+1)$ for $n \geqslant 1$.

2 Let $S(n)$ be the statement:

For a sequence defined by $a_1 = 1, a_n = 3a_{n-1}, n \geqslant 1$, it is valid that $a_n = 3^{n-1}$.

Basis step:

$S(1)$:

LHS $\Rightarrow a_1 = 1$

RHS $\Rightarrow a_1 = 3^{1-1} = 1$

Inductive step:

Assume $S(k)$ is true, i.e. assume that $a_k = 3^{k-1}$ (*).

Then $S(k + 1)$: $a_{k+1} = 3a_k = (*) = 3 \cdot 3^{k-1} = 3^k$

This shows that $S(k + 1)$ is true whenever $S(k)$ is true.

Therefore: $a_n = 3^{n-1}$ for $n \geqslant 1$.

3 Let $S(n)$ be the statement:

For a sequence defined by $a_1 = 1, a_n = a_{n-1} + 4, n \geqslant 2$, it is valid that $a_n = 4n - 3, n > 1$.

Basis step:

$S(1)$:

LHS $\Rightarrow a_1 = 1$

RHS $\Rightarrow a_1 = 4 \cdot 1 - 3 = 1$

Inductive step:

Assume $S(k)$ is true, i.e. assume that $a_k = 4k - 3$ (*).

Then $S(k + 1)$: $a_{k+1} = a_k + 4 = (*) = 4k - 3 + 4 = 4(k+1) - 3$

This shows that $S(k + 1)$ is true whenever $S(k)$ is true.

Therefore: $a_n = 4n - 3, n > 1$.

4 Let $S(n)$ be the statement:

For a sequence defined by $a_1 = 1, a_n = 2a_{n-1} + 1, n \geqslant 2$, it is valid that $a_n = 2^n - 1, n > 1$.

Basis step:
$S(1)$:

LHS $\Rightarrow a_1 = 1$

RHS $\Rightarrow a_1 = 2^1 - 1 = 1$

Inductive step:
Assume $S(k)$ is true, i.e. assume that $a_k = 2^k - 1$ (*).

Then $S(k + 1)$: $a_{k+1} = 2a_k + 1 = (*) = 2(2^k - 1) + 1 = 2^{k+1} - 2 + 1 = 2^{k+1} + 1$

This shows that $S(k + 1)$ is true whenever $S(k)$ is true.

Therefore: $a_n = 2^n - 1, n > 1$.

5 Let $S(n)$ be the statement:

For a sequence defined by $a_1 = \frac{1}{2}, a_n = a_{n-1} + \frac{1}{n(n+1)}, n \geqslant 2$, it is valid that $a_n = \frac{n}{n+1}, n > 1$.

Basis step:
$S(1)$:

LHS $\Rightarrow a_1 = \frac{1}{2}$

RHS $\Rightarrow a_1 = \frac{1}{1 \cdot (1+1)} = \frac{1}{2}$

Inductive step:
Assume $S(k)$ is true, i.e. assume that $a_k = \frac{k}{k+1}$ (*).

Then $S(k + 1)$: $a_{k+1} = a_k + \frac{1}{(k+1)(k+2)} = (*) = \frac{k}{k+1} + \frac{1}{(k+1)(k+2)} = \frac{1}{k+1}\left(k + \frac{1}{(k+2)}\right)$

$$= \frac{1}{k+1} \cdot \frac{k^2 + 2k + 1}{k+2} = \frac{(k+1)^2}{(k+1)(k+2)} = \frac{k+1}{k+2}$$

This shows that $S(k + 1)$ is true whenever $S(k)$ is true.

Therefore: $a_n = \frac{n}{n+1}, n > 1$.

6 For $a_1 = \frac{1}{2}, r = \frac{1}{2} \Rightarrow S_n = \frac{1}{2} \cdot \frac{1 - \left(\frac{1}{2}\right)^n}{1 - \frac{1}{2}} = 1 - \left(\frac{1}{2}\right)^n$

Proof:
Let $S(n)$ be the statement: $\frac{1}{2} + \frac{1}{4} + \frac{1}{8} + \ldots + \frac{1}{2^n} = 1 - \left(\frac{1}{2}\right)^n$ for $n \geqslant 1$.

Basis step:
$S(1)$:

LHS $= \frac{1}{2^1} = \frac{1}{2}$

RHS $= 1 - \frac{1}{2^1} = \frac{1}{2}$

Inductive step:
Assume $S(k)$ is true, i.e. assume that $\frac{1}{2}+\frac{1}{4}+\frac{1}{8}+\ldots+\frac{1}{2^k}=1-\left(\frac{1}{2}\right)^k$ (*).

Then $S(k+1)$:

$$\underbrace{\frac{1}{2}+\frac{1}{4}+\frac{1}{8}+\ldots+\frac{1}{2^k}}_{*}+\frac{1}{2^{k+1}}=1-\left(\frac{1}{2}\right)^k+\frac{1}{2^{k+1}}=1-\left(\frac{1}{2}\right)^k\left(1-\frac{1}{2}\right)=1-\left(\frac{1}{2}\right)^k\cdot\frac{1}{2}=1-\left(\frac{1}{2}\right)^{k+1}$$

This shows that $S(k+1)$ is true whenever $S(k)$ is true.

Therefore: $\frac{1}{2}+\frac{1}{4}+\frac{1}{8}+\ldots+\frac{1}{2^n}=1-\left(\frac{1}{2}\right)^n$ for $n \geqslant 1$.

7 Let $S(n)$ be the statement: $1+2+2^2+\ldots+2^n=2^{n+1}-1$ for $n \geqslant 0$.

Basis step:
$S(0)$:

LHS $= 1$

RHS $= 2^{0+1}-1=1$

Inductive step:
Assume $S(k)$ is true, i.e. assume that $1+2+2^2+\ldots+2^k=2^{k+1}-1$ (*).

Then $S(k+1)$: $\underbrace{1+2+2^2+\ldots+2^k}_{*}+2^{k+1}=2^{k+1}-1+2^{k+1}=2\cdot 2^{k+1}-1=2^{k+2}-1$

This shows that $S(k+1)$ is true whenever $S(k)$ is true.

Therefore: $1+2+2^2+\ldots+2^n=2^{n+1}-1$ for $n \geqslant 0$.

8 Let $S(n)$ be the statement:

For a sequence defined by a_1, $a_n=ra_{n-1}$, $n \geqslant 2$, it is valid that $a_n=a_1r^{n-1}$, $n>1$.

Basis step:
$S(1)$:

LHS $\Rightarrow a_1=a_1$

RHS $\Rightarrow a_1=a_1r^{1-1}=a_1$

Inductive step:
Assume $S(k)$ is true, i.e. assume that $a_k=a_1r^{k-1}$ (*).

Then $S(k+1)$: $a_{k+1}=ra_k=(*)=ra_1r^{k-1}=a_1r^k$

This shows that $S(k+1)$ is true whenever $S(k)$ is true.

Therefore: $a_n=a_1r^{n-1}$, $n>1$.

9 Let $S(n)$ be the statement: $a+ar+ar^2+\ldots+ar^{n-1}=\frac{a-ar^n}{1-r}$ for $n \geqslant 1$.

Basis step:
$S(1)$:

LHS $= a$

RHS $=\frac{a-ar^1}{1-r}=\frac{a(1-r)}{1-r}=a$

Inductive step:
Assume $S(k)$ is true, i.e. assume that $a + ar + ar^2 + ... + ar^{k-1} = \dfrac{a - ar^k}{1 - r}$.

Then $S(k+1)$: $\underbrace{a + ar + ar^2 + ... + ar^{k-1}}_{*} + ar^k$

$$= \frac{a - ar^k}{1 - r} + ar^k = \frac{a - ar^k + ar^k(1 - r)}{1 - r} = \frac{a - ar^k + ar^k - ar^{k+1}}{1 - r} = \frac{a - ar^{k+1}}{1 - r}$$

This shows that $S(k+1)$ is true whenever $S(k)$ is true.

Therefore: $a + ar + ar^2 + ... + ar^{n-1} = \dfrac{a - ar^n}{1 - r}$ for $n \geqslant 1$.

10 Let $S(n)$ be the statement: $2^n < n!, n > 3$.

Basis step:
$S(4)$:
LHS $\Rightarrow 2^4 = 16$
RHS $\Rightarrow 4! = 24$
True!

Inductive step:
Assume $S(k)$ is true, i.e. assume that $2^k < k!$ (*).

Then $S(k+1)$: (*) $\Rightarrow 2 \cdot 2^k < 2 \cdot k!$ and since $k > 3 \Rightarrow k + 1 > 2$

So: $2 \cdot 2^k < 2 \cdot k! < (k+1)k! = (k+1)! \Rightarrow 2^{k+1} < (k+1)!$

This shows that $S(k+1)$ is true whenever $S(k)$ is true.

Therefore: $2^n < n!, n > 3$.

11 Let $S(n)$ be the statement: $2^n > n^2, n > 4$.

Basis step:
$S(5)$:
LHS $\Rightarrow 2^5 = 32$
RHS $\Rightarrow 5^2 = 25$
True!

Inductive step:
Assume $S(k)$ is true, i.e. assume that $2^k > k^2$ (*).

Then $S(k+1)$: (*) $\Rightarrow 2 \cdot 2^k > 2 \cdot k^2$ and since $k > 4 \Rightarrow k^2 > 4k \Rightarrow k^2 > 2k + 2k \Rightarrow k^2 > 2k + 1$

So: $2 \cdot 2^k > 2k^2 = k^2 + k^2 > k^2 + 2k + 1 = (k+1)^2 \Rightarrow 2^{k+1} > (k+1)^2$
This shows that $S(k+1)$ is true whenever $S(k)$ is true.

Therefore: $2^n > n^2, n > 4$.

12 Let $S(n)$ be the statement: $1 \cdot 1! + 2 \cdot 2! + 3 \cdot 3! + ... + n \cdot n! = (n+1)! - 1$ for $n \geqslant 1$.

Basis step:
$S(1)$:
LHS $= 1 \cdot 1! = 1$
RHS $= (1+1)! - 1 = 2 - 1 = 1$

Inductive step:
Assume $S(k)$ is true, i.e. assume that $1\cdot 1!+2\cdot 2!+3\cdot 3!+\ldots+k\cdot k!=(k+1)!-1$ (*).

Then $S(k+1)$:

$$\underbrace{1\cdot 1!+2\cdot 2!+3\cdot 3!+\ldots+k\cdot k!}_{*}+(k+1)\cdot(k+1)!$$

$$=(k+1)!-1+(k+1)\cdot(k+1)!=(k+1)!(1+k+1)-1=(k+1)!(k+2)-1=(k+2)!-1$$

This shows that $S(k+1)$ is true whenever $S(k)$ is true.

Therefore: $1\cdot 1!+2\cdot 2!+3\cdot 3!+\ldots+n\cdot n!=(n+1)!-1$ for $n\geqslant 1$.

13 Let $S(n)$ be the statement: $\frac{1}{1\cdot 2}+\frac{1}{2\cdot 3}+\frac{1}{3\cdot 4}+\ldots+\frac{1}{n\cdot(n+1)}=\frac{n}{n+1}$ for $n\geqslant 1$.

Basis step:
$S(1)$:

$$\text{LHS}=\frac{1}{1\cdot 2}=\frac{1}{2}$$

$$\text{RHS}=\frac{1}{1+1}=\frac{1}{2}$$

Inductive step:
Assume $S(k)$ is true, i.e. assume that $\frac{1}{1\cdot 2}+\frac{1}{2\cdot 3}+\frac{1}{3\cdot 4}+\ldots+\frac{1}{k\cdot(k+1)}=\frac{k}{k+1}$ (*).

Then $S(k+1)$:

$$\underbrace{\frac{1}{1\cdot 2}+\frac{1}{2\cdot 3}+\frac{1}{3\cdot 4}+\ldots+\frac{1}{k\cdot(k+1)}}_{*}+\frac{1}{(k+1)(k+2)}$$

$$=\frac{k}{k+1}+\frac{1}{(k+1)(k+2)}=\frac{1}{k+1}\left[k+\frac{1}{(k+2)}\right]=\frac{1}{k+1}\cdot\frac{k^2+2k+1}{k+2}=\frac{(k+1)^2}{(k+1)(k+2)}=\frac{k+1}{k+2}$$

This shows that $S(k+1)$ is true whenever $S(k)$ is true.

Therefore: $\frac{1}{1\cdot 2}+\frac{1}{2\cdot 3}+\frac{1}{3\cdot 4}+\ldots+\frac{1}{n\cdot(n+1)}=\frac{n}{n+1}$ for $n\geqslant 1$.

14 Let $S(n)$ be the statement: n^3-n is divisible by 3 for $n\geqslant 1$.

Basis step:
$S(1)$:

$1^3-1=0$ which is divisible by 3.

Inductive step:
Assume $S(k)$ is true, i.e. assume that k^3-k is divisible by 3. So, we assume that $k^3-k=3A,\ A\in\mathbb{N}$ (*).

Then $S(k+1)$:

$$(k+1)^3-(k+1)=k^3+3k^2+3k+1-k-1=\underbrace{k^3-k}_{*}+3k^2+3k=3A+3k^2+3k=3\left(A+k^2+k\right),$$

i.e. $(k+1)^3-(k+1)$ is divisible by 3.

This shows that $S(k+1)$ is true whenever $S(k)$ is true.

Therefore: n^3-n is divisible by 3 for $n\geqslant 1$.

15 Let $S(n)$ be the statement: n^5-n is divisible by 5 for $n\geqslant 1$.

Basis:
$S(1)$:

$1^5-1=0$ which is divisible by 5.

Inductive step:
Assume $S(k)$ is true, i.e. assume that $k^5 - k$ is divisible by 5. So, we assume that $k^3 - k = 5A,\ A \in \mathbb{N}$ (*).

Then $S(k + 1)$:
$(k+1)^5 - (k+1)$
$= k^5 + 5k^4 + 10k^3 + 10k^2 + 5k + 1 - k - 1 = \underbrace{k^5 - k}_{*} + 5k^4 + 10k^3 + 10k^2 + 5k$
$= 5A + 5k^4 + 10k^3 + 10k^2 + 5k = 5\left(A + k^4 + 2k^3 + 2k^2 + k\right)$, i.e. $(k+1)^5 - (k+1)$ is divisible by 5.
This shows that $S(k + 1)$ is true whenever $S(k)$ is true.

Therefore: $n^5 - n$ is divisible by 5 for $n \geqslant 1$.

16 Let $S(n)$ be the statement: $n^3 - n$ is divisible by 6 for $n \geqslant 1$.

Basis step:
$S(1)$:
$1^3 - 1 = 0$ which is divisible by 6.

Inductive step:
Assume $S(k)$ is true, i.e. assume that $k^3 - k$ is divisible by 6. So, we assume that $k^3 - k = 6A,\ A \in \mathbb{N}$ (*).

Then $S(k + 1)$:
$(k+1)^3 - (k+1)$
$= k^3 + 3k^2 + 3k + 1 - k - 1 = \underbrace{k^3 - k}_{*} + 3k^2 + 3k$
$= 6A + 3k^2 + 3k = 3\left(2A + k^2 + k\right) = 3\left(2A + k(k+1)\right)$,
which shows that $(k+1)^3 - (k+1)$ is divisible by 3. But, $2A + k(k+1)$ is even, because either k or $k + 1$ is even, and $2A$ is also even. So, $(k+1)^3 - (k+1)$ is divisible by 6.

This shows that $S(k + 1)$ is true whenever $S(k)$ is true.

Therefore: $n^3 - n$ is divisible by 6 for $n \geqslant 1$.

17 Let $S(n)$ be the statement: $n^2 + n$ is even for all integers n.

Basis step:
$S(1)$:
$1^2 + 1 = 2$ which is even.

Inductive step:
Assume $S(k)$ is true, i.e. assume that $k^2 + k$ is even. So, we assume that $k^2 + k = 2A,\ A \in \mathbb{Z}$ (*).

Then $S(k + 1)$:
$(k+1)^2 + (k+1) = k^2 + 2k + 1 + k + 1 = \underbrace{k^2 + k}_{*} + 2k + 2 = 2A + 2k + 2 = 2\left(A + k + 1\right)$,
i.e. $(k+1)^2 + (k+1)$ is even.

This shows that $S(k + 1)$ is true whenever $S(k)$ is true.

Therefore: $n^2 + n$ is even for all integers n.

18 Let $S(n)$ be the statement: $5^n - 1$ is divisible by 4 for $n \geqslant 1$.

Basis step:
$S(1)$:
$5^1 - 1 = 4$ which is divisible by 4.

Inductive step:
Assume $S(k)$ is true, i.e. assume that $5^k - 1$ is divisible by 4. So, we assume that $5^k - 1 = 4A \Rightarrow 5^k = 4A + 1,\ A \in \mathbb{N}\ \ (*)$.

Then $S(k + 1)$: $5^{k+1} - 1 = 5 \cdot 5^k - 1 = (*) = 5(4A + 1) - 1 = 20A + 5 - 1 = 20A + 4 = 4(5A + 1)$,
i.e. $5^{k+1} - 1$ is divisible by 4.

This shows that $S(k + 1)$ is true whenever $S(k)$ is true.

Therefore: $5^n - 1$ is divisible by 4 for $n \geqslant 1$.

19 Let $S(n)$ be the statement: $\begin{pmatrix} a & 0 \\ 0 & b \end{pmatrix}^n = \begin{pmatrix} a^n & 0 \\ 0 & b^n \end{pmatrix}$ for $n \geqslant 1$, $a, b \in \mathbb{R}$.

Basis step:
$S(1)$:

$$\text{LHS} = \begin{pmatrix} a & 0 \\ 0 & b \end{pmatrix}^1 = \begin{pmatrix} a & 0 \\ 0 & b \end{pmatrix}$$

$$\text{RHS} = \begin{pmatrix} a^1 & 0 \\ 0 & b^1 \end{pmatrix} = \begin{pmatrix} a & 0 \\ 0 & b \end{pmatrix}$$

Inductive step:
Assume $S(k)$ is true, i.e. assume that $\begin{pmatrix} a & 0 \\ 0 & b \end{pmatrix}^k = \begin{pmatrix} a^k & 0 \\ 0 & b^k \end{pmatrix}$ (*).

Then $S(k + 1)$:

$$\begin{pmatrix} a & 0 \\ 0 & b \end{pmatrix}^{k+1} = \begin{pmatrix} a & 0 \\ 0 & b \end{pmatrix}^k \cdot \begin{pmatrix} a & 0 \\ 0 & b \end{pmatrix} = (*) = \begin{pmatrix} a^k & 0 \\ 0 & b^k \end{pmatrix} \cdot \begin{pmatrix} a & 0 \\ 0 & b \end{pmatrix} = \begin{pmatrix} a^k \cdot a + 0 \cdot 0 & a^k \cdot 0 + 0 \cdot 0 \\ 0 \cdot a + b^k \cdot 0 & 0 \cdot 0 + b^k \cdot b \end{pmatrix} = \begin{pmatrix} a^{k+1} & 0 \\ 0 & b^{k+1} \end{pmatrix}$$

This shows that $S(k + 1)$ is true whenever $S(k)$ is true.

Therefore: $\begin{pmatrix} a & 0 \\ 0 & b \end{pmatrix}^n = \begin{pmatrix} a^n & 0 \\ 0 & b^n \end{pmatrix}$ for $n \geqslant 1$, $a, b \in \mathbb{R}$.

20 **a)** Let $S(n)$ be the statement: $\sum_{i=1}^{n} (2i + 4) = n^2 + 5n$ for $n \geqslant 1$.

Basis step:
$S(1)$:

$$\text{LHS} = \sum_{i=1}^{1} (2i + 4) = 2 \cdot 1 + 4 = 6$$

$$\text{RHS} = 1^2 + 5 \cdot 1 = 6$$

True!

Inductive step:
Assume $S(k)$ is true, i.e. assume that $\sum_{i=1}^{k} (2i + 4) = k^2 + 5k$ (*).

Then $S(k + 1)$:

$$\sum_{i=1}^{k+1} (2i + 4) = \sum_{i=1}^{k} (2i + 4) + 2(k + 1) + 4 = (*) = k^2 + 5k + 2k + 2 + 4$$

$$= k^2 + 2k + 1 + 5k + 5 = (k + 1)^2 + 5(k + 1)$$

This shows that $S(k + 1)$ is true whenever $S(k)$ is true.

Therefore: $\sum_{i=1}^{n} (2i + 4) = n^2 + 5n$ for $n \geqslant 1$.

b) Let $S(n)$ be the statement: $\sum_{i=1}^{n}(2\cdot 3^{i-1}) = 3^n - 1$ for $n \geqslant 1$.

Basis step:

$S(1)$:

$$\text{LHS} = \sum_{i=1}^{1}(2\cdot 3^{i-1}) = 2\cdot 3^{1-1} = 2$$

$$\text{RHS} = 3^1 - 1 = 2$$

True!

Inductive step:

Assume $S(k)$ is true, i.e. assume that $\sum_{i=1}^{k}(2\cdot 3^{i-1}) = 3^k - 1$ (*).

Then $S(k+1)$: $\sum_{i=1}^{k+1}(2\cdot 3^{i-1}) = \sum_{i=1}^{k}(2\cdot 3^{i-1}) + 2\cdot 3^k = (*) = 3^k - 1 + 2\cdot 3^k = 3\cdot 3^k - 1 = 3^{k+1} - 1$

This shows that $S(k+1)$ is true whenever $S(k)$ is true.

Therefore: $\sum_{i=1}^{n}(2\cdot 3^{i-1}) = 3^n - 1$ for $n \geqslant 1$.

c) Let $S(n)$ be the statement: $\sum_{i=1}^{n}\frac{1}{(2i-1)(2i+1)} = \frac{n}{2n+1}$ for $n \geqslant 1$.

Basis step:

$S(1)$:

$$\text{LHS} = \sum_{i=1}^{1}\frac{1}{(2i-1)(2i+1)} = \frac{1}{(2-1)(2+1)} = \frac{1}{3}$$

$$\text{RHS} = \frac{1}{2\cdot 1+1} = \frac{1}{3}$$

True!

Inductive step:

Assume $S(k)$ is true, i.e. assume that $\sum_{i=1}^{k}\frac{1}{(2i-1)(2i+1)} = \frac{k}{2k+1}$ (*).

Then $S(k+1)$:

$$\sum_{i=1}^{k+1}\frac{1}{(2i-1)(2i+1)} = \sum_{i=1}^{k}\frac{1}{(2i-1)(2i+1)} + \frac{1}{[2(k+1)-1][2(k+1)+1]} = (*)$$

$$= \frac{k}{2k+1} + \frac{1}{(2k+1)(2k+3)} = \frac{1}{2k+1}\left(k + \frac{1}{2k+3}\right) = \frac{2k^2+3k+1}{(2k+1)(2k+3)}$$

$$= \frac{(2k+1)(k+1)}{(2k+1)(2k+3)} = \frac{k+1}{2k+3}$$

This shows that $S(k+1)$ is true whenever $S(k)$ is true.

Therefore: $\sum_{i=1}^{n}\frac{1}{(2i-1)(2i+1)} = \frac{n}{2n+1}$ for $n \geqslant 1$.

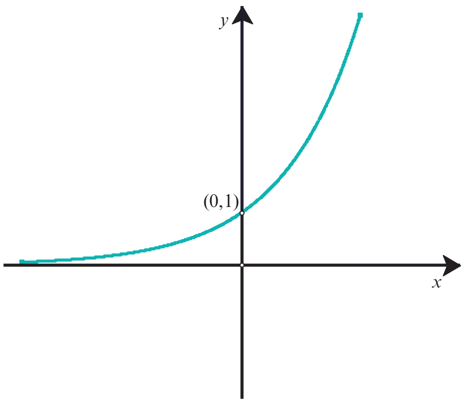
Chapter 5

Exercise 5.1 and 5.2

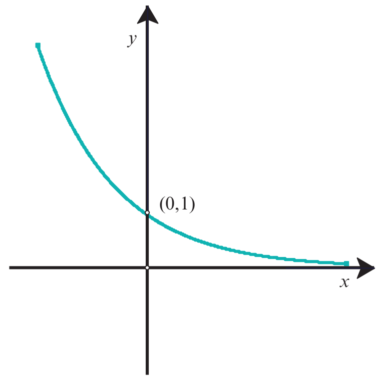
1 **a)** $y = b^x$

b) The domain is $\{x : x \in \mathbb{R}\}$; the range is $\{y : y > 0\}$.

c) **i)** $b > 1$:

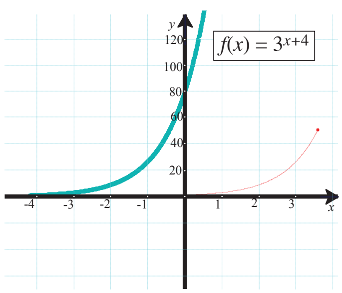


ii) $0 < b < 1$:



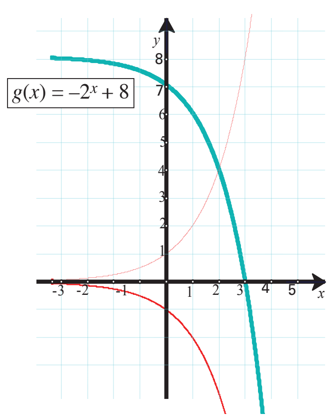
2 The graph of $f(x) = 3^{x+4}$ can be obtained by translating the graph of $f_1(x) = 3^x$ four units to the left.

For function f, the domain is $x \in \mathbb{R}$, the range is $y > 0$, the y-intercept is $(0, 81)$, and the horizontal asymptote is $y = 0$ (x-axis).



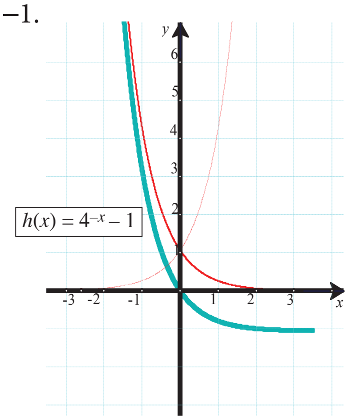
3 The graph of $g(x) = -2^x + 8$ can be obtained by first reflecting the graph of $g_1(x) = 2^x$ in the x-axis, and then translating the graph of $g_2(x) = -2^x$ vertically eight units up.

For function g, the domain is $x \in \mathbb{R}$, the range is $y < 8$, the y-intercept is $(0, 7)$, and the horizontal asymptote is $y = 8$.

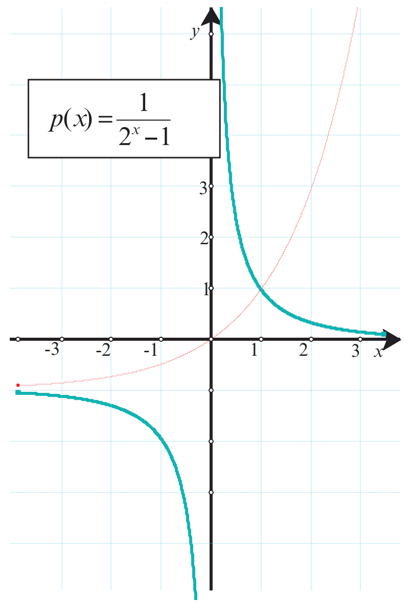


4 The graph of $h(x) = 4^{-x} - 1$ can be obtained by first reflecting the graph of $h_1(x) = 4^x$ in the y-axis, and then translating the graph of $h_2(x) = 4^{-x}$ vertically one unit down.

For function h, the domain is $x \in \mathbb{R}$, the range is $y > -1$, the y-intercept is (0, 0), and the horizontal asymptote is $y = -1$.

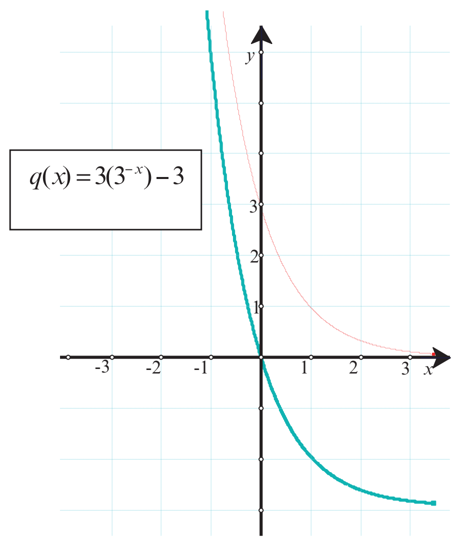


5 The graph of $p(x) = \dfrac{1}{2^x - 1}$ can be obtained by first translating the graph of $p_1(x) = 2^x$ vertically one unit down, and then finding the reciprocal function of $p_2(x) = 2^x - 1$. Consequently, the graph of p will have a vertical asymptote when $2^x - 1 = 0 \Rightarrow 2^x = 1 \Rightarrow x = 0$. When x becomes very large, values of $p_2(x)$ also become very large, so values of $p(x)$ become very small positive numbers approaching zero. When $x \to -\infty$, values of $p_2(x)$ approach -1, so values of $p(x)$ approach -1. For function p, the domain is $x \in \mathbb{R}$, $x \neq 0$, the range is $y < -1$ or $y > 0$, the y-intercept does not exist, and the horizontal asymptotes are $y = -1$ and $y = 0$.



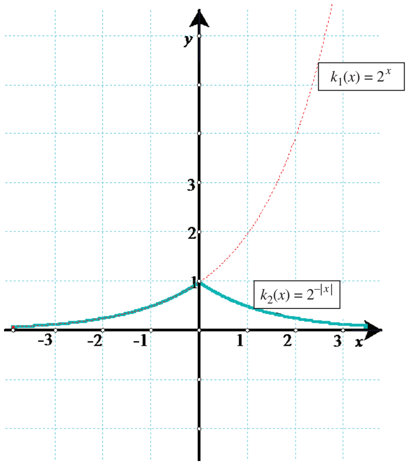
6 The graph of $q(x) = 3(3^{-x}) - 3$ can be obtained by first reflecting the graph of $q_1(x) = 3^x$ in the y-axis, followed by a vertical stretch of $q_2(x) = 3^{-x}$ by scale factor 3, and, finally, translating the graph of $q_3(x) = 3\left(3^{-x}\right)$ vertically three units down.

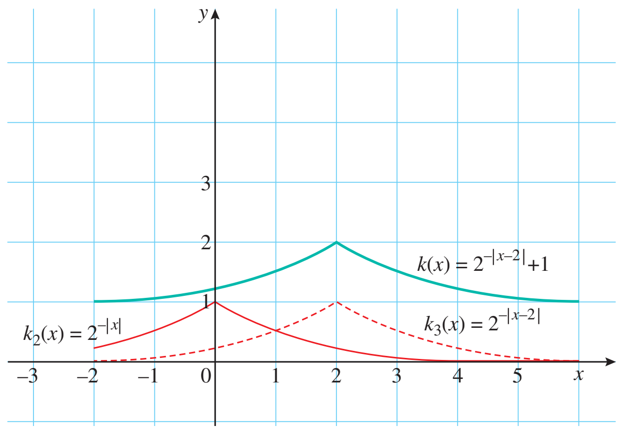
So, the domain of q is $x \in \mathbb{R}$, the range is $y > -3$, the y-intercept is $(0, 3 \cdot 3^{-0} - 3) = (0, 0)$, and the horizontal asymptote is $y = -3$.



7 The graph of $k(x) = 2^{-|x-2|} + 1$ can be obtained by first reflecting the negative part (negative x-values) of the graph of $k_1(x) = 2^x$ in the y-axis. In this way we obtain the graph of $k_2(x) = 2^{-|x|}$. Then we translate the graph of $k_2(x) = 2^{-|x|}$ two units to the right, and, finally, we translate the graph of $k_3(x) = 2^{-|x-2|}$ vertically one unit up.

So, the domain of k is $x \in \mathbb{R}$, the range is $1 < y \leqslant 2$, the y-intercept is $(0, 2^{-|-2|} + 1) = \left(0, \frac{1}{4} + 1\right) = \left(0, \frac{5}{4}\right)$, and the horizontal asymptote is $y = 1$.





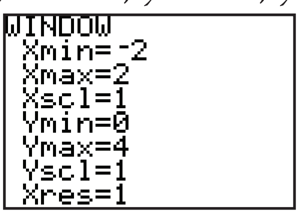
8 The graph of $f(x) = a(b)^{x-c} + d$ can be obtained by first translating the graph of $f_1(x) = b^x$ c units to the left/right (this will not influence the domain, range, or asymptote). Secondly, stretch/shrink the graph of $f_2(x) = b^{x-c}$ by scale factor $|a|$ (this will also not influence the domain, range, or asymptote). If a is negative, reflect the graph in the x-axis (this will influence the range). And, finally, translate the graph of $f_2(x) = a \cdot b^{x-c}$ vertically d units up/down (this will influence the range and asymptote).

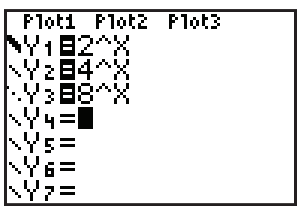
So, for all parameters, the domain of f is $x \in \mathbb{R}$, the y-intercept is $(0, a \cdot b^{-c} + d)$, and the horizontal asymptote is $y = d$.

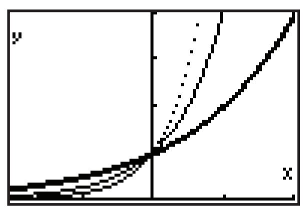
The range of function f depends on the sign of a. If $a > 0$ (no reflection), the range is $y > d$. If $a < 0$ (reflection), the range is $y < d$.

[**Note:** If $c > 0$, the graph will move to the right; if $d > 0$, the graph will move up.]

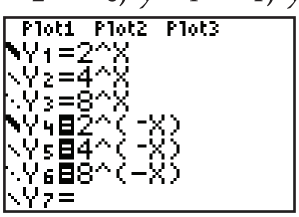
9 **a)** $y = 2^x$ **b)** $y = 4^x$ **c)** $y = 8^x$

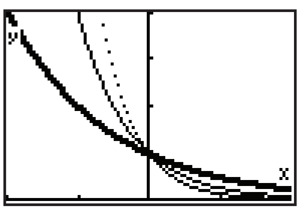


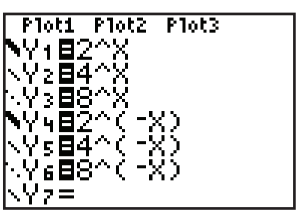


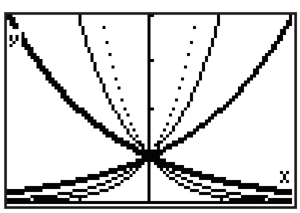


d) $y = 2^{-x}$ **e)** $y = 4^{-x}$ **f)** $y = 8^{-x}$









10 (Q9d) $y = 2^{-x} = \frac{1}{2^x} = \left(\frac{1}{2}\right)^x$ (Q9e) $y = 4^{-x} = \frac{1}{4^x} = \left(\frac{1}{4}\right)^x$ (Q9f) $y = 8^{-x} = \frac{1}{8^x} = \left(\frac{1}{8}\right)^x$

11 Comparing the graphs from question 9 **a**, **b** and **c**, we see that $y = 4^x$ is steeper than $y = 2^x$, and that $y = 8^x$ is steeper than both $y = 4^x$ and $y = 2^x$. Similarly, we can conclude that for $1 < a < b$, $y = b^x$ is steeper than $y = a^x$.

12 The population will follow an exponential model: $P(t) = P_0 b^t$. The initial value is 100 000, so $P_0 = 100\,000$. The population triples every 25 years, so the growth factor is $3^{\frac{1}{25}}$ when time is expressed in years. So, the population function is:

$$P(t) = 100\,000 \cdot \left(3^{\frac{1}{25}}\right)^t = 100\,000 \cdot 3^{\frac{t}{25}}$$

a) $P(50) = 100\,000 \cdot 3^{\frac{50}{25}} = 900\,000$

b) $P(70) = 100\,000 \cdot 3^{\frac{70}{25}} = 2\,167\,402 \approx 2\,170\,000$

c) $P(100) = 100\,000 \cdot 3^{\frac{100}{25}} = 8\,100\,000$

(**Note:** Population should be a natural number.)

13 The number of bacteria will follow an exponential model: $N(t) = N_0 b^t$. The initial value is 10^4, so $N_0 = 10^4$. The population doubles every 3 minutes, so the growth factor is $2^{\frac{1}{3}}$ when time is expressed in minutes. So, the function for the number of bacteria is: $N(t) = 10^4 \cdot \left(2^{\frac{1}{3}}\right)^t = 10^4 \cdot 2^{\frac{t}{3}}$.

a) $N(3) = 10^4 \cdot 2^{\frac{3}{3}} = 10^4 \cdot 2 = 20\,000$

b) $N(9) = 10^4 \cdot 2^{\frac{9}{3}} = 10^4 \cdot 2^3 = 80\,000$

c) $N(27) = 10^4 \cdot 2^{\frac{27}{3}} = 10^4 \cdot 2^9 = 5\,120\,000$

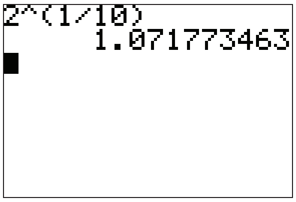
d) Firstly, express one hour in minutes, and then find the value of the function for $t = 60$:

$N(60) = 10^4 \cdot 2^{\frac{60}{3}} = 10^4 \cdot 2^{20} \approx 1.048\,576 \times 10^{10} \approx 10\,500\,000\,000$

14 **a)** The amount of money will follow an exponential model: $A(t) = A_0 b^t$. The amount doubles every 10 years, so the growth factor is $2^{\frac{1}{10}}$ when time is expressed in minutes. Therefore, the function for the amount of money is: $A(t) = A_0 2^{\frac{t}{10}}$.

b) If interest is compounded once a year, we can find the amount of money using the formula: $A(t) = A_0\,(1 + r)^t$.

Hence, $A(t) = A_0\,(1 + r)^t = A_0 2^{\frac{t}{10}} \Rightarrow (1 + r)^t = 2^{\frac{t}{10}}$. Since it holds for all t, it follows that $(1 + r)^1 = 2^{\frac{1}{10}} \Rightarrow 1 + r = 1.071\,7734... \Rightarrow r \approx 0.0718 = 7.18\%$.



15 We use the exponential function associated with compound interest: $A(t) = P\left(1 + \frac{r}{n}\right)^{nt}$ with values $P = 10\,000$, $r = 0.11$, and $n = 4$. So, $A(t) = 10\,000\left(1 + \frac{0.11}{4}\right)^{4t}$.

a) $A(5) = 10\,000\left(1 + \frac{0.11}{4}\right)^{20} = 17\,204.28$

The value of the investment will be \$17 204.28 after 5 years.

b) $A(10) = 10\,000\left(1 + \dfrac{0.11}{4}\right)^{40} = 29\,598.74$

The value of the investment will be \$29 598.74 after 10 years.

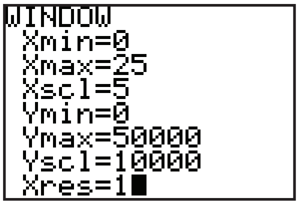
c) $A(15) = 10\,000\left(1 + \dfrac{0.11}{4}\right)^{60} = 50\,922.51$

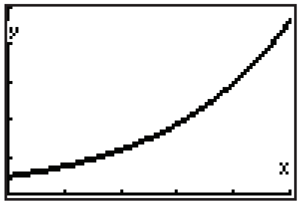
The value of the investment will be \$50 922.51 after 15 years.

(**Note:** Since we are working with money, answers are rounded to two decimal places.)

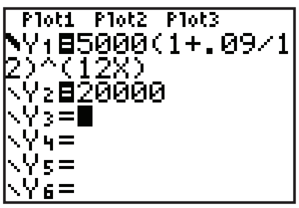
16 a) We use the exponential function associated with compound interest: $A(t) = P\left(1 + \dfrac{r}{n}\right)^{nt}$ with $P = 5000$, $r = 0.09$, and $n = 12$. So, $A(t) = 5000\left(1 + \dfrac{0.09}{12}\right)^{12t}$.

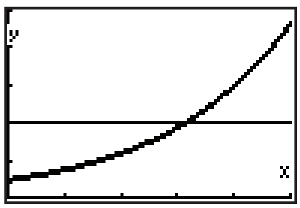
b) To determine the scale on the y-axis, we find $A(25) \approx 47\,000$. So, we choose our window as follows:

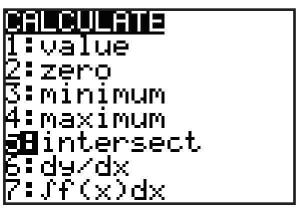


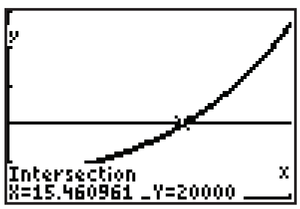


c) Firstly, we have to find the time at which the investment has a value of \$20 000; that means we have to solve the equation $A(t) = 20\,000$.









So, at 15.46 years, the value is approximately \$20 000. After that time the value of the investment will be greater than \$20 000. Therefore, the investment has a value greater than \$20 000 after a minimum of 16 years.

17 We use the exponential function associated with compound interest: $A(t) = P\left(1 + \dfrac{r}{n}\right)^{nt}$ with $P = 10\,000$, $r = 0.11$, and $t = 5$. So, $A(5) = 10\,000\left(1 + \dfrac{0.11}{n}\right)^{n \cdot 5}$.

a) Annually means $n = 1$, so $A(5) = 10\,000\,(1.11)^5 = 16\,850.58$.

The value of the investment will be \$16 850.58.

b) Monthly means $n = 12$, so $A(5) = 10\,000\left(1 + \dfrac{0.11}{12}\right)^{12 \cdot 5} = 17\,289.16$.

The value of the investment will be \$17 289.16.

c) Daily means $n = 365$, so $A(5) = 10\,000\left(1 + \dfrac{0.11}{365}\right)^{365 \cdot 5} = 17\,331.09$.

The value of the investment will be \$17 331.09.

d) Hourly means $n = 365 \cdot 24 = 8760$, so $A(5) = 10\,000\left(1 + \dfrac{0.11}{365 \cdot 24}\right)^{265 \cdot 24 \cdot 5} = 17\,332.47$.

The value of the investment will be \$17 332.47.

18 We use the exponential function associated with compound interest: $A(t) = P\left(1 + \dfrac{r}{n}\right)^{nt}$ with $P = 1$, $r = 1$, and $t = 1$. So $A(1) = 1\left(1 + \dfrac{1}{n}\right)^{n \cdot 1}$.

a) Annually means $n = 1$, so $A(1) = 1\,(2)^1 = 2$.

b) Monthly means $n = 12$, so $A(1) = 1\left(1 + \dfrac{1}{12}\right)^{12} \approx 2.61$.

c) Daily means $n = 365$, so $A(1) = 1\left(1 + \dfrac{1}{365}\right)^{365} \approx 2.71$.

d) Hourly means $n = 365 \cdot 24 = 8760$, so $A(1) = 1\left(1 + \dfrac{1}{8760}\right)^{8760} \approx 2.72$.

e) Every minute means $n = 365 \cdot 24 \cdot 60 = 525\,600$, so $A(1) = 1\left(1 + \dfrac{1}{525\,600}\right)^{525\,600} \approx 2.72$.

19 The population of deer behaves according to the exponential function associated with compound interest: $A(t) = P\left(1 + \dfrac{r}{n}\right)^{nt}$, with values $r = 0.032$ and $n = 1$. So, $A(t) = P \cdot (1.032)^t$.

a) One year ago the population was 248 000 / 1.032 = 240 310.

b) Let P be the initial population eight years ago. So, the present population is $248\,000 = A(8) = P \cdot (1.032)^8$. Therefore, $P = \dfrac{248\,000}{1.032^8} \approx 192\,759$.

20 The exponential decay model for carbon-14 is $A(t) = A_0\left(\dfrac{1}{2}\right)^{\frac{t}{5730}}$, where time is given in years. After 20 000 years, the relation between the original amount and the actual amount is

$A(20\,000) = A_0\left(\dfrac{1}{2}\right)^{\frac{20\,000}{5730}}$, so their ratio is: $\dfrac{A(20\,000)}{A_0} = \dfrac{A_0\left(\frac{1}{2}\right)^{\frac{20\,000}{5730}}}{A_0} = \left(\dfrac{1}{2}\right)^{\frac{20\,000}{5730}} = 0.088\,978\ldots \approx 0.0890$.

Approximately 8.90% of the original amount remains.

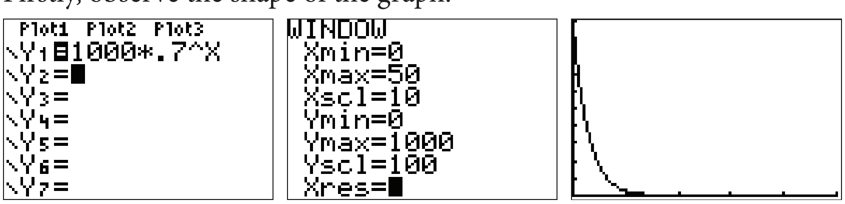
21 The exponential decay model is $A(t) = A_0\left(\dfrac{1}{2}\right)^{\frac{t}{36}}$, where A_0 is the initial value and time (t) is given in hours. After the drug has been in the bloodstream for 5 days: $A(120) = A_0\left(\dfrac{1}{2}\right)^{\frac{120}{36}}$, so

$\dfrac{A(120)}{A_0} = \left(\dfrac{1}{2}\right)^{\frac{120}{36}} \approx 0.0992$.

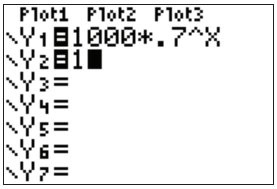
Approximately 9.92% of the original concentration.

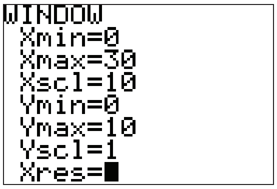
22 a) The exponential decay model for the amount of fluid is $A(w) = A_0 b^w$, where $A_0 = 1000$ ml and $b = 0.7$, where time is given in weeks. So, the function is: $A(w) = 1000\,(0.7)^w$.

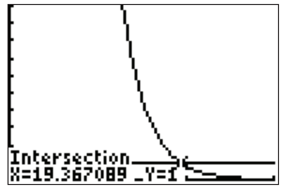
b) Firstly, observe the shape of the graph:



Now solve for $A(w) < 1$:

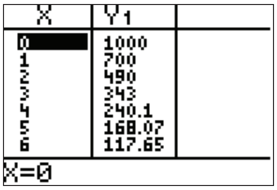




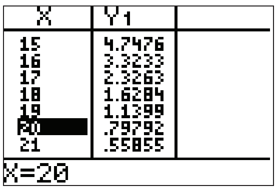


We can see that when $w > 19.4$ the volume is less than 1 ml; therefore, it will take 20 weeks for the volume to be less than 1 ml.

Note: We can find the solution using a table of values of the function:



Work down the list looking for the y_1 that is less than 1.



23 When $b = 0$, $f(x) = 0^x = 0$ is a constant function.
When $b < 0$ and x is an even integer, $b^x > 0$.
When $b < 0$ and x is an odd integer, $b^x < 0$.
When $b < 0$ and x is not an integer, there are cases when b^x is not defined (for example, $b^{\frac{1}{2}}$).

24 Payment plan I behaves as an arithmetic series with first term 1 and a difference of 1:

$$1 + 2 + 3 + ... + 30 = \frac{30}{2}(1 + 30) = 465$$

Payment plan II behaves as a geometric series with first term 0.01 and ratio $r = 2$:

$$0.01 + 0.02 + 0.04 + ... = 0.01\frac{2^{30} - 1}{2 - 1} = 10\,737\,418.23$$

Therefore, payment plan II would give the largest salary.

25 **a)** From the graph, we can see that for k and a:

$$f(1) = 6 \Rightarrow ka = 6$$

$$f(3) = 24 \Rightarrow ka^3 = 24$$

So, using $k = \frac{6}{a}$, we have: $\frac{6}{a}a^3 = 24 \Rightarrow a^2 = 4 \Rightarrow a = 2$ (since a has to be positive), and $k = 3$.

b) From the graph, we can see that for k and a:

$$f(0) = 2 \Rightarrow k = 2$$

$$f(2) = \frac{2}{9} \Rightarrow ka^2 = \frac{2}{9} \Rightarrow 2a^2 = \frac{2}{9} \Rightarrow a = \frac{1}{3}$$ (since a has to be positive).

c) From the graph, we can see that for k and a:

$f(1) = -12 \Rightarrow ka = -12$

$f(-1) = -\frac{4}{3} \Rightarrow ka^{-1} = -\frac{4}{3}$

So, using $k = -\frac{12}{a}$, we have: $-\frac{12}{a}\frac{1}{a} = -\frac{4}{3} \Rightarrow \frac{1}{a^2} = \frac{1}{9} \Rightarrow a = 3$ (since a has to be positive), and $k = -\frac{12}{3} = -4$.

d) From the graph, we can see that for k and a:

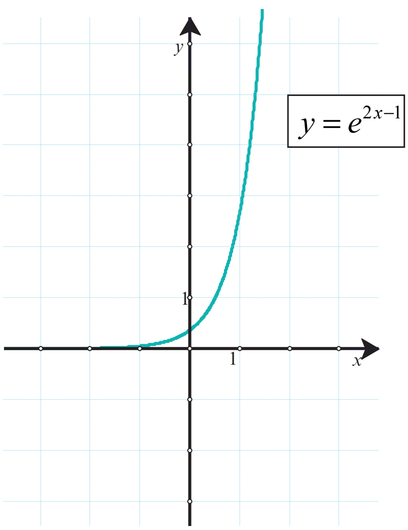
$f(1) = 15 \Rightarrow ka = 15$

$f(2) = 150 \Rightarrow ka^2 = 150$

So, using $k = \frac{15}{a}$, we have: $\frac{15}{a}a^2 = 150 \Rightarrow a = 10$ and $k = \frac{15}{10} = \frac{3}{2}$.

Exercise 5.3

1

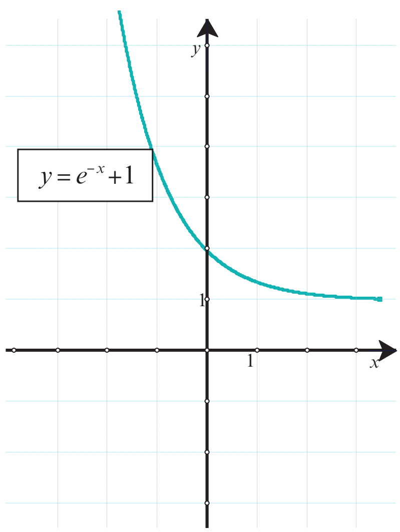


a) The domain is $\{x : x \in \mathbb{R}\}$; the range is $\{y : y > 0\}$.

b) There is no x-intercept; the y-intercept is $(0, e^{0-1}) = \left(0, \frac{1}{e}\right)$.

c) Horizontal asymptote: $y = 0$

2

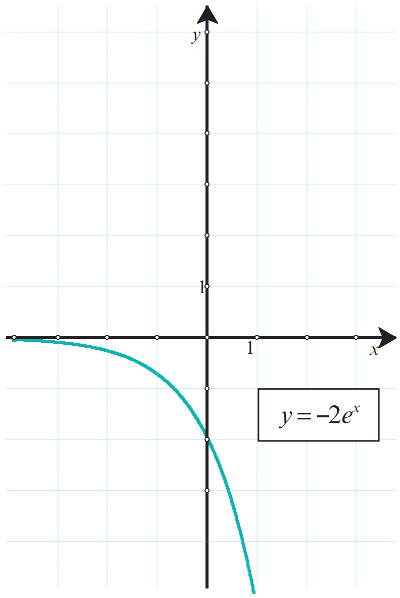


a) The domain is $\{x : x \in \mathbb{R}\}$; the range is $\{y : y > 1\}$.

b) There is no x-intercept; the y-intercept is $(0, e^{-0} + 1) = (0, 1 + 1) = (0, 2)$.

c) Horizontal asymptote: $y = 1$

3

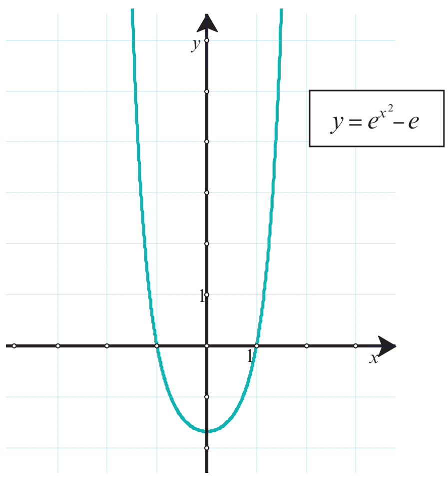


a) The domain is $\{x : x \in \mathbb{R}\}$; the range is $\{y : y < 0\}$.

b) There is no x-intercept; the y-intercept is $\left(0, -2e^0\right) = (0, -2)$.

c) Horizontal asymptote: $y = 0$

4

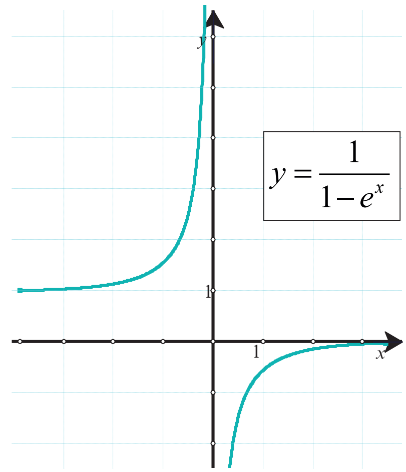


a) The domain is $\{x : x \in \mathbb{R}\}$; the range is $\{y : y \geqslant 1 - e\}$.

b) x-intercepts:
$e^{x^2} - e = 0 \Rightarrow e^{x^2} = e \Rightarrow x^2 = 1 \Rightarrow x = \pm 1$;
hence, x-intercepts are: $(-1, 0), (1, 0)$.
y-intercept is $\left(0, e^0 - e\right) = (0, 1 - e)$.

c) There is no horizontal asymptote.

5

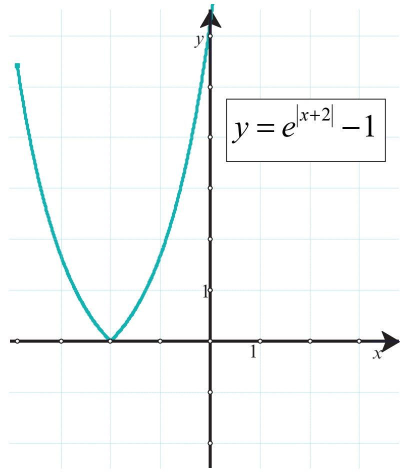


a) The domain is $\{x : x \in \mathbb{R}, x \neq 0\}$; the range is $\{y : y < 0, y > 1\}$.

b) There are no x-intercepts; there is no y-intercept.

c) Horizontal asymptotes: $y = 0$, $y = 1$

6



a) The domain is $\{x : x \in \mathbb{R}\}$; the range is $\{y \geqslant 0\}$.

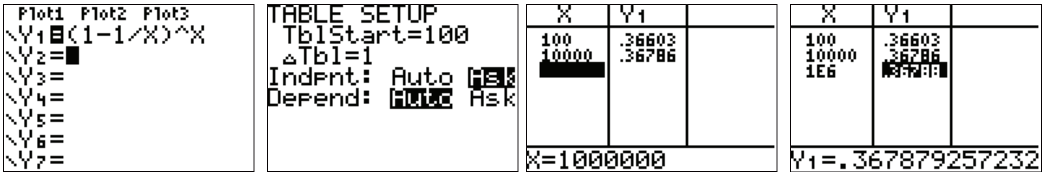
b) x-intercepts: $e^{|x+2|} - 1 = 0 \Rightarrow e^{|x+2|} = 1 \Rightarrow$
$|x + 2| = 0 \Rightarrow x = -2$; hence,
x-intercept is: $(-2, 0)$. y-intercept is:
$\left(0, e^{|0+2|} - 1\right) = \left(0, e^2 - 1\right)$.

c) Horizontal asymptote: none

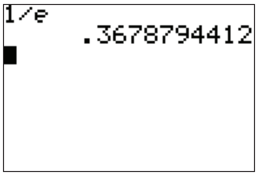
7 a) $e = \lim_{n \to \infty} \left(1 + \frac{1}{n}\right)^n$

b)

100	0.366 032 341 273
10 000	0.367 861 046 433
1 000 000	0.367 879 257 232



c) 0.367 88; this number is the reciprocal of e.



8

As $x \to \infty, \left(1+\frac{1}{x}\right)^x \to e = 2.718\,281\,828\,46...$

The curve $y = \left(1+\frac{1}{x}\right)^x$ is below the line $y = 2.72$ and will not intersect it.

9 For Bank A, we use the formula

$A(t) = P\left(1+\frac{r}{n}\right)^{nt}$ with $P = 500$, $r = 0.0685$, $n = 12$, and $t = 3$. So,

$A(3) = 500\left(1+\frac{0.0685}{12}\right)^{12\cdot 3} = 613.71$ and we earn 113.71 euros in interest.

For Bank B, we use the formula $A(t) = Pe^{rt}$ with $P = 500$, $r = 0.0685$, and $t = 3$. So,

$A(3) = 500e^{0.0685\cdot 3} = 614.07$ and we earn 114.07 euros in interest.

Bank B's account earns 0.36 euros more in interest.

10 For *Blue Star*, we use the formula

$A(t) = P\left(1+\frac{r}{n}\right)^{nt}$ with $P = 1000$, $r = 0.0613$, $n = 52$, and $t = 5$.

So, $A(5) = 1000\left(1+\frac{0.0613}{52}\right)^{52\cdot 3} = 1358.42$.

For *Red Star*, we use the formula $A(t) = Pe^{rt}$ with $P = 1000$, $r = 0.0595$, and $t = 5$. So,
$A(5) = 1000e^{0.0595\cdot 3} = 1346.49$.

Blue Star has the greater total of \$1358.42, which is \$11.93 more than *Red Star*.

11 We will use the formula $A(t) = Ce^{-0.0239t}$ with $C = 1$.

a) $A(1) = e^{-0.0239\cdot 1} = 0.976$ kg

b) $A(10) = e^{-0.0239\cdot 10} = 0.787$ kg

c) $A(100) = e^{-0.0239\cdot 100} = 0.0916$ kg

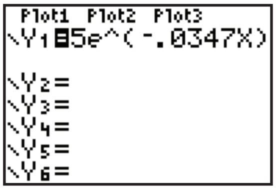
d) $A(250) = e^{-0.0239\cdot 250} = 0.002\,54$ kg

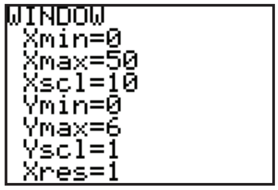
12 a) In the formula for continuous exponential growth/decay, $A(t) = Ce^{rt}$, C is the initial quantity (when $t = 0$). So, from $A(t) = 5e^{-0.0347t}$ kg, we can see that the initial mass is 5 kg.

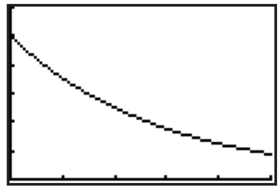
(**Note:** The initial amount can also be obtained by finding $A(0) = 5e^{-0.0347 \cdot 0} = 5e^0 = 5$.)

b) $\dfrac{A(10)}{A(0)} = \dfrac{3.534...}{5} \approx 0.707 = 70.7\%$

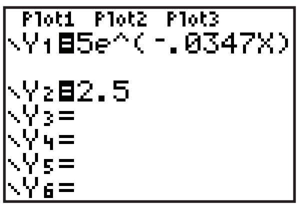
c)

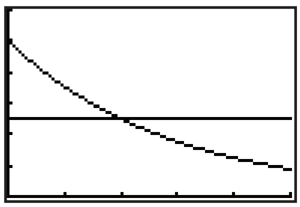




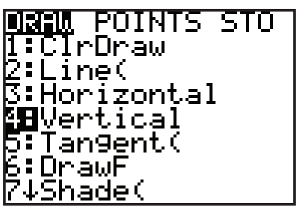


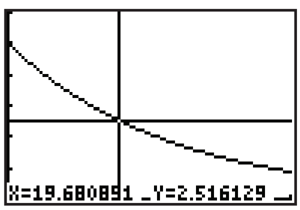
d) The half-life is the amount of time it takes for a given amount of material to decay to half of its original amount. The initial mass is 5 kg, so we have to find the time at which the mass is 2.5 kg.





To approximate the time value, we can use a vertical line from the Draw menu and adjust the line at the intersection.





From the graph we can see that the value of x is approximately 19.7, so the half-life is around 20 days.

13 a) We use the compound interest formula $A(t) = P\left(1 + \dfrac{r}{n}\right)^{nt}$ with $r = 0.085$ and $n = 2$.

$A(t) = P\left(1 + \dfrac{0.085}{2}\right)^{2t} = P\left((1.0425)^2\right)^t = P(1.0868)^t$

b) We use the compound interest formula $A(t) = P\left(1 + \dfrac{r}{n}\right)^{nt}$ with $r = 0.0825$ and $n = 4$.

$A(t) = P\left(1 + \dfrac{0.0825}{4}\right)^{4t} = P\left((1.020\,625)^4\right)^t = P(1.0851)^t$

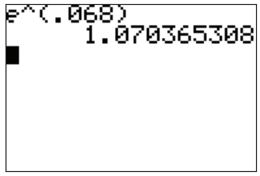
c) We use the continuous interest formula $A(t) = Pe^{rt}$ with $r = 0.08$.

$A(t) = Pe^{0.08t} = P\left(e^{0.08}\right)^t = P(1.0833)^t$

By comparing the bases of the exponential functions, we can see that in **a** we have the largest number, so $8\frac{1}{2}\%$ compounded semi-annually is the better investment.

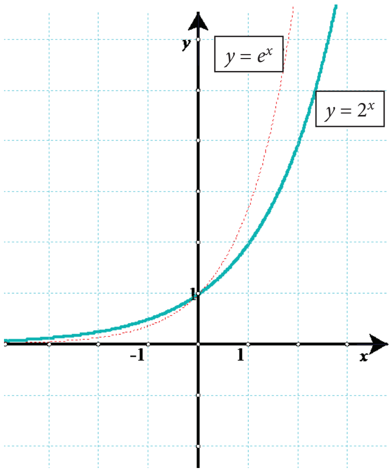
14 a) $20r^t = 20e^{0.068t} \Rightarrow r^t = e^{0.068t}$

Since it holds for all t, it will hold for $t = 0$;
hence, $r = e^{0.068} \approx 1.070\,37$.

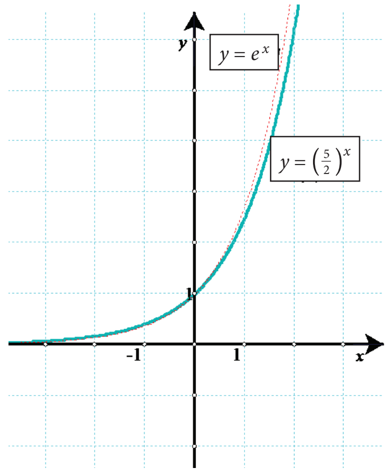


b) $\dfrac{B(t+1)}{B(t)} = \dfrac{20r^{t+1}}{20r^t} = r \approx 1.070\,37$; hence, 7.037%.

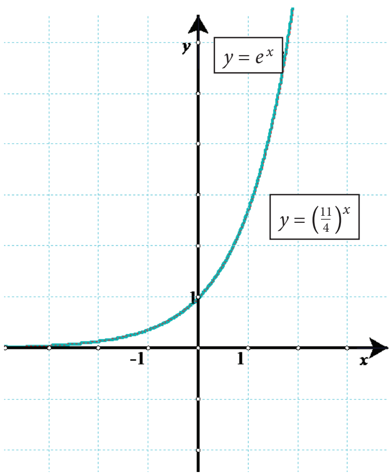
15 a) Less than 1.



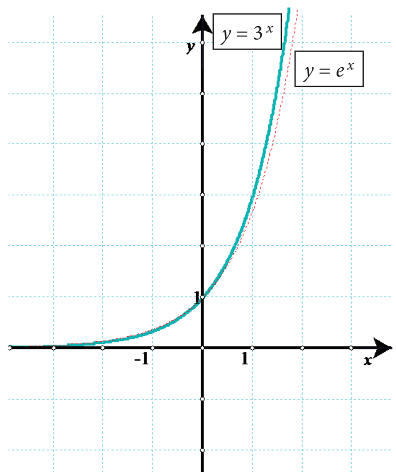
b) Less than 1.



c) Greater than 1.



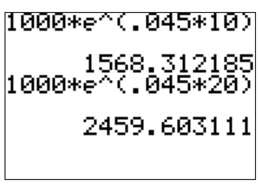
d) Greater than 1.



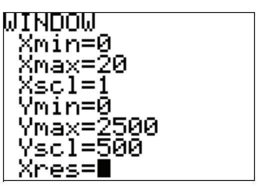
16 a) $A(t) = 1000e^{0.045t}$

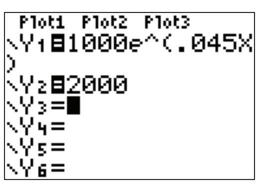
$A(10) = 1000e^{0.045\cdot 10} \approx 1568.31$

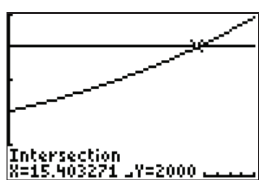
$A(20) = 1000e^{0.045\cdot 20} \approx 2459.60$



b)

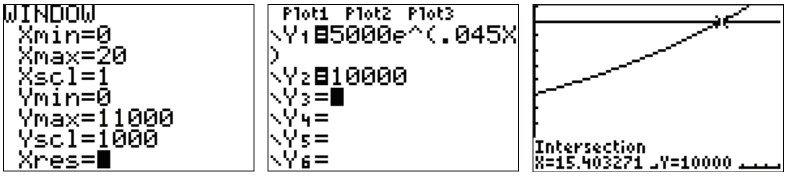






So, the initial amount will have doubled to £2000 after 15.4 years.

c) $A(t) = 5000e^{0.045t}$



So, the initial amount will have doubled to £10 000 after 15.4 years.

d) The same! Time is the solution of the equation $2A_0 = A_0e^{0.045t}$, which is equivalent to $2 = e^{0.045t}$; this equation obviously does not depend on the initial amount A_0.

Exercise 5.4

In questions 1–9, we use the definition of a logarithmic function: $y = \log_b x$ if and only if $x = b^y$

1 $\log_2 16 = 4$ is equivalent to $2^4 = 16$.

2 $\ln 1 = 0$ is equivalent to $e^0 = 1$.

3 $\log 100 = 2$ is equivalent to $10^2 = 100$.

4 $\log 0.01 = -2$ is equivalent to $10^{-2} = 0.01$.

5 $\log_7 343 = 3$ is equivalent to $7^3 = 343$.

6 $\ln\left(\frac{1}{e}\right) = -1$ is equivalent to $e^{-1} = \frac{1}{e}$.

7 $\log 50 = y$ is equivalent to $10^y = 50$.

8 $\ln x = 12$ is equivalent to $e^{12} = x$.

9 $\ln(x + 2) = 3$ is equivalent to $e^3 = x + 2$.

In questions 10–18, we use the definition of a logarithmic function (with rearranged terms):

$b^y = x$ if and only if $\log_b x = y$

10 $2^{10} = 1024$ is equivalent to $\log_2 1024 = 10$.

11 $10^{-4} = 0.0001$ is equivalent to $\log 0.0001 = -4$.

12 $4^{-\frac{1}{2}} = \frac{1}{2}$ is equivalent to $\log_4\left(\frac{1}{2}\right) = -\frac{1}{2}$.

13 $3^4 = 81$ is equivalent to $\log_3 81 = 4$.

14 $10^0 = 1$ is equivalent to $\log 1 = 0$.

15 $e^x = 5$ is equivalent to $\ln 5 = x$.

16 $2^{-3} = 0.125$ is equivalent to $\log_2 0.125 = -3$.

17 $e^4 = y$ is equivalent to $\ln y = 4$.

18 $10^{x+1} = y$ is equivalent to $\log y = x + 1$.

19 $\log_2 64 = y$ is equivalent to the exponential equation $2^y = 64$. Since $2^6 = 64$, then $2^y = 2^6$. Therefore, $y = 6 \Rightarrow \log_2 64 = 6$.

20 $\log_4 64 = y$ is equivalent to the exponential equation $4^y = 64$. Since $4^3 = 64$, then $4^y = 4^3$. Therefore, $y = 3 \Rightarrow \log_4 64 = 3$.

21 $\log_2\left(\frac{1}{8}\right) = y$ is equivalent to the exponential equation $2^y = \frac{1}{8}$. Since $2^{-3} = \frac{1}{8}$, then $2^y = 2^{-3}$. Therefore, $y = -3 \Rightarrow \log_2\left(\frac{1}{8}\right) = -3$.

22 $\log_3(3^5) = y$ is equivalent to the exponential equation $3^y = 3^5$. Therefore, $y = 5 \Rightarrow \log_3 3^5 = 5$.

23 $\log_{16} 8 = y$ is equivalent to the exponential equation $16^y = 8$. Since $16^y = 2^{4y}$ and $8 = 2^3$, then $2^{4y} = 2^3$. Therefore, $4y = 3 \Rightarrow y = \frac{3}{4} \Rightarrow \log_{16} 8 = \frac{3}{4}$.

24 $\log_{27} 3 = y$ is equivalent to the exponential equation $27^y = 3$. Since $27^y = 3^{3y}$, then $3^{3y} = 3^1$. Therefore, $3y = 1 \Rightarrow y = \frac{1}{3} \Rightarrow \log_{27} 3 = \frac{1}{3}$.

25 $\log_{10} 0.001 = y$ is equivalent to the exponential equation $10^y = 0.001$. Since $10^{-3} = 0.001$, then $10^y = 10^{-3}$. Therefore, $y = -3 \Rightarrow \log 0.001 = -3$.

26 $\ln e^{13} = y$ is equivalent to the exponential equation $e^y = e^{13}$. Therefore, $y = 13 \Rightarrow \ln e^{13} = 13$.

27 $\log_8 1 = y$ is equivalent to the exponential equation $8^y = 1$. Since $8^0 = 1$, then $8^y = 8^0$. Therefore, $y = 0 \Rightarrow \log_8 1 = 0$.

28 $10^{\log 6} = y$ is equivalent to the logarithmic equation $\log_{10} y = \log_{10} 6$. Therefore, $y = 6 \Rightarrow 10^{\log 6} = 6$.

29 $\log_3\left(\frac{1}{27}\right) = y$ is equivalent to the exponential equation $3^y = \frac{1}{27}$. Since $3^{-3} = \frac{1}{27}$, then $3^y = 3^{-3}$. Therefore, $y = -3 \Rightarrow \log_3\left(\frac{1}{27}\right) = -3$.

30 $e^{\ln\sqrt{2}} = y$ is equivalent to the logarithmic equation $\ln y = \ln\sqrt{2}$. Therefore, $y = \sqrt{2} \Rightarrow e^{\ln\sqrt{2}} = \sqrt{2}$.

31 $\log 1000 = y$ is equivalent to the exponential equation $10^y = 1000$. Since $10^3 = 1000$, then $10^y = 10^3$. Therefore, $y = 3 \Rightarrow \log 1000 = 3$.

32 $\ln\left(\sqrt{e}\right) = y$ is equivalent to the exponential equation $e^y = \sqrt{e}$. Since $e^{\frac{1}{2}} = \sqrt{e}$, then $e^y = e^{\frac{1}{2}}$. Therefore, $y = \frac{1}{2} \Rightarrow \ln\sqrt{e} = \frac{1}{2}$.

33 $\ln\left(\frac{1}{e^2}\right) = y$ is equivalent to the exponential equation $e^y = \frac{1}{e^2}$. Since $e^{-2} = \frac{1}{e^2}$, then $e^y = e^{-2}$. Therefore, $y = -2 \Rightarrow \ln\left(\frac{1}{e^2}\right) = -2$.

34 $\log_3\left(81^{22}\right) = y$ is equivalent to the exponential equation $3^y = 81^{22}$. Since $81^{22} = \left(3^4\right)^{22} = 2^{88}$, then $3^y = 3^{88}$. Therefore, $y = 88 \Rightarrow \log_3\left(81^{22}\right) = 88$.

35 $\log_4 2 = y$ is equivalent to the exponential equation $4^y = 2$. Since $4^{\frac{1}{2}} = 2$, then $4^y = 4^{\frac{1}{2}}$. Therefore, $y = \frac{1}{2} \Rightarrow \log_4 2 = \frac{1}{2}$.

36 $3^{\log_3 18} = y$ is equivalent to the logarithmic equation $\log_3 y = \log_3 18$. Therefore, $y = 18 \Rightarrow 3^{\log_3 18} = 18$.

37 $\log_5(\sqrt[3]{5}) = y$ is equivalent to the exponential equation $5^y = \sqrt[3]{5}$. Since $5^{\frac{1}{3}} = \sqrt[3]{5}$, then $5^y = 5^{\frac{1}{3}}$. Therefore, $y = \frac{1}{3} \Rightarrow \log_5 \sqrt[3]{5} = \frac{1}{3}$.

38 $10^{\log\pi} = y$ is equivalent to the logarithmic equation $\log y = \log\pi$. Therefore, $y = \pi \Rightarrow 10^{\log\pi} = \pi$.

39 1.699 **40** 0.2386 **41** 3.912 **42** 0.5493

43 1.398 **44** 0.2090 **45** 4.605 **46** 13.82

47 From the definition of a logarithmic function, the domain of $y = \log_b x$ is $x > 0$; thus, for $f(x)$ it follows that $x - 2 > 0 \Rightarrow x > 2$.

48 From the definition of a logarithmic function, the domain of $y = \log_b x$ is $x > 0$; thus, for $g(x)$ it follows that $x^2 > 0 \Rightarrow x \neq 0$. So, $x \in \mathbb{R}$.

49 From the definition of a logarithmic function, the domain of $y = \log_b x$ is $x > 0$; thus, for $h(x)$ it follows that $x > 0$.

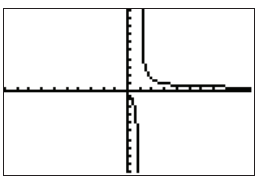
50 From the definition of a logarithmic function, the domain of $y = \log_b x$ is $x > 0$; thus, for $y = \log_7 (8 - 5x)$ it follows that $8 - 5x > 0 \Rightarrow x < \frac{8}{5}$.

51 From the definition of a logarithmic function, the domain of $y = \log_b x$ is $x > 0$; thus, for $\log_3 (9 - 3x)$ it follows that $9 - 3x > 0 \Rightarrow x < 3$. Function $\sqrt{x+2}$ is defined when $x + 2 \geqslant 0 \Rightarrow x \geqslant -2$. Hence, for $y = \sqrt{x+2} - \log_3 (9 - 3x)$, it follows that $-2 \leqslant x < 3$.

52 The function $\sqrt{\ln(1-x)}$ is defined when $\ln(1-x) \geqslant 0 \Rightarrow 1 - x \geqslant 1 \Rightarrow -x \geqslant 0 \Rightarrow x \leqslant 0$.

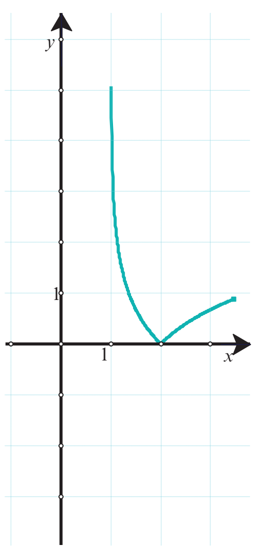
53 The function $y = \frac{1}{\ln x}$ is defined when $\ln x$ is defined $(x > 0)$ and when $\ln x \neq 0$ $(x \neq 1)$. So, the domain is $\{x : x \in \mathbb{R}, x > 0, x \neq 1\}$. Since the range of $y = \ln x$ is $\mathbb{R}$, the range of $y = \frac{1}{\ln x}$ is the same as the range of $y = \frac{1}{x}$; hence, the range is $\{y : y \in \mathbb{R}, y \neq 0\}$.

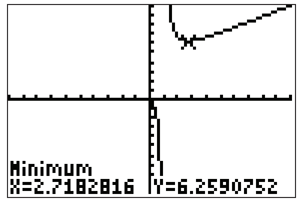
Check:



54 The function $y = |\ln(x-1)|$ is defined when $\ln(x-1)$ is defined $(x - 1 > 0)$. So, the domain is $\{x : x \in \mathbb{R}, x > 1\}$. Since the range of $y = \ln(x-1)$ is $\mathbb{R}$, the range of $y = |\ln(x-1)|$ is the same as the range of $y = |x|$; hence, the range is $\{y : y \in \mathbb{R}, y \geqslant 0\}$.

Check:





55 The function $y = \dfrac{x}{\log x}$ is defined when $\log x$ is defined $(x > 0)$ and when $\log x \neq 0$ $(x \neq 1)$. So, the domain is $\{x : x \in \mathbb{R}, x > 0, x \neq 1\}$. To determine the range, we need to use our GDC. From the graph, we can see that the range is $\{y : y \in \mathbb{R}, y < 0, \text{or } y \geqslant 6.26\}$.

56 We can see that the point $(4, 1)$ is on the graph. So, $\log_b 4 = 1$, which is equivalent to $b^1 = 4$. Therefore, $b = 4$ and the equation of the function that is graphed is $f(x) = \log_4 x$.

57 We can see that the point $\left(\dfrac{1}{2}, -1\right)$ is on the graph. So, $\log_b\left(\dfrac{1}{2}\right) = -1$, which is equivalent to $b^{-1} = \dfrac{1}{2}$. Using $\dfrac{1}{2} = 2^{-1}$, we have $b^{-1} = 2^{-1}$. Therefore, $b = 2$ and the equation of the function that is graphed is $f(x) = \log_2 x$.

58 We can see that the point $(10, 1)$ is on the graph. So, $\log_b 10 = 1$, which is equivalent to $b^1 = 10$. Therefore, $b = 10$ and the equation of the function that is graphed is $f(x) = \log x$.

59 We can see that the point $(9, 2)$ is on the graph. So, $\log_b 9 = 2$, which is equivalent to $b^2 = 9$. Using $9 = 3^2$, we have $b^2 = 3^2$. Therefore, $b = 3$ and the equation of the function that is graphed is $f(x) = \log_3 x$.

60 Using the property for the log of a product, we have: $\log_2(2m) = \log_2 2 + \log_2 m = 1 + \log_2 m$.

(**Note:** We have used the property $\log_b b = 1$.)

61 Using the property for the log of a quotient, we have: $\log\left(\dfrac{9}{x}\right) = \log 9 - \log x$.

62 To use the property for the log of a number raised to an exponent, we firstly have to write the surd as a power: $\sqrt[5]{x} = x^{\frac{1}{5}}$. Now, we have: $\ln \sqrt[5]{x} = \ln x^{\frac{1}{5}} = \dfrac{1}{5}\ln x$.

63 First using the property for the log of a product and then the log of a number raised to an exponent, we have: $\log_3\left(ab^3\right) = \log_3 a + \log_3 b^3 = \log_3 a + 3\log_3 b$.

64 First using the property for the log of a product and then the log of a number raised to an exponent, we have: $\log\left[10x(1+r)^t\right] = \log 10 \log x + \log(1+r)^t$

$$= \log 10 + \log x + t\log(1+r) = 1 + \log x + t\log(1+r).$$

(**Note:** We have used the property $\log_b b = 1$.)

65 First using the property for the log of a quotient and then the log of a number raised to an exponent, we have: $\ln\left(\dfrac{m^3}{n}\right) = \ln m^3 - \ln n = 3\ln m - \ln n$.

66 Using the property for the log of a product, we have: $\log_b pqr = \log_b p + \log_b q + \log_b r$.

67 First using the property for the log of a quotient and then the log of a product, and, finally, the log of a number raised to an exponent, we have:

$$\log_b \frac{p^2q^3}{r} = \log_b\left(p^2q^3\right) - \log_b r = \log_b p^2 + \log_b q^3 - \log_b r = 2\log_b p + 3\log_b q - \log_b r.$$

68 First writing the root as a power, then using the property of a number raised to an exponent, and, finally, the log of a product, we have:

$$\log_b \sqrt[4]{pq} = \log_b (pq)^{\frac{1}{4}} = \frac{1}{4}\log_b(pq) = \frac{1}{4}(\log_b p + \log_b q).$$

Note: We can change the order in which we apply the properties and still obtain the same final result.

$$\log_b \sqrt[4]{pq} = \log_b (pq)^{\frac{1}{4}} = \log_b \ p^{\frac{1}{4}}q^{\frac{1}{4}} = \log_b p^{\frac{1}{4}} + \log_b q^{\frac{1}{4}} = \frac{1}{4}\log_b p + \frac{1}{4}\log_b q$$

69 First writing the root as a power, then using the property of a number raised to an exponent, and, finally, the log of a product and quotient, we have:

$$\log_b \sqrt{\frac{qr}{p}} = \log_b \left(\frac{qr}{p}\right)^{\frac{1}{2}} = \frac{1}{2}\log_b \frac{qr}{p} = \frac{1}{2}(\log_b q + \log_b r - \log_b p).$$

Note: We can change the order in which we apply the properties and still obtain the same final result.

70 Using the properties of a product, quotient and of a number raised to an exponent, we have:

$$\log_b \frac{p\sqrt{q}}{r} = \log_b p + \log_b \sqrt{q} - \log_b r = \log_b p + \frac{1}{2}\log_b q - \log_b r.$$

71 Using the properties of a product, quotient and of a number raised to an exponent, we have:

$$\log_b \frac{(pq)^3}{\sqrt{r}} = \log_b (pq)^3 - \log_b \sqrt{r} = 3\log_b pq - \log_b r^{\frac{1}{2}} = 3(\log_b p + \log_b q) - \frac{1}{2}\log_b r.$$

72 Using the property for the sum of logarithms, we have: $\log(x^2) + \log\left(\frac{1}{x}\right) = \log\left(x^2 \cdot \frac{1}{x}\right) = \log x.$

73 First using the property for the product of a number and logarithm, and then the sum of logarithms, we have: $\log_3 9 + 3\log_3 2 = \log_3 9 + \log_3 (2^3) = \log_3 9 + \log_3 8 = \log_3 (9 \cdot 8) = \log_3 72.$

74 First using the property for the product of a number and logarithm, and then the difference of logarithms, we have: $4\ln y - \ln 4 = \ln(y^4) - \ln 4 = \ln\left(\frac{y^4}{4}\right).$

75 First using the property for the product of a number and logarithm, and then the difference of logarithms, we have: $\log_b 12 - \frac{1}{2}\log_b 9 = \log_b 12 - \log_b (9)^{\frac{1}{2}} = \log_b 12 - \log_b \sqrt{9}$

$$= \log_b 12 - \log_b 3 = \log_b \frac{12}{3} = \log_b 4.$$

76 Using the properties for the sum and difference of logarithms, we have:

$$\log x - \log y - \log z = \log x - (\log y + \log z) = \log x - \log(yz) = \log\left(\frac{x}{yz}\right).$$

Note: We would have the same result if we combined the terms differently. We can use the property for the quotient of logarithms only:

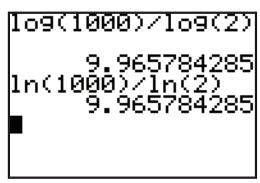
$$\log x - \log y - \log z = (\log x - \log y) - \log z = \log\frac{x}{y} - \log z = \log\left(\frac{\frac{x}{y}}{z}\right) = \log\left(\frac{x}{yz}\right)$$

77 First using the property for the product of a number and logarithm, then using the property $\log_b b = 1$, and, finally, using the property for the quotient of logarithms, we have:

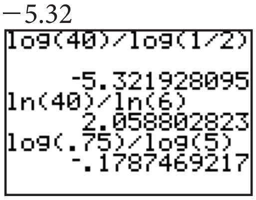
$$2\ln 6 - 1 = \ln(6^2) - 1 = \ln 36 - \ln e = \ln\left(\frac{36}{e}\right).$$

78 $\log_2 1000 = \frac{\log 1000}{\log 2} = \frac{3}{0.301\,029\,99\ldots} \approx 9.97$

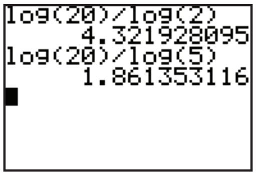
If we use natural logarithms, we get: $\log_2 1000 = \frac{\ln 1000}{\ln 2} = \frac{6.907\,755\ldots}{0.693\,1471\ldots} \approx 9.97.$



79 −5.32 **80** 2.06 **81** −0.179



82 4.32 **83** 1.86



84 Using the change of base formula and the property $\log_a a = 1$, we have: $\log_b a = \dfrac{\log_a a}{\log_a b} = \dfrac{1}{\log_a b}$.

85 Using the change of base formula and the property $\log_b b = 1$, we have: $\log e = \dfrac{\ln e}{\ln 10} = \dfrac{1}{\ln 10}$.

We can establish the property from question 68 using $a = e$ and $b = 10$.

86 Using the property for the quotient of logarithms, then the property for the log of a number raised to an exponent, and, finally, the property $\log_b b = 1$, we have:

$$\text{dB} = 10\log\left(\frac{I}{10^{-16}}\right) = 10\left(\log I - \log 10^{-16}\right) = 10\left(\log I - (-16)\log 10\right)$$

$$= 10\left(\log I + 16\right) = 10\log I + 160$$

For $I = 10^{-4}$ we have: $10\log 10^{-4} + 160 = 10\cdot(-4) + 160 = 120$ dB.

87 **a)** Given $f(x) = 5(2)^x$, if we take the logarithm of both sides, we will have:

$\log_2(f(x)) = \log_2\left(5(2)^x\right)$. Using the property of a product on the right-hand side of the equation, we will have: $\log_2(f(x)) = \log_2 5 + \log_2 2^x = \log_2 5 + x$. Hence, $\log_2(f(x)) = x + \log_2 5$.

b) Given $f(x) = ab^x$, if we take the logarithm of both sides, we will have:

$\log f(x) = \log ab^x = \log a + \log b^x = \log a + x\log b$. So: $\log f(x) = x\log b + \log a$.

Hence, the slope of the line is $m = \log b$ and the y-intercept is $c = \log a$.

Exercise 5.5

1 $10^x = 5$

$\log 10^x = \log 5$ Take the logarithm of both sides.

$x = \log 5 \approx 0.699$ Use the fact that $\log 10^x = x$.

2 $4^x = 32$

$\log_4 4^x = \log_4 32$ Take the logarithm of both sides.

$x = \log_4 32 = \dfrac{\log 32}{\log 4} = 2.5$ Use the fact that $\log_4 4^x = x$ and change of base formula.

Note (1): We can take the logarithm (base 10 or e) of both sides: $\log 4^x = \log 32$

Now use the property $\log_b\left(M^k\right) = k\log_b M$: $x\log 4 = \log 32$

Then dividing the equation by log 4: $x\log 4 = \log 32 \Rightarrow x = \dfrac{\log 32}{\log 4} = 2.5$

Note (2): We can also solve the equation in a third way.

We can write both sides as powers with base 2: $4^x = 32 \Rightarrow \left(2^2\right)^x = 2^5 \Rightarrow 2^{2x} = 2^5 \Rightarrow 2x = 5 \Rightarrow x = 2.5$

3 $8^{x-6} = 60$

$\log 8^{x-6} = \log 60$ Take the logarithm of both sides.

$(x-6)\log 8 = \log 60 \Rightarrow x - 6 = \dfrac{\log 60}{\log 8} \Rightarrow x = \dfrac{\log 60}{\log 8} + 6 \approx 7.97$ Use the property $\log_b(M^k) = k\log_b M$.

4 $2^{x+3} = 100$

$\log 2^{x+3} = \log 100$ Take the logarithm of both sides.

$(x+3)\log 2 = 2 \Rightarrow x + 3 = \dfrac{2}{\log 2} \Rightarrow x = \dfrac{2}{\log 2} - 3 \approx 3.64$ Use the property $\log_b(M^k) = k\log_b M$.

5 $\left(\dfrac{1}{5}\right)^x = 22$

$\log\left(\dfrac{1}{5}\right)^x = \log 22$ Take the logarithm of both sides.

$x\log\left(\dfrac{1}{5}\right) = \log 22 \Rightarrow x = \dfrac{\log 22}{\log\left(\dfrac{1}{5}\right)} \approx -1.92$ Use the property $\log_b(M^k) = k\log_b M$.

6 $e^x = 15$

$\ln e^x = \ln 15$ Take the natural logarithm of both sides.

$x = \ln 15 \approx 2.71$ Use the property $\ln e^x = x$.

7 $10^x = e$

$\log 10^x = \log e$ Take the logarithm of both sides.

$x = \log e \approx 0.434$ Use the fact that $\log 10^x = x$.

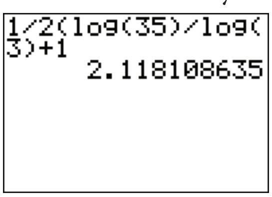
8 $3^{2x-1} = 35$

$\log 3^{2x-1} = \log 35$ Take the logarithm of both sides.

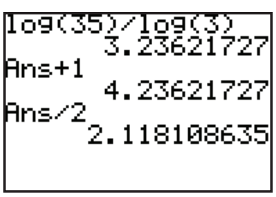
$(2x-1)\log 3 = \log 35 \Rightarrow 2x - 1 = \dfrac{\log 35}{\log 3} \Rightarrow$ Use the property $\log_b(M^k) = k\log_b M$.

$$2x = \frac{\log 35}{\log 3} + 1 \Rightarrow x = \frac{1}{2}\left(\frac{\log 35}{\log 3} + 1\right) \approx 2.12$$

Note: You can use your calculator to work out the final answer:



Or, you can perform step-by-step calculations; firstly for $2x-1$, then for $2x$, and, finally, for x:



9 $2^{x+1} = 3^{x-1}$

$\log\left(2^{x+1}\right) = \log\left(3^{x-1}\right)$ Take the logarithm of both sides.

$(x+1)\log 2 = (x-1)\log 3$ Use the property $\log_b(M^k) = k\log_b M$.

$x\log 2 + \log 2 = x\log 3 - \log 3$ Multiply out both sides.

$x\log 2 - x\log 3 = -\log 3 - \log 2$ Rearrange and collect like terms.

$x(\log 2 - \log 3) = -\log 3 - \log 2$

$x = \dfrac{-\log 3 - \log 2}{\log 2 - \log 3} \approx 4.42$

Note: We can simplify the equation at the beginning: $2^{x+1} = 3^{x-1} \Rightarrow 2\cdot 2^x = \frac{1}{3}\cdot 3^x \Rightarrow \frac{2^x}{3^x} = \frac{1}{6} \Rightarrow \left(\frac{2}{3}\right)^x = \frac{1}{6}$

Now we can take the logarithm of both sides:

$$\log\left(\frac{2}{3}\right)^x = \log\left(\frac{1}{6}\right) \Rightarrow x\log\left(\frac{2}{3}\right) = \log\left(\frac{1}{6}\right) \Rightarrow x = \frac{\log\left(\frac{1}{6}\right)}{\log\left(\frac{2}{3}\right)} \approx 4.42$$

10 $2e^{10x} = 19$

$e^{10x} = 9.5$ First isolate the exponential expression.

$\ln e^{10x} = \ln(9.5)$ Take the natural logarithm of both sides.

$10x\ln e = \ln(9.5) \Rightarrow 10x = \ln 9.5 \Rightarrow$ Use the property $\log_b(M^k) = k\log_b M$ and $\ln e = 1$.

$x = \dfrac{1}{10}\ln 9.5 \approx 0.225$

11 $6^{\frac{x}{2}} = 5^{1-x}$

$\log\left(6^{\frac{x}{2}}\right) = \log\left(5^{1-x}\right)$ Take the logarithm of both sides.

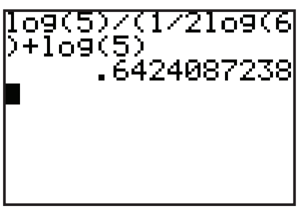
$\frac{x}{2}\log 6 = (1-x)\log 5$ Use the property $\log_b(M^k) = k\log_b M$.

$\frac{x}{2}\log 6 = \log 5 - x\log 5$

$\frac{x}{2}\log 6 + x\log 5 = \log 5$

$x\left(\frac{1}{2}\log 6 + \log 5\right) = \log 5$

$x = \dfrac{\log 5}{\frac{1}{2}\log 6 + \log 5} \approx 0.642$



Note: We can simplify the equation at the beginning:

$$6^{\frac{x}{2}} = 5^{1-x} \Rightarrow \left(6^{\frac{1}{2}}\right)^x = \frac{5}{5^x} \Rightarrow 5^x\left(\sqrt{6}\right)^x = 5 \Rightarrow \left(5\sqrt{6}\right)^x = 5$$

Now we can take the logarithm of both sides:

$$\log\left(5\sqrt{6}\right)^x = \log 5 \Rightarrow x\log\left(5\sqrt{6}\right) = \log 5 \Rightarrow x = \frac{\log 5}{\log\left(5\sqrt{6}\right)} \approx 0.642$$

12 $\left(1+\dfrac{0.05}{12}\right)^{12x}=3$

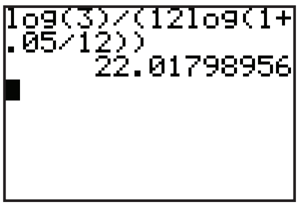
$\log\left(1+\dfrac{0.05}{12}\right)^{12x}=\log 3$ Take the logarithm of both sides.

$12x\log\left(1+\dfrac{0.05}{12}\right)=\log 3$ Use the property $\log_b(M^k)=k\log_b M$.

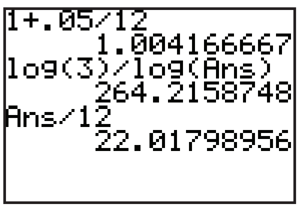
$$12x=\frac{\log 3}{\log\left(1+\dfrac{0.05}{12}\right)}$$

$$x=\frac{\log 3}{12\log\left(1+\dfrac{0.05}{12}\right)}\approx 22.0$$

Note: You can use your calculator to work out the final answer:



Or you can perform step-by-step calculations:



13 We can write 4^x as: $\left(2^2\right)^x=2^{2x}=\left(2^x\right)^2$. So, the equation can be written in the form: $\left(2^x\right)^2-2\cdot 2^x=48$.
Using the substitution $2^x=t$, we will obtain a quadratic equation: $t^2-2\cdot t-48=0\Rightarrow t_1=-6, t_2=8$

Hence, there are two possible solutions:

$2^x=-6$ has no solution (since 2^x has to be positive)

$2^x=8$ has the solution $x=3$ (since $8=2^3$)

14 Using the properties of powers, we can transform the equation:

$$2^{2x+1}-2^{x+1}+1=2^x\Rightarrow 2\cdot 2^{2x}-2\cdot 2^x+1-2^x=0\Rightarrow 2\left(2^x\right)^2-3\cdot 2^x+1=0$$

Using the substitution $2^x=t$: $2t^2-3\cdot t+1=0\Rightarrow t_1=\dfrac{1}{2}, t_2=1$

Hence, the solutions are:

$2^x=\dfrac{1}{2}\Rightarrow x=-1$ (since $\dfrac{1}{2}=2^{-1}$)

$2^x=1\Rightarrow x=0$ (since $1=2^0$)

15 Using the properties of powers, we can transform the equation:

$$6^{2x+1}-17\left(6^x\right)+12=0\Rightarrow 6\left(6^x\right)^2-17\left(6^x\right)+12=0$$

Using the substitution $6^x=t$: $6t^2-17\cdot t+12=0\Rightarrow t_1=\dfrac{4}{3}, t_2=\dfrac{3}{2}$

Hence, the solutions are:

$$6^x = \frac{4}{3} \Rightarrow \log 6^x = \log\left(\frac{4}{3}\right) \Rightarrow x \log 6 = \log\left(\frac{4}{3}\right) \Rightarrow x = \frac{\log\left(\frac{4}{3}\right)}{\log 6}$$

$$6^x = \frac{3}{2} \Rightarrow \log 6^x = \log\left(\frac{3}{2}\right) \Rightarrow x \log 6 = \log\left(\frac{3}{2}\right) \Rightarrow x = \frac{\log\left(\frac{3}{2}\right)}{\log 6}$$

16 Using the properties of powers, we can transform the equation:

$$3^{2x+1} + 3 = 10\left(3^x\right) \Rightarrow 3\left(3^x\right)^2 - 10\left(3^x\right) + 3 = 0$$

Using the substitution $3^x = t$: $3t^2 - 10 \cdot t + 3 = 0 \Rightarrow t_1 = \frac{1}{3}, t_2 = 3$

Hence, the solutions are:

$3^x = \frac{1}{3} \Rightarrow x = -1$ (since $\frac{1}{3} = 3^{-1}$)

$3^x = 3 \Rightarrow x = 1$ (since $3 = 3^1$)

17 We use the exponential function associated with compound interest: $A(t) = P\left(1 + \frac{r}{n}\right)^{nt}$ with values $P = 5000$, $r = 0.075$, and $n = 4$. So, $A(t) = 5000\left(1 + \frac{0.075}{4}\right)^{4t}$.

a) $A(3) = 5000\left(1 + \frac{0.075}{4}\right)^{4 \cdot 3} \approx 6248.58$

There will be \$6248.58 in the account.

b) We have to solve the equation $A(t) = 2 \cdot 5000$.

$$5000\left(1 + \frac{0.075}{4}\right)^{4t} = 2 \cdot 5000$$

$$\left(1 + \frac{0.075}{4}\right)^{4t} = 2$$ First divide both sides by 5000.

$$\log\left(\left(1 + \frac{0.075}{4}\right)^{4t}\right) = \log 2$$ Take the logarithm of both sides.

$$4t \log\left(1 + \frac{0.075}{4}\right) = \log 2$$ Use the property $\log_b(M^k) = k \log_b M$.

$$t = \frac{\log 2}{4 \log\left(1 + \frac{0.075}{4}\right)} = 9.3283\ldots$$

Since 9.3283 is closer to 9.25 than 9.5, we conclude that it will take about nine-and-a-quarter years for the money in the account to double.

18 We use the continuous interest formula $A(t) = Pe^{rt}$ with $P = 500$ and $r = 0.085$. We have to solve the equation $A(t) = 3 \cdot 500$.

$$500e^{0.085t} = 3 \cdot 500$$

$$e^{0.085t} = 3$$ First divide both sides by 500.

$$\ln e^{0.085t} = \ln 3$$ Take the natural logarithm of both sides.

$$0.085t = \ln 3$$ Use the property $\log_b(M^k) = k \log_b M$.

$$t = \frac{\ln 3}{0.085} \approx 12.9 \text{ years}$$

19 The population will follow an exponential model: $P(t) = P_0 b^t$. The initial value is 1, so $P_0 = 1$. The population doubles every hour, so the growth factor is 2 when time is expressed in hours. Therefore, the population function is $P(t) = 2^t$.

We have to find the minimum time needed for $P(t) > 1\,000\,000$. As P is an increasing function, we will solve the equation $P(t) = 1\,000\,000$ and take an appropriate greater value.

$2^t = 1\,000\,000$

$\log\left(2^t\right) = \log(1\,000\,000)$ — Take the logarithm of both sides.

$t \log 2 = 6$ — Use the property $\log_b(M^k) = k \log_b M$.

$t = \dfrac{6}{\log 2} \approx 19.9$

The colony will have more than one million bacteria after 20 hours.

20 We use the exponential function associated with compound interest: $A(t) = P\left(1 + \frac{r}{n}\right)^{nt}$ with $n = 1$. So, $A(t) = P(1+r)^t$. We have to solve the equation $A(t) = 2P$, i.e. $P(1+r)^t = 2 \cdot P \Rightarrow (1+r)^t = 2$.

a) For $r = 0.03$:

$(1.03)^t = 2 \Rightarrow \log(1.03)^t = \log 2 \Rightarrow t \log 1.03 = \log 2 \Rightarrow t = \dfrac{\log 2}{\log 1.03} \approx 23.45$

The money will double in 24 years.

b) For $r = 0.06$:

$(1.06)^t = 2 \Rightarrow \log(1.06)^t = \log 2 \Rightarrow t \log 1.06 = \log 2 \Rightarrow t = \dfrac{\log 2}{\log 1.06} \approx 11.9$

The money will double in 12 years.

c) For $r = 0.09$:

$(1.09)^t = 2 \Rightarrow \log(1.09)^t = \log 2 \Rightarrow t \log 1.09 = \log 2 \Rightarrow t = \dfrac{\log 2}{\log 1.09} \approx 8.04$

The money will double in 9 years.

21 The value of the car will follow an exponential model: $A(t) = P \cdot b^t$ with $b = 1 - 0.11 = 0.89$ and P as the original value. We have to solve the equation $A(t) = \frac{1}{2}P$, i.e. $P \cdot (0.89)^t = \frac{1}{2}P$.

$(0.89)^t = 0.5$

$\log(0.89)^t = \log 0.5$ — Take the logarithm of both sides.

$t \log(0.89) = \log 0.5$ — Use the property $\log_b(M^k) = k \log_b M$.

$t = \dfrac{\log 0.5}{\log(0.89)} \approx 5.95$

The car is worth less than one-half of its original value after 6 years.

22 The exponential decay model for uranium-235 is $A(t) = A_0\left(\frac{1}{2}\right)^{\frac{t}{2.7 \cdot 10^5}}$, where $A_0 = 1$ and time is given in years.

a) $A(1000) = \left(\frac{1}{2}\right)^{\frac{1000}{2.7 \cdot 10^5}} \approx 0.997\,436\ldots$

After 1000 years, 0.997 g of uranium-235 remains (99.7% of the original material).

b) We have to solve the equation $A(t) = 0.7$.

$\left(\frac{1}{2}\right)^{\frac{t}{2.7\cdot 10^5}} = 0.7$

$\log\left(\frac{1}{2}\right)^{\frac{t}{2.7\cdot 10^5}} = \log 0.7$ Take the logarithm of both sides.

$\frac{t}{2.7\cdot 10^5}\log\left(\frac{1}{2}\right) = \log 0.7$ Use the property $\log_b(M^k) = k\log_b M$.

$\frac{t}{2.7\cdot 10^5} = \frac{\log 0.7}{\log\left(\frac{1}{2}\right)}$

$t = 2.7\cdot 10^5\,\frac{\log 0.7}{\log\left(\frac{1}{2}\right)} = 138\,934.755... \approx 139\,000$

It will take approximately 139 000 years for the sample to decompose until its mass is 0.7 g.

23 The population will follow an exponential model: $P(t) = P_0 b^t$. The initial value is 16, so $P_0 = 16$, and $b = 1 + 0.18$.

a) $P(5) = 16\cdot(1.18)^5 \approx 36.6$

The projected population is 37 stray dogs.

b) We have to find the minimum time needed for $P(t) > 70$. As P is an increasing function, we will solve the equation $P(t) = 70$ and take an appropriate greater value.

$16\cdot(1.18)^t = 70$

$(1.18)^t = \frac{70}{16}$ Divide both sides by 16.

$\log(1.18)^t = \log\left(\frac{35}{8}\right)$ Take the logarithm of both sides.

$t\log(1.18) = \log\left(\frac{35}{8}\right)$ Use the property $\log_b(M^k) = k\log_b M$.

$t = \frac{\log\left(\frac{35}{8}\right)}{\log(1.18)} \approx 8.92$

After 9 years the number of stray dogs will be greater than 70.

24 a) $V(10) = 1000\,(0.925)^{10} \approx 458.58... \approx 459$ litres

b) We have to solve the equation $V(t) = \frac{1}{2}\cdot 1000$.

$1000\,(0.925)^t = \frac{1}{2}\cdot 1000$

$(0.925)^t = \frac{1}{2}$ Divide both sides by 1000.

$\log(0.925)^t = \log\left(\frac{1}{2}\right)$ Take the logarithm of both sides.

$t\log(0.925) = \log\left(\frac{1}{2}\right)$ Use the property $\log_b(M^k) = k\log_b M$.

$t = \frac{\log\left(\frac{1}{2}\right)}{\log(0.925)} \approx 8.89$

It takes 8.89 minutes (8 minutes 53 seconds).

c) We have to find the minimum time needed for $V(t) < 0.05 \cdot 1000$. As V is a decreasing function, we will solve the equation $V(t) = 0.05 \cdot 1000$ and take an appropriate greater value.

$1000\,(0.925)^t = 0.05 \cdot 1000$

$(0.925)^t = 0.05$ — Divide both sides by 1000.

$\log\,(0.925)^t = \log 0.05$ — Take the logarithm of both sides.

$t \log\,(0.925) = \log 0.05$ — Use the property $\log_b(M^k) = k \log_b M$.

$t = \dfrac{\log 0.05}{\log\,(0.925)} \approx 38.4$

39 minutes have passed before the tank can first be considered empty.

25 **a)** For $t = 0 : m = 5e^0 = 5$ kg

b) We have to solve the equation $5e^{-0.13t} = 0.5$.

$5e^{-0.13t} = 0.5$

$e^{-0.13t} = 0.1$ — Divide both sides by 5.

$\ln\left(e^{-0.13t}\right) = \ln 0.1$ — Take the natural logarithm of both sides.

$-0.13t = \ln 0.1$ — Use the property $\log_b(M^k) = k \log_b M$.

$t = \dfrac{\ln 0.1}{-0.13} \approx 17.7$

It takes approximately 17.7 days for the substance to decay to 0.5 kg.

26 $\log_2(3x - 4) = 4$

$3x - 4 = 2^4$ — Rewrite in exponential form.

$3x - 4 = 16$

$x = \dfrac{20}{3}$

27 $\log(x - 4) = 2$

$10^{\log(x-4)} = 10^2$ — Exponentiate both sides by base = 10.

$x - 4 = 100$ — Use the property $b^{\log_b x} = x$.

$x = 104$

28 $\ln x = -3$

$e^{\ln x} = e^{-3}$ — Exponentiate both sides by base = e.

$x = e^{-3} = \dfrac{1}{e^3}$ — Use the property $b^{\log_b x} = x$.

29 $\log_{16} x = \dfrac{1}{2}$

$16^{\log_{16} x} = 16^{\frac{1}{2}}$ — Exponentiate both sides by base = 16.

$x = 16^{\frac{1}{2}} = \sqrt{16} = 4$ — Use the property $b^{\log_b x} = x$.

30 $\log \sqrt{x + 2} = 1$

$10^{\log \sqrt{x+2}} = 10^1$ — Exponentiate both sides by base = 10.

$\sqrt{x + 2} = 10$ — Use the property $b^{\log_b x} = x$.

$x + 2 = 100$

$x = 98$

31 $\ln\left(x^2\right) = 16$

$e^{\ln\left(x^2\right)} = e^{16}$ Exponentiate both sides by base = e.

$x^2 = e^{16}$ Use the property $b^{\log_b x} = x$.

$x = \pm\sqrt{e^{16}} = \pm e^8$

32 $\log_2\left(x^2 + 8\right) = \log_2 x + \log_2 6$

$\log_2\left(x^2 + 8\right) = \log_2 (x \cdot 6)$ Use property for the sum of logarithms.

$x^2 + 8 = 6x$ Exponentiate both sides by base = 2.

$x^2 - 6x + 8 = 0 \Rightarrow x_{1,2} = 4, 2$

$x = 2$, or $x = 4$

Both of the solutions produce true statements when substituted into the original equation; therefore, $x = 2$ or $x = 4$.

33 $\log_3 (x - 8) + \log_3 x = 2$

$\log_3 ((x - 8)\, x) = 2$ Use property for the sum of logarithms.

$3^{\log_3 ((x-8)x)} = 3^2$ Exponentiate both sides by base = 3.

$(x - 8)x = 9$ Use the property $b^{\log_b x} = x$.

$x^2 - 8x - 9 = 0 \Rightarrow x_{1,2} = -1, 9$

$x = -1$ is not a solution, because, when substituted into the original equation, both $\log_3 (x - 8)$ and $\log_3 x$ are undefined.

$x = 9$ produces a true statement when substituted into the original equation. Therefore, the solution is $x = 9$.

34 $\log 7 - \log (4x + 5) + \log (2x - 3) = 0$

$\log 7 = \log (4x + 5) - \log (2x - 3)$

$\log 7 = \log\left(\dfrac{4x + 5}{2x - 3}\right)$ Use property for the difference of logarithms.

$7 = \dfrac{4x + 5}{2x - 3} \Rightarrow 7(2x - 3) = 4x + 5 \Rightarrow 10x = 26 \Rightarrow x = \dfrac{13}{5}$ Exponentiate both sides by base = 10.

$x = \dfrac{13}{5}$ produces a true statement when substituted into the original equation. Therefore, the solution is $x = \dfrac{13}{5}$.

35 $\log_3 x + \log_3 (x - 2) = 1$

$\log_3 [x (x - 2)] = 1$

$3^{\log_3 (x(x-2))} = 3^1$ Exponentiate both sides by base = 3.

$x(x - 2) = 3 \Rightarrow x^2 - 2x - 3 = 0 \Rightarrow x_{1,2} = -1, 3$ Use the property $b^{\log_b x} = x$.

$x = -1$ is not a solution, because, when substituted into the original equation, both $\log_3 (x - 2)$ and $\log_3 x$ are undefined.

$x = 3$ produces a true statement when substituted into the original equation. Therefore, the solution is $x = 3$.

36 $\log x^8 = (\log x)^4$

Use the property $\log_b\left(M^k\right) = k\log_b M$:

$8\log x = (\log x)^4$

Let $\log x = t$, and solve for t:

$8t = t^4 \Rightarrow t(t^3 - 8) = 0 \Rightarrow t = 0 \text{ or } t^3 = 8 \Rightarrow t = 0, t = 2$

Now, solving for x:

$\log x = 0$ or $\log x = 2$

$\log x = 0 \Rightarrow x = 1$

$\log x = 2 \Rightarrow x = 10^2 = 100$

Both $x = 1$ and $x = 100$ are solutions.

37 $5\log x > -2 \Rightarrow \log x > -\dfrac{2}{5} \Rightarrow x > 10^{-\frac{2}{5}} = 100^{-\frac{1}{5}} = \dfrac{1}{\sqrt[5]{100}}$

38 Transform using the properties of logarithms:

$2\log x^2 - 3\log x < \log 8x - \log 4x \Rightarrow 2\cdot 2\log x - 3\log x < \log\dfrac{8x}{4x} \Rightarrow \log x < \log 2 \Rightarrow x < 2$

Since logarithm is defined only for positive numbers, the solution is $0 < x < 2$.

39 Multiply out the left-hand side: $\left(e^x\right)^2 - 3\cdot e^x - 2\cdot e^x + 6 < 2\cdot e^x \Rightarrow \left(e^x\right)^2 - 7\cdot e^x + 6 < 0$

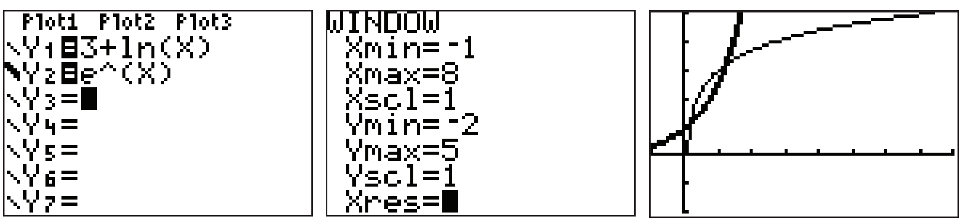
Using the substitution $e^x = t$: $t^2 - 7\cdot t + 6 < 0$

The solutions of the quadratic equation are $t = 1$ and $t = 6$; hence, the solution set of the quadratic inequality is: $1 < t < 6$. Therefore, for x, it holds that: $1 < e^x < 6 \Rightarrow \ln 1 < \ln e^x < \ln 6 \Rightarrow 0 < x < \ln 6$.

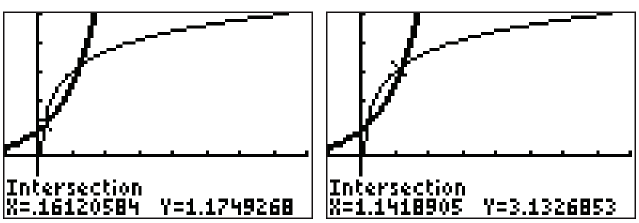
Note: We can solve the quadratic inequality by factorizing:

$t^2 - 7\cdot t + 6 < 0 \Rightarrow (t-1)(t-6) < 0 \Rightarrow 1 < t < 6$, and then proceeding as above.

40 The inequality $3 + \ln x > e^x$ cannot be solved exactly. We can use a GDC:



We can see that the graph of e^x is below the graph of $3 + \ln x$ on the interval between the intersections.



Hence, the solution set is: $0.161 < x < 1.14$ (to 3 s.f.).

Chapter 6

Exercise 6.1 and 6.2

1 For matrices $A = \begin{pmatrix} -2 & x \\ y-1 & 3 \end{pmatrix}$ and $B = \begin{pmatrix} x+1 & -3 \\ 4 & y-2 \end{pmatrix}$, we have:

a) **i)** $A + B = \begin{pmatrix} x-1 & x-3 \\ y+3 & y+1 \end{pmatrix}$

ii) $3A - B = \begin{pmatrix} -6 & 3x \\ 3y-3 & 9 \end{pmatrix} - \begin{pmatrix} x+1 & -3 \\ 4 & y-2 \end{pmatrix} = \begin{pmatrix} -x-7 & 3x+3 \\ 3y-7 & 11-y \end{pmatrix}$

b) If $A = B$, their corresponding elements are equal; this leads to a system of equations:

$x + 1 = -2$

$x = -3$

$y - 1 = 4$

$y - 2 = 3$

From each of the first two equations we get $x = -3$, and from the last two $y = 5$.

c) If $A + B$ is a diagonal matrix, the elements on the opposite diagonal are zeros; hence,

$x - 3 = 0$

$y + 3 = 0$

Therefore, $x = 3$, $y = -3$.

d) $AB = \begin{pmatrix} -2(x+1)+4x & -2(-3)+x(y-2) \\ (y-1)(x+1)+3\cdot 4 & (y-1)(-3)+3(y-2) \end{pmatrix} = \begin{pmatrix} 2x-2 & xy-2x+6 \\ xy-x+y+11 & -3 \end{pmatrix}$

$BA = \begin{pmatrix} (x+1)(-2)-3(y-1) & (x+1)x-3\cdot 3 \\ 4(-2)+(y-2)(y-1) & 4x+3(y-2) \end{pmatrix} = \begin{pmatrix} -2x-3y+1 & x^2+x-9 \\ y^2-3y-6 & 4x+3y-6 \end{pmatrix}$

2 **a)** $\begin{pmatrix} 3 & 0 \\ 4 & 2 \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 6 \\ -12 \end{pmatrix} \Rightarrow \begin{pmatrix} 3x \\ 4x+2y \end{pmatrix} = \begin{pmatrix} 6 \\ -12 \end{pmatrix}$

By comparing the corresponding elements, we obtain the following equations:

$$\left.\begin{array}{l} 3x = 6 \\ 4x + 2y = -12 \end{array}\right\} \Rightarrow \begin{array}{r} x = 2 \\ 8 + 2y = -12 \end{array}$$

Therefore, $x = 2$, $y = -10$.

b) $\begin{pmatrix} 2 & p \\ 3 & q \end{pmatrix}\begin{pmatrix} 4 \\ 5 \end{pmatrix} = \begin{pmatrix} 18 \\ -8 \end{pmatrix} \Rightarrow \begin{pmatrix} 8+5p \\ 12+5q \end{pmatrix} = \begin{pmatrix} 18 \\ -8 \end{pmatrix}$

By comparing the corresponding elements, we obtain the following equations:

$8 + 5p = 18$

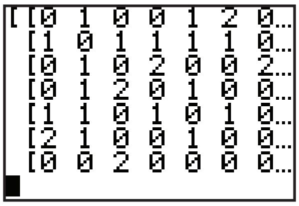
$12 + 5q = -8$

Therefore, $p = 2$, $q = -4$.

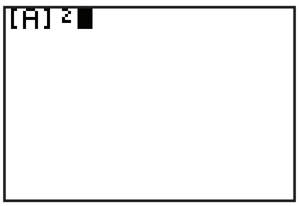
3 a)

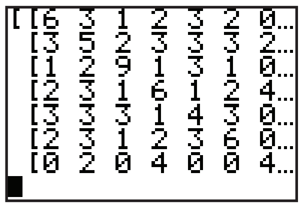
$$\begin{array}{c} \\ V \\ M \\ F \\ S \\ Z \\ L \\ P \end{array}\begin{array}{c} \begin{array}{ccccccc} V & M & F & S & Z & L & P \end{array} \\ \begin{bmatrix} 0 & 1 & 0 & 0 & 1 & 2 & 0 \\ 1 & 0 & 1 & 1 & 1 & 1 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 & 2 \\ 0 & 1 & 2 & 1 & 1 & 0 & 0 \\ 1 & 1 & 0 & 0 & 0 & 1 & 0 \\ 2 & 1 & 0 & 1 & 1 & 0 & 0 \\ 0 & 0 & 2 & 0 & 0 & 0 & 0 \end{bmatrix} \end{array}$$

b) Matrix A showing the number of direct routes between each pair of cities is input into a GDC.



We square matrix A and obtain the answer shown in the GDC screenshot below right.





Matrix A^2 represents the number of routes between each pair of cities that go via another city. For example, the entry $a_{1,2} = 3$ means that there are three different routes from Vienna to Munich: two of them through Milano and one through Zurich. Also, $a_{7,7} = 4$ means that there are four different routes from Paris to another city (all to Frankfurt) and back to Paris.

4 a) $$A + C = \begin{pmatrix} 2 & 5 & 1 \\ 0 & -3 & 2 \\ 7 & 0 & -1 \end{pmatrix} + \begin{pmatrix} x-1 & 5 & y \\ 0 & -x & y+1 \\ 2x+y & x-3y & 2y-x \end{pmatrix} = \begin{pmatrix} x+1 & 10 & y+1 \\ 0 & -x-3 & y+3 \\ 2x+y+7 & x-3y & -x+2y-1 \end{pmatrix}$$

b) $$AB = \begin{pmatrix} 2 & 5 & 1 \\ 0 & -3 & 2 \\ 7 & 0 & -1 \end{pmatrix} \cdot \begin{pmatrix} m & -2 \\ 3m & -1 \\ 2 & 3 \end{pmatrix} = \begin{pmatrix} 17m+2 & -6 \\ -9m+4 & 9 \\ 7m-2 & -17 \end{pmatrix}$$

c) BA cannot be found because B is a 3×2 matrix and A is a 3×3 matrix; as such, the number of columns of matrix B does not match the number of rows of matrix A, so the product is not defined.

d) If $A = C$, their corresponding elements are equal; this leads to a system of equations:

$x - 1 = 2$

$y = 1$

$x = 3$

$y + 1 = 2$

$2x + y = 7$

$x - 3y = 0$

$2y - x = -1$

These equations all give the same solution: $x = 3$, $y = 1$.

This system of equations can be input into a GDC as an augmented matrix and then, by selecting reduced row echelon form, it can be solved. Alternatively, it can be input directly into the PolySmlt application as a system of seven equations with two variables and then solved. Both processes are very long and time consuming, and so, on the final exam, students would not be encouraged to use a GDC in such a question.

e) $B + C$ cannot be found because B is a 3×2 matrix and C is a 3×3 matrix; the sum of two matrices of different order is not defined.

f) $3B + 2\begin{pmatrix} -1 & m^2 \\ -5 & 2 \\ 1 & -1 \end{pmatrix} = \begin{pmatrix} 7 & 12 \\ 17 & 1 \\ 2m+2 & 7 \end{pmatrix}$

By comparing the corresponding elements from $3\begin{pmatrix} m & -2 \\ 3m & -1 \\ 2 & 3 \end{pmatrix} + 2\begin{pmatrix} -1 & m^2 \\ -5 & 2 \\ 1 & -1 \end{pmatrix} = \begin{pmatrix} 7 & 12 \\ 17 & 1 \\ 2m+2 & 7 \end{pmatrix}$, we obtain the following system of equations:

$3m - 2 = 7$

$-6 + 2m^2 = 12$

$9m - 10 = 17$

$8 = 2m + 2$

All of the equations give us the unique solution: $m = 3$.

5 By comparing the corresponding elements from $2\begin{pmatrix} a-1 & b \\ c+2 & 3 \end{pmatrix} + \begin{pmatrix} 3 & -1 \\ 0 & 5 \end{pmatrix} = \begin{pmatrix} -5 & 5 \\ 8 & c+9 \end{pmatrix}$, we obtain the following system of equations:

$2(a-1) + 3 = -5$

$2b - 1 = 5$

$2(c+2) = 8$

$2 \cdot 3 + 5 = c + 9$

Therefore, $a = -3$, $b = 3$, $c = 2$.

6 $\begin{pmatrix} 2 & -3 \\ -5 & 7 \end{pmatrix}\begin{pmatrix} x-11 & 1-x \\ -5 & x+2y \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$

After performing matrix multiplication, we get:

$\begin{pmatrix} 2x-7 & -5x-6y+2 \\ -5x+20 & 12x+14y-5 \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$

By comparing the corresponding elements, we obtain the following system of equations:

$2x - 7 = 1$

$-5x - 6y + 2 = 0$

$-5x + 20 = 0$

$12x + 14y - 5 = 1$

The first and third equations give us $x = 4$.

Substituting $x = 4$ into the second and fourth equation gives us the same solution: $y = -3$.

7 By comparing the corresponding elements, we obtain the following system of equations:

$m^2 - 1 = 3$

$m + 2 = n + 1$

$n - 5 = -2$

Therefore, $m = 2, n = 3$.

8 Let matrix $L = \begin{pmatrix} 2 & 5 & 3 \end{pmatrix}$ represent the quantities from your shopping list, and $P = \begin{pmatrix} 1.66 & 1.58 \\ 2.55 & 2.6 \\ 0.90 & 0.95 \end{pmatrix}$ represent the prices in shops A and B.

Then $LP = \begin{pmatrix} 2 & 5 & 3 \end{pmatrix}\begin{pmatrix} 1.66 & 1.58 \\ 2.55 & 2.6 \\ 0.90 & 0.95 \end{pmatrix} = \begin{pmatrix} 18.77 & 19.01 \end{pmatrix}$ gives us the total cost of the shopping in shop A (€18.77) and B (€19.01) respectively.

Therefore, you should go to shop A because the total cost is cheaper.

9 **a)** $$A+(B+C) = \begin{pmatrix} 2 & 0 \\ -5 & 1 \end{pmatrix} + \left(\begin{pmatrix} 3 & -1 \\ 1 & 4 \end{pmatrix} + \begin{pmatrix} -3 & 5 \\ 2 & 7 \end{pmatrix}\right) = \begin{pmatrix} 2 & 0 \\ -5 & 1 \end{pmatrix} + \begin{pmatrix} 0 & 4 \\ 3 & 11 \end{pmatrix} = \begin{pmatrix} 2 & 4 \\ -2 & 12 \end{pmatrix}$$

$$(A+B)+C = \left(\begin{pmatrix} 2 & 0 \\ -5 & 1 \end{pmatrix} + \begin{pmatrix} 3 & -1 \\ 1 & 4 \end{pmatrix}\right) + \begin{pmatrix} -3 & 5 \\ 2 & 7 \end{pmatrix} = \begin{pmatrix} 5 & -1 \\ -4 & 5 \end{pmatrix} + \begin{pmatrix} -3 & 5 \\ 2 & 7 \end{pmatrix} = \begin{pmatrix} 2 & 4 \\ -2 & 12 \end{pmatrix}$$

b) We conclude that the addition of 2×2 matrices is associative, which can be proved as follows:

$$A+(B+C) = \begin{pmatrix} a & b \\ c & d \end{pmatrix} + \left(\begin{pmatrix} m & n \\ p & q \end{pmatrix} + \begin{pmatrix} r & s \\ t & u \end{pmatrix}\right) = \begin{pmatrix} a & b \\ c & d \end{pmatrix} + \begin{pmatrix} m+r & n+s \\ p+t & q+u \end{pmatrix} = \begin{pmatrix} a+m+r & b+n+s \\ c+p+t & d+q+u \end{pmatrix}$$

$$(A+B)+C = \left(\begin{pmatrix} a & b \\ c & d \end{pmatrix} + \begin{pmatrix} m & n \\ p & q \end{pmatrix}\right) + \begin{pmatrix} r & s \\ t & u \end{pmatrix} = \begin{pmatrix} a+m & b+n \\ c+p & d+q \end{pmatrix} + \begin{pmatrix} r & s \\ t & u \end{pmatrix} = \begin{pmatrix} a+m+r & b+n+s \\ c+p+t & d+q+u \end{pmatrix}$$

So, $A+(B+C) = (A+B)+C$ for all real numbers $a, b, c, d, m, n, p, q, r, s, t, u$.

c) $$A(BC) = \begin{pmatrix} 2 & 0 \\ -5 & 1 \end{pmatrix}\left(\begin{pmatrix} 3 & -1 \\ 1 & 4 \end{pmatrix}\begin{pmatrix} -3 & 5 \\ 2 & 7 \end{pmatrix}\right) = \begin{pmatrix} 2 & 0 \\ -5 & 1 \end{pmatrix}\begin{pmatrix} -11 & 8 \\ 5 & 33 \end{pmatrix} = \begin{pmatrix} -22 & 16 \\ 60 & -7 \end{pmatrix}$$

$$(AB)C = \left(\begin{pmatrix} 2 & 0 \\ -5 & 1 \end{pmatrix}\begin{pmatrix} 3 & -1 \\ 1 & 4 \end{pmatrix}\right)\begin{pmatrix} -3 & 5 \\ 2 & 7 \end{pmatrix} = \begin{pmatrix} 6 & -2 \\ -14 & 9 \end{pmatrix}\begin{pmatrix} -3 & 5 \\ 2 & 7 \end{pmatrix} = \begin{pmatrix} -22 & 16 \\ 60 & -7 \end{pmatrix}$$

d) We conclude that the multiplication of 2×2 matrices is associative, which can be proved as follows:

$$A(BC) = \begin{pmatrix} a & b \\ c & d \end{pmatrix}\left(\begin{pmatrix} m & n \\ p & q \end{pmatrix}\begin{pmatrix} r & s \\ t & u \end{pmatrix}\right) = \begin{pmatrix} a & b \\ c & d \end{pmatrix}\begin{pmatrix} mr+nt & ms+nu \\ pr+qt & ps+qu \end{pmatrix}$$

$$= \begin{pmatrix} amr+ant+bpr+bqt & ams+anu+bps+bqu \\ cmr+cnt+dpr+dqt & cms+cnu+dps+dqu \end{pmatrix}$$

$$(AB)C = \left(\begin{pmatrix} a & b \\ c & d \end{pmatrix}\begin{pmatrix} m & n \\ p & q \end{pmatrix}\right)\begin{pmatrix} r & s \\ t & u \end{pmatrix} = \begin{pmatrix} am+bp & an+bq \\ cm+dp & cn+dq \end{pmatrix}\begin{pmatrix} r & s \\ t & u \end{pmatrix}$$

$$= \begin{pmatrix} amr+ant+bpr+bqt & ams+anu+bps+bqu \\ cmr+cnt+dpr+dqt & cms+cnu+dps+dqu \end{pmatrix}$$

So, $A(BC) = (AB)C$ for all real numbers $a, b, c, d, m, n, p, q, r, s, t, u$.

10 $AB = \begin{pmatrix} 235 & 562 & 117 \end{pmatrix}\begin{pmatrix} 120 \\ 95 \\ 56 \end{pmatrix} = (235 \cdot 120 + 562 \cdot 95 + 117 \cdot 56) = (88\,142)$

AB represents the total profit (€88 142).

11 $rA + B = A \Rightarrow r\begin{pmatrix} 2 & 3 \\ 5 & 7 \end{pmatrix} + \begin{pmatrix} -4 & -6 \\ s-8 & -37 \end{pmatrix} = \begin{pmatrix} 2 & 3 \\ 5 & 7 \end{pmatrix} \Rightarrow \begin{pmatrix} 2r-4 & 3r-6 \\ 5r+s-8 & 7r-37 \end{pmatrix} = \begin{pmatrix} 2 & 3 \\ 5 & 7 \end{pmatrix}$

By comparing the corresponding elements, we obtain the following system of equations:

$2r - 4 = 2$

$3r - 6 = 3$

$5r + s - 8 = 5$

$7r - 37 = 7$

From the first two equations we get $r = 3$, but the fourth give us $r = \frac{44}{7}$; therefore, there is no solution.

12 For $A = \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}$:

a) i) $A^2 = A \cdot A = \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}\begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix} = \begin{pmatrix} 1 & 2 \\ 0 & 1 \end{pmatrix}$

ii) $A^3 = A^2 \cdot A = \begin{pmatrix} 1 & 2 \\ 0 & 1 \end{pmatrix}\begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix} = \begin{pmatrix} 1 & 3 \\ 0 & 1 \end{pmatrix}$

iii) $A^4 = A^3 \cdot A = \begin{pmatrix} 1 & 3 \\ 0 & 1 \end{pmatrix}\begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix} = \begin{pmatrix} 1 & 4 \\ 0 & 1 \end{pmatrix}$

iv) By observing the pattern of entries in the resulting matrices, we would conclude that

$A^n = \begin{pmatrix} 1 & n \\ 0 & 1 \end{pmatrix}$.

For $B = \begin{pmatrix} 3 & 3 \\ 0 & 3 \end{pmatrix}$:

b) i) $B^2 = B \cdot B = \begin{pmatrix} 3 & 3 \\ 0 & 3 \end{pmatrix}\begin{pmatrix} 3 & 3 \\ 0 & 3 \end{pmatrix} = \begin{pmatrix} 9 & 18 \\ 0 & 9 \end{pmatrix} = \begin{pmatrix} 9 & 2 \cdot 9 \\ 0 & 9 \end{pmatrix}$

ii) $B^3 = B^2 \cdot B = \begin{pmatrix} 9 & 18 \\ 0 & 9 \end{pmatrix}\begin{pmatrix} 3 & 3 \\ 0 & 3 \end{pmatrix} = \begin{pmatrix} 27 & 81 \\ 0 & 27 \end{pmatrix} = \begin{pmatrix} 27 & 3 \cdot 27 \\ 0 & 27 \end{pmatrix}$

iii) $B^4 = B^3 \cdot B = \begin{pmatrix} 27 & 81 \\ 0 & 27 \end{pmatrix}\begin{pmatrix} 3 & 3 \\ 0 & 3 \end{pmatrix} = \begin{pmatrix} 81 & 324 \\ 0 & 81 \end{pmatrix} = \begin{pmatrix} 81 & 4 \cdot 81 \\ 0 & 81 \end{pmatrix}$

iv) By rewriting the result matrices and observing the pattern of entries, we can conclude that

$B^n = \begin{pmatrix} 3^n & n \cdot 3^n \\ 0 & 3^n \end{pmatrix}$.

13 For matrices A and B we have:

$AB = \begin{pmatrix} 2 & 3 \\ 4 & 1 \end{pmatrix}\begin{pmatrix} x & 2 \\ y & 3 \end{pmatrix} = \begin{pmatrix} 2x+3y & 13 \\ 4x+y & 11 \end{pmatrix}$

$BA = \begin{pmatrix} x & 2 \\ y & 3 \end{pmatrix}\begin{pmatrix} 2 & 3 \\ 4 & 1 \end{pmatrix} = \begin{pmatrix} 2x+8 & 3x+2 \\ 2y+12 & 3y+3 \end{pmatrix}$

From $AB = BA$ we obtain the following system of equations:

$2x + 3y = 2x + 8$

$13 = 3x + 2$

$4x + y = 2y + 12$

$11 = 3y + 3$

Solutions: $x = \frac{11}{3}, y = \frac{8}{3}$.

14 For matrices A and B we have:

$$AB = \begin{pmatrix} 3 & x \\ -2 & 1 \end{pmatrix}\begin{pmatrix} 5 & 2 \\ y & 1 \end{pmatrix} = \begin{pmatrix} 15 + xy & 6 + x \\ -10 + y & -3 \end{pmatrix}$$

$$BA = \begin{pmatrix} 5 & 2 \\ y & 1 \end{pmatrix}\begin{pmatrix} 3 & x \\ -2 & 1 \end{pmatrix} = \begin{pmatrix} 11 & 5x + 2 \\ 3y - 2 & xy + 1 \end{pmatrix}$$

From $AB = BA$ we get the following system of equations:

$15 + xy = 11$

$6 + x = 5x + 2$

$-10 + y = 3y - 2$

$-3 = xy + 1$

From the second equation we get $x = 1$, and, from the third, $y = -4$. This solution satisfies the remaining two equations as well, so we have the unique solution: $x = 1, y = -4$.

15 For matrices A and B we have:

$$AB = \begin{pmatrix} 1 & 2 & 3 \\ x & 2 & -3 \\ 1 & 0 & 4 \end{pmatrix}\begin{pmatrix} -8 & x+3 & 12 \\ 23 & x-6 & -18 \\ 2 & -2 & 8 \end{pmatrix} = \begin{pmatrix} 44 & 3x - 15 & 0 \\ -8x + 40 & x^2 + 5x - 6 & 12x - 60 \\ 0 & x - 5 & 44 \end{pmatrix}$$

$$BA = \begin{pmatrix} -8 & x+3 & 12 \\ 23 & x-6 & -18 \\ 2 & -2 & 8 \end{pmatrix}\begin{pmatrix} 1 & 2 & 3 \\ x & 2 & -3 \\ 1 & 0 & 4 \end{pmatrix} = \begin{pmatrix} x^2 + 3x + 4 & 2x - 10 & -3x + 15 \\ x^2 - 6x + 5 & 2x + 34 & -3x + 15 \\ -2x + 10 & 0 & 44 \end{pmatrix}$$

From $AB = BA$, and by comparing the corresponding elements, we get a system of eight equations. The only solution that satisfies all of the equations is $x = 5$. Some of the equations have two solutions, but our final solution must satisfy all the equations and therefore we cannot accept them.

16 For matrices A and B we have:

$$AB = \begin{pmatrix} y & 2 & y+2 \\ x & 2 & -3 \\ 1 & y-1 & 4 \end{pmatrix}\begin{pmatrix} -8 & x+3 & 12 \\ 23 & x-6 & -18 \\ 2 & -2 & 8 \end{pmatrix} = \begin{pmatrix} -6y + 50 & xy + y + 2x - 16 & 20y - 20 \\ -8x + 40 & x^2 + 5x - 6 & 12x - 60 \\ 23y - 23 & xy - 6y + 1 & -18y + 62 \end{pmatrix}$$

$$BA = \begin{pmatrix} -8 & x+3 & 12 \\ 23 & x-6 & -18 \\ 2 & -2 & 8 \end{pmatrix}\begin{pmatrix} y & 2 & y+2 \\ x & 2 & -3 \\ 1 & y-1 & 4 \end{pmatrix} = \begin{pmatrix} -8y + x^2 + 3x + 12 & 2x + 12y - 22 & -8y - 3x + 23 \\ 23y + x^2 - 6x - 18 & 2x - 18y + 52 & 23y - 3x - 8 \\ 2y - 2x + 8 & 8y - 8 & 2y + 42 \end{pmatrix}$$

From $AB = BA$, and by comparing the corresponding elements, we get a system of nine equations. The last equation, $-18y + 62 = 2y + 42$, gives us $y = 1$. Then, for example, by substituting $y = 1$ into the penultimate equation ($xy - 6y + 1 = 8y - 8$) we get $x = 5$. By checking all of the remaining equations, we verify that (5, 1) is the unique solution of this system.

Exercise 6.3

Solution Paper 1 type

1 **a** To find the inverse of matrix $A = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$, we use the formula $A^{-1} = \frac{1}{\det(A)}\begin{pmatrix} d & -b \\ -c & a \end{pmatrix}$, where $\det(A) = \begin{vmatrix} a & b \\ c & d \end{vmatrix} = ad - bc$.

$$\begin{vmatrix} 3 & 7 \\ -4 & -9 \end{vmatrix} = 3(-9) - 7(-4) = 1$$

$$\begin{pmatrix} 3 & 7 \\ -4 & -9 \end{pmatrix}^{-1} = \frac{1}{1}\begin{pmatrix} -9 & -7 \\ 4 & 3 \end{pmatrix} = \begin{pmatrix} -9 & -7 \\ 4 & 3 \end{pmatrix}$$

b As the product of two matrices, $M = \begin{pmatrix} 3 & 7 \\ -4 & -9 \end{pmatrix}^{-1}\begin{pmatrix} 2 & 1 \\ 3 & 5 \end{pmatrix} = \begin{pmatrix} -9 & -7 \\ 4 & 3 \end{pmatrix}\begin{pmatrix} 2 & 1 \\ 3 & 5 \end{pmatrix}$.

c $M = \begin{pmatrix} (-9)\cdot 2 + (-7)\cdot 3 & (-9)\cdot 1 + (-7)\cdot 5 \\ 4\cdot 2 + 3\cdot 3 & 4\cdot 1 + 3\cdot 5 \end{pmatrix} = \begin{pmatrix} -39 & -44 \\ 17 & 19 \end{pmatrix}$

d **i** As the product of two matrices, $N = \begin{pmatrix} 2 & 1 \\ 3 & 5 \end{pmatrix}\begin{pmatrix} 3 & 7 \\ -4 & -9 \end{pmatrix}^{-1} = \begin{pmatrix} 2 & 1 \\ 3 & 5 \end{pmatrix}\begin{pmatrix} -9 & -7 \\ 4 & 3 \end{pmatrix}$.

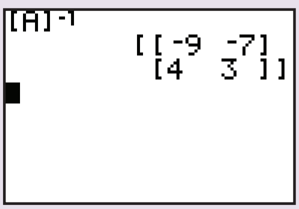
ii $N = \begin{pmatrix} -14 & -11 \\ -7 & -6 \end{pmatrix}$

e In parts **b**–**c** we multiplied $\begin{pmatrix} 2 & 1 \\ 3 & 5 \end{pmatrix}$ by the inverse from the left to determine matrix M, whilst in **d** we multiplied it from the right to obtain matrix N. The results are two different matrices as multiplication of matrices is not commutative.

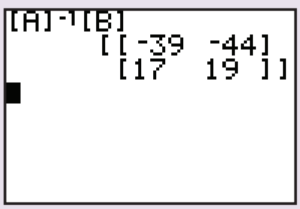
Solution Paper 2 type

1 We input the matrices $A = \begin{pmatrix} 3 & 7 \\ -4 & -9 \end{pmatrix}$ and $B = \begin{pmatrix} 2 & 1 \\ 3 & 5 \end{pmatrix}$ into a GDC.

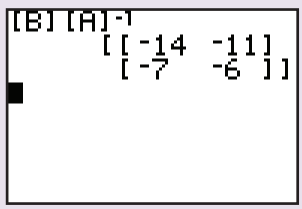
For A^{-1}:



For M:



For N:



2 $E = \begin{pmatrix} 1 & 0 \\ 3 & 1 \end{pmatrix}^{-1}\begin{pmatrix} 1 & 3 \\ 3 & 4 \end{pmatrix}\begin{pmatrix} 1 & 0 \\ 0 & -5 \end{pmatrix}^{-1} = \frac{1}{1}\begin{pmatrix} 1 & 0 \\ -3 & 1 \end{pmatrix}\begin{pmatrix} 1 & 3 \\ 3 & 4 \end{pmatrix}\left(-\frac{1}{5}\right)\begin{pmatrix} -5 & 0 \\ 0 & 1 \end{pmatrix}$

$$= -\frac{1}{5}\begin{pmatrix} 1 & 3 \\ 0 & -5 \end{pmatrix}\begin{pmatrix} -5 & 0 \\ 0 & 1 \end{pmatrix} = -\frac{1}{5}\begin{pmatrix} -5 & 3 \\ 0 & -5 \end{pmatrix} = \begin{pmatrix} 1 & -\frac{3}{5} \\ 0 & 1 \end{pmatrix}$$

3 **a)** Matrix A should have an inverse if $\det(A) \neq 0$.

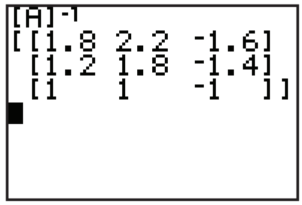
$$\det(A) = \begin{vmatrix} 2 & -3 & 1 \\ 1 & 1 & -3 \\ 3 & -2 & -3 \end{vmatrix} = 2(1\cdot(-3) - (-3)(-2)) + 3(1\cdot(-3) + 3\cdot 3) + 1\cdot(1\cdot(-2) - 1\cdot 3)$$

$$= 2(-9) + 3\cdot 6 + (-5) = -5$$

Or, with a GDC:

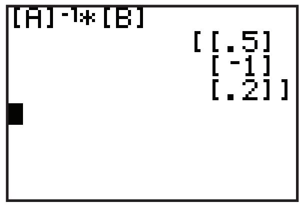
Therefore, A is regular, i.e. it has an inverse.

b) For matrix $A = \begin{pmatrix} 2 & -3 & 1 \\ 1 & 1 & -3 \\ 3 & -2 & -3 \end{pmatrix}$, we have:



c) For $X = \begin{pmatrix} x \\ y \\ z \end{pmatrix}$ and $B = \begin{pmatrix} 4.2 \\ -1.1 \\ 2.9 \end{pmatrix}$, the system can be written as $AX = B$; therefore, the solution will be $X = A^{-1}B$.

Using a GDC:



So, the solution is: $x = \frac{1}{2}$, $y = -1$, $z = \frac{1}{5}$.

4 a) $\det(A) = \begin{vmatrix} \frac{\sqrt{3}}{2} & -\frac{1}{2} \\ \frac{1}{2} & \frac{\sqrt{3}}{2} \end{vmatrix} = \left(\frac{\sqrt{3}}{2}\right)^2 + \left(\frac{1}{2}\right)^2 = \frac{3}{4} + \frac{1}{4} = 1$

$$A^{-1} = \begin{pmatrix} \frac{\sqrt{3}}{2} & \frac{1}{2} \\ -\frac{1}{2} & \frac{\sqrt{3}}{2} \end{pmatrix}$$

b) $\det B = \begin{vmatrix} a & 1 \\ a+2 & \frac{3}{a}+1 \end{vmatrix} = a\left(\frac{3}{a}+1\right) - (a+2) = 3 + a - a - 2 = 1$

$$B^{-1} = \begin{pmatrix} \frac{3}{a}+1 & -1 \\ -a-2 & a \end{pmatrix}$$

5 Matrix A is singular if $\det(A) = 0$. We have:

$$\det(A) = \begin{vmatrix} x+1 & 3 \\ 3x-1 & x+3 \end{vmatrix} = (x+1)(x+3) - 3(3x-1) = x^2 - 5x + 6$$

$$x^2 - 5x + 6 = 0 \Rightarrow x_{1,2} = \frac{5 \pm \sqrt{(-5)^2 - 4 \cdot 6}}{2} = 2, 3$$

Therefore, A is singular for $x = 2$ or $x = 3$.

6 If one matrix is the inverse of the other, it must satisfy:

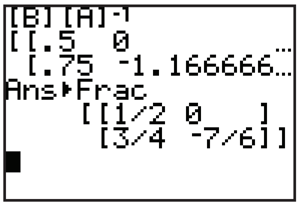
$$\begin{pmatrix} 2 & -1 & 4 \\ 2n & 2 & 0 \\ 2 & 1 & 4n \end{pmatrix}\begin{pmatrix} -2 & -3 & 4 \\ 1 & 2 & -2 \\ 3n & 2 & -5n \end{pmatrix} = \begin{pmatrix} -2 & -3 & 4 \\ 1 & 2 & -2 \\ 3n & 2 & -5n \end{pmatrix}\begin{pmatrix} 2 & -1 & 4 \\ 2n & 2 & 0 \\ 2 & 1 & 4n \end{pmatrix} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

$$\begin{pmatrix} 12n-5 & 0 & 10-20n \\ 2-4n & 4-6n & 8n-4 \\ -3+12n^2 & 8n-4 & 6-20n^2 \end{pmatrix} = \begin{pmatrix} 4-6n & 0 & 16n-8 \\ 4n-2 & 1 & 4-8n \\ 0 & 4-8n & 12n-20n^2 \end{pmatrix} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

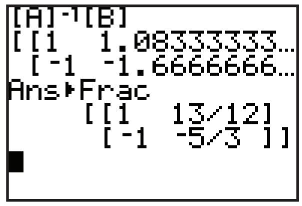
By comparing the corresponding elements, we can see that all equations are satisfied for $n = \frac{1}{2}$.

7 For $A = \begin{pmatrix} 4 & 2 \\ 0 & -3 \end{pmatrix}$ and $B = \begin{pmatrix} 2 & 1 \\ 3 & 5 \end{pmatrix}$, we have:

a) $XA = B \Rightarrow X = BA^{-1}$



b) $AY = B \Rightarrow Y = A^{-1}B$



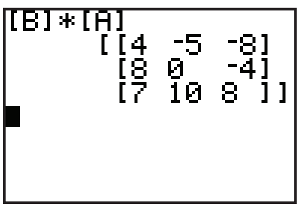
c) We can see that $X \neq Y$ because, in general, multiplication of matrices is not commutative.

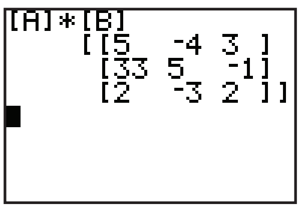
8 For this question we will use a GDC and, as such, we need to rename our matrices:

$$P = A = \begin{pmatrix} 2 & 0 & -1 \\ 3 & 5 & 4 \\ 1 & 0 & -1 \end{pmatrix} \text{ and } Q = B = \begin{pmatrix} 3 & -1 & 1 \\ 4 & 0 & 0 \\ 1 & 2 & -1 \end{pmatrix}$$

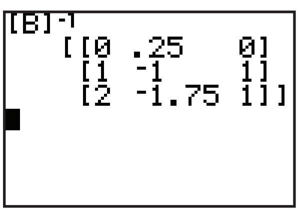
Then:

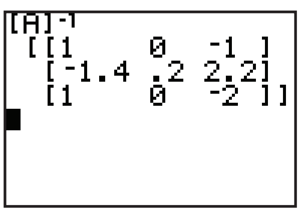
a)

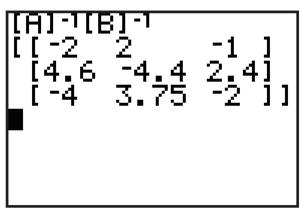


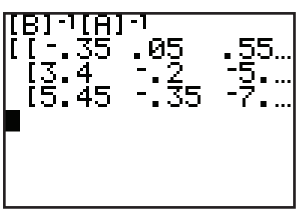


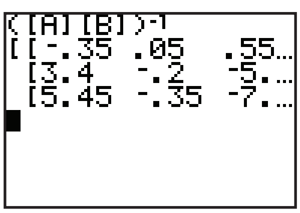
b)

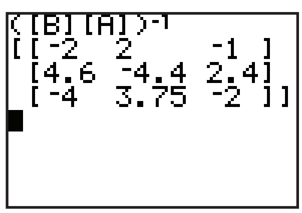










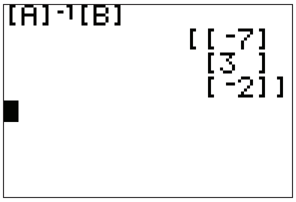


c) We can see that the multiplication of matrices in general is not commutative: $AB \neq BA$.

For the inverse matrices, the following is valid: $(AB)^{-1} = B^{-1}A^{-1}, (BA)^{-1} = A^{-1}B^{-1}$.

9 For matrices $A = \begin{pmatrix} 3 & -2 & 1 \\ -4 & 1 & -3 \\ 1 & -5 & 1 \end{pmatrix}$ and $B = \begin{pmatrix} -29 \\ 37 \\ -24 \end{pmatrix}$:

a) If $AC = B$ then $C = A^{-1}B$.



b) By multiplying the second and third equation by -1 the system is equivalent to the matrix notation $AX = B$, where $X = \begin{pmatrix} x \\ y \\ z \end{pmatrix}$. So, $X = A^{-1}B$, which means that the solution is: $x = -7$, $y = 3$, $z = -2$.

10 $\begin{pmatrix} 2 & 2+x \\ 5 & 4+x \end{pmatrix}\begin{pmatrix} 3 & x \\ x-4 & 2 \end{pmatrix} = \begin{pmatrix} 3 & x \\ x-4 & 2 \end{pmatrix}\begin{pmatrix} 2 & 2+x \\ 5 & 4+x \end{pmatrix} \Rightarrow$

$\begin{pmatrix} x^2-2x-2 & 4x+4 \\ x^2-1 & 7x+8 \end{pmatrix} = \begin{pmatrix} 5x+6 & x^2+7x+6 \\ 2x+2 & x^2 \end{pmatrix}$

By comparing the corresponding elements and rearranging the equations, we get:

$x^2 - 7x - 8 = 0 \Rightarrow x_{1,2} = -1, 8$

$x^2 + 3x + 2 = 0 \Rightarrow x_{3,4} = -1, -2$

$x^2 - 2x - 3 = 0 \Rightarrow x_{5,6} = -1, 3$

$x^2 - 7x - 8 = 0 \Rightarrow x_{7,8} = -1, 8$

$x = -1$ is the only common solution for all the equations, so it is the solution.

11 $AB = BA \Rightarrow$

$\begin{pmatrix} 2 & 1 \\ 5 & 3 \end{pmatrix}\begin{pmatrix} 2-x & 1 \\ 5x & y \end{pmatrix} = \begin{pmatrix} 2-x & 1 \\ 5x & y \end{pmatrix}\begin{pmatrix} 2 & 1 \\ 5 & 3 \end{pmatrix} \Rightarrow$

$\begin{pmatrix} 3x+4 & y+2 \\ 10x+10 & 3y+5 \end{pmatrix} = \begin{pmatrix} 9-2x & 5-x \\ 10x+5y & 5x+3y \end{pmatrix}$

By comparing the corresponding elements and rearranging the equations, we get:

$5x = 5$

$x + y = 3$

$5y = 10$

$5x = 5$

The solution that satisfies all the equations is: $x = 1$, $y = 2$.

12 $AB = BA \Rightarrow$

$\begin{pmatrix} 3 & 1 \\ -5 & 2 \end{pmatrix}\begin{pmatrix} 1-x & x \\ 5x & y \end{pmatrix} = \begin{pmatrix} 1-x & x \\ 5x & y \end{pmatrix}\begin{pmatrix} 3 & 1 \\ -5 & 2 \end{pmatrix} \Rightarrow$

$$\begin{pmatrix} 2x+3 & 3x+y \\ 15x-5 & -5x+2y \end{pmatrix} = \begin{pmatrix} 3-8x & x+1 \\ 15x-5y & 5x+2y \end{pmatrix}$$

By comparing the corresponding elements and rearranging the equations, we get:

$10x = 0$

$2x + y = 1$

$5y = 5$

$10x = 0$

The solution that satisfies all the equations is: $x = 0, y = 1$.

13 $AB = BA \Rightarrow$

$$\begin{pmatrix} 3+x & 1 \\ -5 & 2 \end{pmatrix}\begin{pmatrix} y-x & x \\ 5x-y+1 & y+x \end{pmatrix} = \begin{pmatrix} y-x & x \\ 5x-y+1 & y+x \end{pmatrix}\begin{pmatrix} 3+x & 1 \\ -5 & 2 \end{pmatrix} \Rightarrow$$

$$\begin{pmatrix} xy-x^2+2x+2y+1 & x^2+4x+y \\ 15x-7y+2 & 2y-3x \end{pmatrix} = \begin{pmatrix} xy-x^2-8x+3y & x+y \\ 5x^2-xy+11x-8y+3 & 7x+y+1 \end{pmatrix}$$

By comparing the corresponding elements and rearranging the equations, we get:

$10x - y = -1$

$x^2 + 3x = 0$

$5x^2 - xy - 4x - y = -1$

$10x - y = -1$

From the second equation, we get the solutions $x = 0$ or $x = -3$.

Substituting our solutions for x into the first equation, we get: $x = 0, y = 1$, and $x = -3, y = -29$.

Both solutions satisfy the third equation, so both pairs are valid.

14 a) $\begin{vmatrix} x & y & 1 \\ -5 & -6 & 1 \\ 3 & 11 & 1 \end{vmatrix} = 0 \Rightarrow x(-6-11) - y(-5-3) + (-55+18) = 0 \Rightarrow 17x - 8y + 37 = 0$

b) $\begin{vmatrix} x & y & 1 \\ 5 & -2 & 1 \\ 3 & -2 & 1 \end{vmatrix} = 0 \Rightarrow x(-2+2) - y(5-3) + (-10+6) = 0 \Rightarrow 2y + 4 = 0 \Rightarrow y + 2 = 0$

c) $\begin{vmatrix} x & y & 1 \\ -5 & 3 & 1 \\ -5 & 8 & 1 \end{vmatrix} = 0 \Rightarrow x(3-8) - y(-5+5) + (-40+15) = 0 \Rightarrow -5x - 25 = 0 \Rightarrow x + 5 = 0$

15 a) Area $= \left\| \begin{matrix} -5 & -6 & 1 \\ 3 & 11 & 1 \\ 8 & 1 & 1 \end{matrix} \right\| = |[-5(11-1) + 6(3-8) + (3-88)]| = |-165| = 165$

b) Area $= \left\| \begin{matrix} 3 & -5 & 1 \\ 3 & 11 & 1 \\ 8 & 11 & 1 \end{matrix} \right\| = |[3(11-11) + 5(3-8) + (33-88)]| = |-80| = 80$

c) Area $= \left\| \begin{matrix} 4 & -6 & 1 \\ -3 & 9 & 1 \\ 7 & 7 & 1 \end{matrix} \right\| = |[4(9-7)+6(-3-7)+(-21-63)]| = |-136| = 136$

16 a) Area $= \left| \frac{1}{2} \begin{vmatrix} x & -6 & 1 \\ 3 & 11 & 1 \\ 8 & 3 & 1 \end{vmatrix} \right| = \left| \frac{1}{2} [x(11-3)+6(3-8)+(9-88)] \right| = \frac{1}{2}|8x-109|$

$\frac{1}{2}|8x-109| = 10 \Rightarrow |8x-109| = 20 \Rightarrow 8x-109 = 20$ or $8x-109 = -20 \Rightarrow x = \frac{129}{8}$ or $x = \frac{89}{8}$

b) Area $= \left| \frac{1}{2} \begin{vmatrix} -5 & x & 1 \\ 3 & x+2 & 1 \\ x^2+2x-3 & 1 & 1 \end{vmatrix} \right| = \left| \frac{1}{2} [-5(x+2-1) - x(3-x^2-2x+3) + (3-(x+2)(x^2+2x-3))] \right|$

$= |-x^2 - 6x + 2|$

We have two possibilities:

$-x^2 - 6x + 2 = 10$ or $-x^2 - 6x + 2 = -10 \Rightarrow$

$x_1 = -2, x_2 = -4, x_3 = -3 + \sqrt{21}, x_4 = -3 - \sqrt{21}$

17 a) $\begin{vmatrix} 2 & -5 & 1 \\ 4 & k & 1 \\ 5 & -2 & 1 \end{vmatrix} = 0 \Rightarrow 2(k+2) + 5(4-5) + 1(-8-5k) = 0 \Rightarrow -3k - 9 = 0 \Rightarrow k = -3$

b) $\begin{vmatrix} -6 & 2 & 1 \\ -5 & k & 1 \\ -3 & 5 & 1 \end{vmatrix} = 0 \Rightarrow -6(k-5) - 2(-5+3) + 1(-25+3k) = 0 \Rightarrow -3k + 9 = 0 \Rightarrow k = 3$

18 For matrix $A = \begin{pmatrix} 2 & 7 \\ 5 & 5 \end{pmatrix}$:

a) $\det(A) = \begin{vmatrix} 2 & 7 \\ 5 & 5 \end{vmatrix} = 2 \cdot 5 - 7 \cdot 5 = -25$

b) $f(x) = \det(xI - A) = \det\left(x\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} - \begin{pmatrix} 2 & 7 \\ 5 & 5 \end{pmatrix} \right) = \begin{vmatrix} x-2 & -7 \\ -5 & x-5 \end{vmatrix} = (x-2)(x-5) - (-7)(-5) = x^2 - 7x - 25$

We can see that the constant term is equal to $\det(A)$.

c) Coefficient of $x = -(a+d)$.

d) $f(A) = \begin{pmatrix} 2 & 7 \\ 5 & 5 \end{pmatrix}^2 - 7\begin{pmatrix} 2 & 7 \\ 5 & 5 \end{pmatrix} - 25\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} = \begin{pmatrix} 39 & 49 \\ 35 & 60 \end{pmatrix} - \begin{pmatrix} 14 & 49 \\ 35 & 35 \end{pmatrix} - \begin{pmatrix} 25 & 0 \\ 0 & 25 \end{pmatrix} = \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}$

e) Generally, we have:

$\det(A) = \begin{vmatrix} a & b \\ c & d \end{vmatrix} = ad - bc$

$$f(x) = \det\left(x\begin{pmatrix}1 & 0\\0 & 1\end{pmatrix} - \begin{pmatrix}a & b\\c & d\end{pmatrix}\right) = \begin{vmatrix}x-a & -b\\-c & x-d\end{vmatrix} = (x-a)(x-d) - bc = x^2 - (a+d)x + ad - bc$$

The constant term is equal to $\det(A)$.

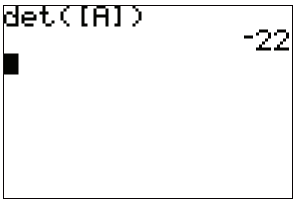
Coefficient of $x = -(a+d)$.

$$f(A) = \begin{pmatrix}a & b\\c & d\end{pmatrix}^2 - (a+d)\begin{pmatrix}a & b\\c & d\end{pmatrix} + (ad-bc)\begin{pmatrix}1 & 0\\0 & 1\end{pmatrix}$$

$$= \begin{pmatrix}a^2+bc & ab+bd\\ac+cd & bc+d^2\end{pmatrix} - \begin{pmatrix}a^2+ad & ab+bd\\ac+cd & ad+d^2\end{pmatrix} + \begin{pmatrix}ad-bc & 0\\0 & ad-bc\end{pmatrix} = \begin{pmatrix}0 & 0\\0 & 0\end{pmatrix}$$

19 For $A = \begin{pmatrix}2 & 7 & 1\\-1 & 3 & 2\\5 & 5 & 4\end{pmatrix}$:

a)



b) $f(x) = \det(xI - A) = \begin{vmatrix}x-2 & -7 & -1\\1 & x-3 & -2\\-5 & -5 & x+4\end{vmatrix}$

$$= (x-2)[(x-3)(x+4)-10] + 7[1(x+4)-10] - 1[-5+5(x-3)]$$

$$= x^3 - x^2 - 22x + 22$$

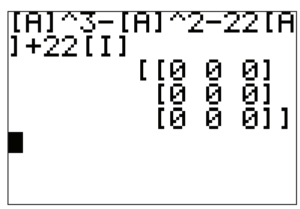
We can see that the constant term in the expansion of $f(x)$ is equal to $-\det(A)$.

c) The coefficient of x^2 is -1, which is the opposite of the sum of the main diagonal.

d) $f(A) = A^3 - A^2 - 22A + 22I$

We can calculate this with a GDC, knowing that $I = \begin{pmatrix}1 & 0 & 0\\0 & 1 & 0\\0 & 0 & 1\end{pmatrix}$.

$f(A)$ is again the zero matrix.



e) For $B = \begin{pmatrix}a & b & c\\d & e & f\\g & h & i\end{pmatrix}$ we have:

$$\det B = \begin{vmatrix}a & b & c\\d & e & f\\g & h & i\end{vmatrix} = a\begin{vmatrix}e & f\\h & i\end{vmatrix} - b\begin{vmatrix}d & f\\g & i\end{vmatrix} + c\begin{vmatrix}d & e\\g & h\end{vmatrix}$$

$$= a(ei - fh) - b(di - fg) + c(dh - eg)$$

$$= aei - afh - bdi + bfg + cdh - ceg$$

$$f(x)=\det(xI-B)=\begin{vmatrix} x-a & -b & -c \\ -d & x-e & -f \\ -g & -h & x-i \end{vmatrix}$$

$$=(x-a)\begin{vmatrix} x-e & -f \\ -h & x-i \end{vmatrix}+b\begin{vmatrix} -d & -f \\ -g & x-i \end{vmatrix}-c\begin{vmatrix} -d & x-e \\ -g & -h \end{vmatrix}$$

$$=(x-a)\left[(x-e)(x-i)-fh\right]+b\left[-d(x-i)-fg\right]-c\left[dh+(x-e)g\right]$$

$$=(x-a)\left[x^2-ex-ix+ei-fh\right]+b\left[-dx+di-fg\right]-c\left[dh+gx-eg\right]$$

$$=x^3-ax^2-ex^2+aex-ix^2+aix+eix-aei-fhx+afh-bdx+bdi-bfg-cdh-cgx+ceg$$

$$=x^3-(a+e+i)x^2+x(ae+ai+ei-bd-cg-fh)+afh-aei+bdi-bfg-cdh+ceg$$

$$=x^3-(a+e+i)x^2+x(ae+ai+ei-bd-cg-fh)-(aei-afh-bdi+bfg+cdh-ceg)$$

We can see that the constant term in the expansion of $f(x)$ is again equal to $-\det(B)$.

The coefficient of x^2 is $-(a+e+i)$, which is the opposite of the sum of the main diagonal.

$f(B)$ is the zero matrix.

Exercise 6.4

1 Given matrix $A=\begin{pmatrix} 5 & 6 \\ -1 & 0 \end{pmatrix}$:

$$\det(A-mI)=\det\left(\begin{pmatrix} 5 & 6 \\ -1 & 0 \end{pmatrix}-m\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}\right)=\begin{vmatrix} 5-m & 6 \\ -1 & -m \end{vmatrix}=(5-m)(-m)+6=m^2-5m+6$$

$m^2-5m+6=0 \Rightarrow m_1=2, m_2=3$

2 **a)** If A is the inverse of matrix B, then $AB=BA=I$ must be satisfied. So, we have:

$$AB=\begin{pmatrix} a & -4 & -6 \\ -8 & 5 & 7 \\ -5 & 3 & 4 \end{pmatrix}\begin{pmatrix} 1 & 2 & -2 \\ 3 & b & 1 \\ -1 & 1 & -3 \end{pmatrix}=\begin{pmatrix} a-6 & 2a-4b-6 & -2a+14 \\ 0 & 5b-9 & 0 \\ 0 & 3b-6 & 1 \end{pmatrix}=\begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

$$BA=\begin{pmatrix} 1 & 2 & -2 \\ 3 & b & 1 \\ -1 & 1 & -3 \end{pmatrix}\begin{pmatrix} a & -4 & -6 \\ -8 & 5 & 7 \\ -5 & 3 & 4 \end{pmatrix}=\begin{pmatrix} a-6 & 0 & 0 \\ 3a-8b-5 & 5b-9 & 7b-14 \\ -a+7 & 0 & 1 \end{pmatrix}=\begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

By comparing the corresponding elements, we find the solution $a=7$, $b=2$ satisfies all the conditions.

b) The system of linear equations for $a=7$, $b=2$ can be written as: $B\begin{pmatrix} x \\ y \\ z \end{pmatrix}=\begin{pmatrix} 5 \\ 0 \\ 6 \end{pmatrix}$.

So, we have: $\begin{pmatrix} x \\ y \\ z \end{pmatrix}=B^{-1}\begin{pmatrix} 5 \\ 0 \\ 6 \end{pmatrix}=A\begin{pmatrix} 5 \\ 0 \\ 6 \end{pmatrix}=\begin{pmatrix} 7 & -4 & -6 \\ -8 & 5 & 7 \\ -5 & 3 & 4 \end{pmatrix}\begin{pmatrix} 5 \\ 0 \\ 6 \end{pmatrix}=\begin{pmatrix} -1 \\ 2 \\ -1 \end{pmatrix}$.

3 If the matrix is singular, its determinant is zero.

$$\begin{vmatrix} 1 & m & 1 \\ 3 & 1-m & 2 \\ m & -3 & m-1 \end{vmatrix} = 1\left[(1-m)(m-1)+6\right] - m\left[3(m-1)-2m\right] + 1\left[-9-m(1-m)\right] = -m^2+4m-4 \Rightarrow$$

$-m^2 + 4m - 4 = 0 \Rightarrow (m-2)^2 = 0 \Rightarrow m = 2$

4 For each of the given systems, we will reduce the augmented matrix of the system to reduced row echelon form. From this, we can conclude whether the solution is unique (and which one it is), or if there are an infinite number of solutions, or no solution. Part **a** will be done algebraically, whilst parts **b**–**h** will be done using a GDC.

a)

$$\left(\begin{array}{ccc|c} 4 & -1 & 1 & -5 \\ 2 & 2 & 3 & 10 \\ 5 & -2 & 6 & 1 \end{array}\right) \begin{cases} 2R_2 - R_1 \\ -2R_3 + 5R_2 \end{cases}$$

$$\left(\begin{array}{ccc|c} 4 & -1 & 1 & -5 \\ 0 & 5 & 5 & 25 \\ 0 & 14 & 3 & 48 \end{array}\right) \left\{ \frac{1}{5} R_2 \right.$$

$$\left(\begin{array}{ccc|c} 4 & -1 & 1 & -5 \\ 0 & 1 & 1 & 5 \\ 0 & 14 & 3 & 48 \end{array}\right) \left\{ -R_3 + 14R_2 \right.$$

$$\left(\begin{array}{ccc|c} 4 & -1 & 1 & -5 \\ 0 & 1 & 1 & 5 \\ 0 & 0 & 11 & 22 \end{array}\right) \left\{ \frac{1}{11} R_3 \right.$$

$$\left(\begin{array}{ccc|c} 4 & -1 & 1 & -5 \\ 0 & 1 & 1 & 5 \\ 0 & 0 & 1 & 2 \end{array}\right) \begin{cases} R_1 - R_3 \\ R_2 - R_3 \end{cases}$$

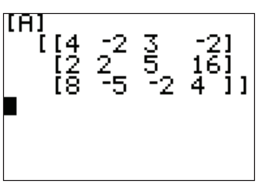
$$\left(\begin{array}{ccc|c} 4 & -1 & 0 & -7 \\ 0 & 1 & 0 & 3 \\ 0 & 0 & 1 & 2 \end{array}\right) \left\{ R_1 + R_3 \right.$$

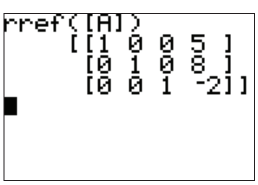
$$\left(\begin{array}{ccc|c} 4 & 0 & 0 & -4 \\ 0 & 1 & 0 & 3 \\ 0 & 0 & 1 & 2 \end{array}\right) \left\{ \frac{1}{4} R_1 \right.$$

$$\left(\begin{array}{ccc|c} 1 & 0 & 0 & -1 \\ 0 & 1 & 0 & 3 \\ 0 & 0 & 1 & 2 \end{array}\right)$$

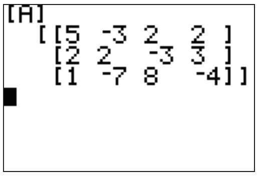
We can read the unique solution as: $x = -1,\ y = 3,\ z = 2.$

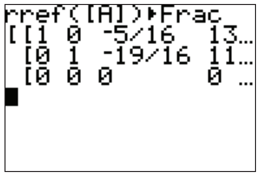
b)

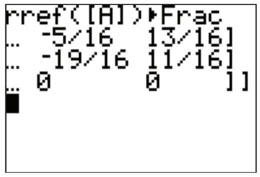




The unique solution is: $x = 5,\ y = 8,\ z = -2.$

c) 

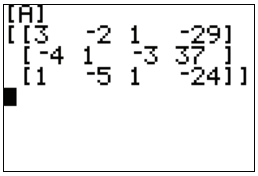


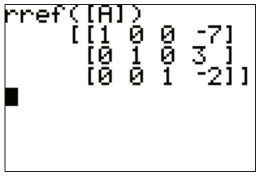


Since the last row is all zeros, the system has an infinite number of solutions. For:

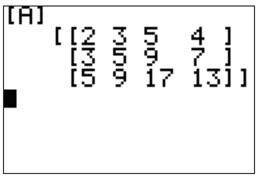
$$z = 16t \Rightarrow y - \frac{19}{16}z = \frac{11}{16} \Rightarrow y = \frac{11}{16} + 19t$$

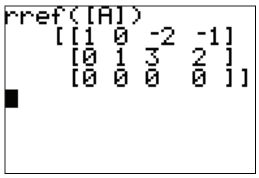
$$x - \frac{5}{16}z = \frac{13}{16} \Rightarrow x = \frac{13}{16} + 5t$$

d) 

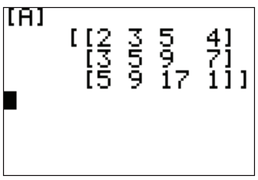


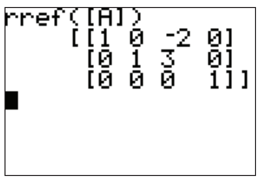
The unique solution is: $x = -7$, $y = 3$, $z = -2$.

e) 

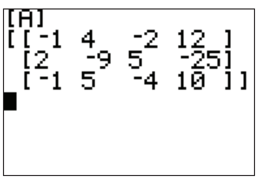


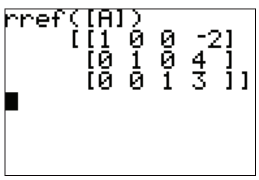
Since the last row is all zeros, the system has an infinite number of solutions. For:
$z = t \Rightarrow y = 2 - 3t,\ x = -1 + 2t$

f) 

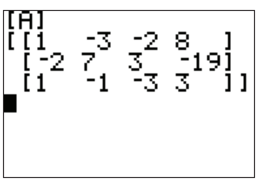


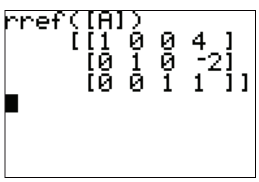
Since the last equation became $0 = 1$, the system is inconsistent; there is no solution.

g) 



The unique solution is: $x = -2$, $y = 4$, $z = 3$.

h) 



The unique solution is: $x = 4$, $y = -2$, $z = 1$.

5 a) $\det(A) = \begin{vmatrix} 1 & 1 & k-1 \\ k & 0 & -1 \\ 6 & 2 & -3 \end{vmatrix} = 1(0+2) - 1(-3k+6) + (k-1)(2k) = 2k^2 + k - 4$

$$\det(A) = 0 \Rightarrow k_{1,2} = \frac{-1 \pm \sqrt{33}}{4}$$

So, the matrix is **not** singular for all $k \neq \frac{-1 \pm \sqrt{33}}{4}$.

b) If A is the inverse of matrix B, then $AB = BA = I$ must be satisfied. So, we have:

$AB = BA = I$

$$\begin{pmatrix} 1 & 1 & k-1 \\ k & 0 & -1 \\ 6 & 2 & -3 \end{pmatrix}\begin{pmatrix} k-3 & -3 & k \\ 3 & k+2 & -1 \\ -2 & -4 & 1 \end{pmatrix} = \begin{pmatrix} k-3 & -3 & k \\ 3 & k+2 & -1 \\ -2 & -4 & 1 \end{pmatrix}\begin{pmatrix} 1 & 1 & k-1 \\ k & 0 & -1 \\ 6 & 2 & -3 \end{pmatrix} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

$$\begin{pmatrix} k-3+3-2k+2 & -3+k+2-4k+4 & k-1+k-1 \\ k^2-3k+2 & -3k+4 & k^2-1 \\ 6k-18+6+6 & -18+2k+4+12 & 6k-2-3 \end{pmatrix} = \begin{pmatrix} k-3-3k+6k & k-3+2k & (k-3)(k-1)+3-3k \\ 3+k^2+2k-6 & 3-2 & 3k-3-k-2+3 \\ -2-4k+6 & -2+2 & -2k+2+4-3 \end{pmatrix}$$

$$\begin{pmatrix} -k+2 & -3k+3 & 2k-2 \\ k^2-3k+2 & -3k+4 & k^2-1 \\ 6k-6 & 2k-2 & 6k-5 \end{pmatrix} = \begin{pmatrix} 4k-3 & 3k-3 & k^2-7k+6 \\ k^2+2k-3 & 1 & 2k-2 \\ 4-4k & 0 & -2k+3 \end{pmatrix} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

By comparing the corresponding elements, we find the solution $k = 1$ satisfies all the conditions.

c) For $k = 1$ we have:

$$\left(\begin{array}{ccc|ccc} 1 & 1 & 0 & 1 & 0 & 0 \\ 1 & 0 & -1 & 0 & 1 & 0 \\ 6 & 2 & -3 & 0 & 0 & 1 \end{array}\right) \begin{cases} R_1 \\ -R_2 + R_1 \\ -R_3 + 6R_1 \end{cases}$$

$$\left(\begin{array}{ccc|ccc} 1 & 1 & 0 & 1 & 0 & 0 \\ 0 & 1 & 1 & 1 & -1 & 0 \\ 0 & 4 & 3 & 6 & 0 & -1 \end{array}\right) \{-R_3 + 4R_2$$

$$\left(\begin{array}{ccc|ccc} 1 & 1 & 0 & 1 & 0 & 0 \\ 0 & 1 & 1 & 1 & -1 & 0 \\ 0 & 0 & 1 & -2 & -4 & 1 \end{array}\right) \{R_2 - R_3$$

$$\left(\begin{array}{ccc|ccc} 1 & 1 & 0 & 1 & 0 & 0 \\ 0 & 1 & 0 & 3 & 3 & -1 \\ 0 & 0 & 1 & -2 & -4 & 1 \end{array}\right) \{R_1 - R_2$$

$$\left(\begin{array}{ccc|ccc} 1 & 0 & 0 & -2 & -3 & 1 \\ 0 & 1 & 0 & 3 & 3 & -1 \\ 0 & 0 & 1 & -2 & -4 & 1 \end{array}\right)$$

The square matrix on the right-hand side of this augmented matrix is exactly $B = A^{-1}$.

6 For $A = \begin{pmatrix} \frac{2}{5} & \frac{-17}{5} & \frac{k+9}{5} \\ \frac{-1}{5} & \frac{21}{5} & \frac{-13}{5} \\ k-2 & 3 & -2 \end{pmatrix}$:

a) A is singular if $\det(A) = 0$.

$$\det(A) = \begin{vmatrix} \frac{2}{5} & \frac{-17}{5} & \frac{k+9}{5} \\ \frac{-1}{5} & \frac{21}{5} & \frac{-13}{5} \\ k-2 & 3 & -2 \end{vmatrix} = \frac{2}{5}\left[-\frac{42}{5}+\frac{39}{5}\right]+\frac{17}{5}\left[\frac{2}{5}+\frac{13}{5}(k-2)\right]+\frac{(k+9)}{5}\left[-\frac{3}{5}-\frac{21}{5}(k-2)\right]$$

$$= \frac{1}{25}\left(-21k^2+71k-63\right) \Rightarrow 21k^2-71k+63=0 \Rightarrow k_{1,2} = \frac{71 \pm \sqrt{-251}}{42}$$

Because there are no real solutions for k, A is regular (not singular) for all real numbers k.

b) If A is the inverse of matrix B, then $AB = BA = I$ must be satisfied. So, we have:

$$AB = \begin{pmatrix} \frac{2}{5} & \frac{-17}{5} & \frac{k+9}{5} \\ \frac{-1}{5} & \frac{21}{5} & \frac{-13}{5} \\ k-2 & 3 & -2 \end{pmatrix}\begin{pmatrix} k+1 & 1 & k \\ 2 & k+2 & -3 \\ 3 & 6 & -5 \end{pmatrix} = \begin{pmatrix} k-1 & \frac{-11k+22}{5} & \frac{-3k+6}{5} \\ \frac{-k+2}{5} & \frac{21k-37}{5} & \frac{-k+2}{5} \\ (k-2)(k+1) & 4k-8 & k^2-2k+1 \end{pmatrix} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

By comparing the corresponding elements, we see that the solution $k = 2$ satisfies all the equations.

c) For $k = 2$:

$$\left(\begin{array}{ccc|ccc} 2 & -17 & 11 & 1 & 0 & 0 \\ -1 & 21 & -13 & 0 & 1 & 0 \\ 0 & 15 & -10 & 0 & 0 & 1 \end{array}\right) \{2R_2 + R_1$$

$$\left(\begin{array}{ccc|ccc} 2 & -17 & 11 & 1 & 0 & 0 \\ 0 & 25 & -15 & 1 & 2 & 0 \\ 0 & 15 & -10 & 0 & 0 & 1 \end{array}\right) \{5R_3 - 3R_2$$

$$\left(\begin{array}{ccc|ccc} 2 & -17 & 11 & 1 & 0 & 0 \\ 0 & 25 & -15 & 1 & 2 & 0 \\ 0 & 0 & -5 & -3 & -6 & 5 \end{array}\right) \begin{cases} 5R_1 + 11R_3 \\ R_2 - 3R_3 \end{cases}$$

$$\left(\begin{array}{ccc|ccc} 10 & -85 & 0 & -28 & -66 & 55 \\ 0 & 25 & 0 & 10 & 20 & -15 \\ 0 & 0 & -5 & -3 & -6 & 5 \end{array}\right) \{5R_1 + 17R_2$$

$$\left(\begin{array}{ccc|ccc} 50 & 0 & 0 & 30 & 10 & 20 \\ 0 & 25 & 0 & 10 & 20 & -15 \\ 0 & 0 & -5 & -3 & -6 & 5 \end{array}\right) \begin{cases} \frac{1}{50}R_1 \\ \frac{1}{25}R_2 \\ -\frac{1}{5}R_3 \end{cases}$$

$$\left(\begin{array}{ccc|ccc} 1 & 0 & 0 & \frac{3}{5} & \frac{1}{5} & \frac{2}{5} \\ 0 & 1 & 0 & \frac{2}{5} & \frac{4}{5} & -\frac{3}{5} \\ 0 & 0 & 1 & \frac{3}{5} & \frac{6}{5} & -1 \end{array}\right)$$

7 **a)**

$$\left(\begin{array}{ccc|ccc} 2 & 0 & 3 & 1 & 0 & 0 \\ -1 & 1 & 1 & 0 & 1 & 0 \\ 2 & -2 & 1 & 0 & 0 & 1 \end{array}\right) \begin{cases} 2R_2 + R_1 \\ R_3 - R_1 \end{cases}$$

$$\left(\begin{array}{ccc|ccc} 2 & 0 & 3 & 1 & 0 & 0 \\ 0 & 2 & 5 & 1 & 2 & 0 \\ 0 & -2 & -2 & -1 & 0 & 1 \end{array}\right) \{R_3 + R_2$$

$$\left(\begin{array}{ccc|ccc} 2 & 0 & 3 & 1 & 0 & 0 \\ 0 & 2 & 5 & 1 & 2 & 0 \\ 0 & 0 & 3 & 0 & 2 & 1 \end{array}\right) \begin{cases} R_1 - R_3 \\ 3R_2 - 5R_3 \end{cases}$$

$$\left(\begin{array}{ccc|ccc} 2 & 0 & 0 & 1 & -2 & -1 \\ 0 & 6 & 0 & 3 & -4 & -5 \\ 0 & 0 & 3 & 0 & 2 & 1 \end{array}\right) \begin{cases} \frac{1}{2}R_1 \\ \frac{1}{6}R_2 \\ \frac{1}{3}R_3 \end{cases}$$

$$\left(\begin{array}{ccc|ccc} 1 & 0 & 0 & \frac{1}{2} & -1 & -\frac{1}{2} \\ 0 & 1 & 0 & \frac{1}{2} & -\frac{2}{3} & -\frac{5}{6} \\ 0 & 0 & 1 & 0 & \frac{2}{3} & \frac{1}{3} \end{array}\right)$$

B is the inverse of *A*.

b)

$$\left(\begin{array}{ccc|ccc} 1 & 4 & 5 & 1 & 0 & 0 \\ 2 & -3 & 1 & 0 & 1 & 0 \\ -1 & 8 & 6 & 0 & 0 & 1 \end{array}\right) \begin{cases} R_2 - 2R_1 \\ R_3 + R_1 \end{cases}$$

$$\left(\begin{array}{ccc|ccc} 1 & 4 & 5 & 1 & 0 & 0 \\ 0 & -11 & -9 & -2 & 1 & 0 \\ 0 & 12 & 11 & 1 & 0 & 1 \end{array}\right) \{11R_3 + 12R_2$$

$$\left(\begin{array}{ccc|ccc} 1 & 4 & 5 & 1 & 0 & 0 \\ 0 & -11 & -9 & -2 & 1 & 0 \\ 0 & 0 & 13 & -13 & 12 & 11 \end{array}\right) \begin{cases} 13R_1 - 5R_3 \\ 13R_2 + 9R_3 \end{cases}$$

$$\left(\begin{array}{ccc|ccc} 13 & 52 & 0 & 78 & -60 & -55 \\ 0 & -143 & 0 & -143 & 121 & 99 \\ 0 & 0 & 13 & -13 & 12 & 11 \end{array}\right) \{11R_1 + 4R_2$$

$$\left(\begin{array}{ccc|ccc} 143 & 0 & 0 & 286 & -176 & -209 \\ 0 & -143 & 0 & -143 & 121 & 99 \\ 0 & 0 & 13 & -13 & 12 & 11 \end{array}\right) \begin{cases} \frac{1}{143}R_1 \\ \frac{-1}{143}R_2 \\ \frac{1}{13}R_3 \end{cases}$$

$$\left(\begin{array}{ccc|ccc} 1 & 0 & 0 & 2 & -\frac{16}{13} & -\frac{19}{13} \\ 0 & 1 & 0 & 1 & -\frac{11}{13} & -\frac{9}{13} \\ 0 & 0 & 1 & -1 & \frac{12}{13} & \frac{11}{13} \end{array}\right)$$

B is the inverse of *A*.

8 **a)** For $f(x) = ax^2 + bx + c$ and $f(-1) = 5$, $f(2) = -1$, $f(4) = 35$, we obtain the following system of equations:

$a - b + c = 5$

$4a + 2b + c = -1$

$16a + 4b + c = 35$

For the augmented matrix of the system, we solve using a GDC:

```
[A]
  [[1   -1  1  5 ]
   [4   2   1  -1]
   [16  4   1  35]]
```

```
rref([A]
    [[1  0  0  4 ]
     [0  1  0  -6]
     [0  0  1  -5]]
```

So, the function is $f(x) = 4x^2 - 6x - 5$.

b) From $f(-1) = 12$, $f(2) = -3$, we have the following system of equations:

$a - b + c = 12$

$4a + 2b + c = -3$

$$\left(\begin{array}{ccc|c} 1 & -1 & 1 & 12 \\ 4 & 2 & 1 & -3 \end{array}\right)\{R_2 - 4R_1$$

$$\left(\begin{array}{ccc|c} 1 & -1 & 1 & 12 \\ 0 & 6 & -3 & -51 \end{array}\right)\{6R_1 + R_2$$

$$\left(\begin{array}{ccc|c} 6 & 0 & 3 & 21 \\ 0 & 6 & -3 & -51 \end{array}\right)\begin{cases} \frac{1}{6}R_1 \\ \frac{1}{6}R_2 \end{cases}$$

$$\left(\begin{array}{ccc|c} 1 & 0 & \frac{1}{2} & \frac{7}{2} \\ 0 & 1 & -\frac{1}{2} & -\frac{17}{2} \end{array}\right)$$

So, for $c = m \Rightarrow a = \frac{1}{2}(7-m), b = \frac{1}{2}(m-17) \Rightarrow f(x) = \frac{1}{2}(7-m)x^2 + \frac{1}{2}(m-17) + m.$

c) From $f(-1) = 5$, $f(1) = -3$, $f(2) = 5$, $f(3) = 45$, we have the following system of equations:

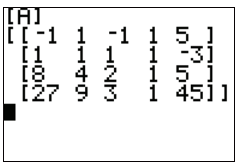
$-a + b - c + d = 5$

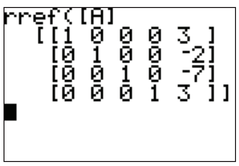
$a + b + c + d = -3$

$8a + 4b + 2c + d = 5$

$27a + 9b + 3c + d = 45$

For the augmented matrix of the system, we solve using a GDC:





So, the function is $f(x) = 3x^3 - 2x^2 - 7x + 3.$

d) From $f(-3) = 4$, $f(-1) = 4$, $f(2) = 4$, we have the following system of equations:

$-27a + 9b - 3c + d = 4$

$-a + b - c + d = 4$

$8a + 4b + 2c + d = 4$

$$\left(\begin{array}{cccc|c} -1 & 1 & -1 & 1 & 4 \\ 8 & 4 & 2 & 1 & 4 \\ -27 & 9 & -3 & 1 & 4 \end{array}\right)\begin{cases} R_2 + 8R_1 \\ R_3 - 27R_1 \end{cases}$$

$$\left(\begin{array}{cccc|c} -1 & 1 & -1 & 1 & 4 \\ 0 & 12 & -6 & 9 & 36 \\ 0 & -18 & 24 & -26 & -104 \end{array}\right)\begin{cases} \frac{1}{3}R_2 \\ 2R_3 + 3R_2 \end{cases}$$

$$\left(\begin{array}{cccc|c} -1 & 1 & -1 & 1 & 4 \\ 0 & 4 & -2 & 3 & 12 \\ 0 & 0 & 30 & -25 & -100 \end{array}\right)\begin{cases} 6R_1 + \frac{1}{5}R_3 \\ 3R_2 + \frac{1}{5}R_3 \\ \frac{1}{5}R_3 \end{cases}$$

$$\left(\begin{array}{cccc|c} -6 & 6 & 0 & 1 & 4 \\ 0 & 12 & 0 & 4 & 16 \\ 0 & 0 & 6 & -5 & -20 \end{array}\right)\{-2R_1 + R_2$$

$$\left(\begin{array}{cccc|c} 12 & 0 & 0 & 2 & 8 \\ 0 & 12 & 0 & 4 & 16 \\ 0 & 0 & 6 & -5 & -20 \end{array}\right)\begin{cases} \frac{1}{12}R_1 \\ \frac{1}{12}R_2 \\ \frac{1}{6}R_3 \end{cases}$$

$$\left(\begin{array}{cccc|c} 1 & 0 & 0 & \frac{1}{6} & \frac{4}{6} \\ 0 & 1 & 0 & \frac{1}{3} & \frac{4}{3} \\ 0 & 0 & 1 & -\frac{5}{6} & -\frac{20}{6} \end{array}\right)$$

$$d = t \Rightarrow a = \frac{1}{6}(4-t), b = \frac{1}{3}(4-t), c = \frac{1}{6}(5t-20) \Rightarrow f(x) = \frac{1}{6}(4-t)x^3 + \frac{1}{3}(4-t)x^2 + \frac{1}{6}(5t-20)x + t$$

9 $\left(\begin{array}{ccc|c} 2 & 1 & 3 & -5 \\ 3 & -1 & 4 & 2 \\ 5 & 0 & 7 & m-5 \end{array}\right) \sim \left(\begin{array}{ccc|c} 2 & 1 & 3 & -5 \\ 0 & 5 & 1 & -19 \\ 0 & 5 & 1 & -2m-15 \end{array}\right) \sim \left(\begin{array}{ccc|c} 2 & 1 & 3 & -5 \\ 0 & 5 & 1 & -19 \\ 0 & 0 & 0 & 2m-4 \end{array}\right)$

The system is consistent if $2m - 4 = 0 \Rightarrow m = 2$.

$\left(\begin{array}{ccc|c} 2 & 1 & 3 & -5 \\ 0 & 5 & 1 & -19 \\ 0 & 0 & 0 & 0 \end{array}\right) \sim \left(\begin{array}{ccc|c} 10 & 0 & 14 & -6 \\ 0 & 5 & 1 & -19 \\ 0 & 0 & 0 & 0 \end{array}\right) \sim \left(\begin{array}{ccc|c} 1 & 0 & \frac{7}{5} & -\frac{3}{5} \\ 0 & 1 & \frac{1}{5} & \frac{-19}{5} \\ 0 & 0 & 0 & 0 \end{array}\right) \Rightarrow$ The general solution is $\begin{pmatrix} -7t - \frac{3}{5} \\ -t - \frac{19}{5} \\ 5t \end{pmatrix}$.

10 $\left(\begin{array}{ccc|c} -3 & 2 & 3 & 1 \\ 4 & -1 & -5 & -5 \\ 1 & 1 & -2 & m-3 \end{array}\right) \sim \left(\begin{array}{ccc|c} -3 & 2 & 3 & 1 \\ 0 & 5 & -3 & -11 \\ 0 & 5 & -3 & 3m-8 \end{array}\right) \sim \left(\begin{array}{ccc|c} -3 & 2 & 3 & 1 \\ 0 & 5 & -3 & -11 \\ 0 & 0 & 0 & 3m+3 \end{array}\right)$

The system is consistent if $3m + 3 = 0 \Rightarrow m = -1$.

$\left(\begin{array}{ccc|c} -3 & 2 & 3 & 1 \\ 0 & 5 & -3 & -11 \\ 0 & 0 & 0 & 0 \end{array}\right) \sim \left(\begin{array}{ccc|c} 15 & 0 & -21 & -27 \\ 0 & 5 & -3 & -11 \\ 0 & 0 & 0 & 0 \end{array}\right) \sim \left(\begin{array}{ccc|c} 1 & 0 & -\frac{7}{5} & -\frac{9}{5} \\ 0 & 1 & -\frac{3}{5} & -\frac{11}{5} \\ 0 & 0 & 0 & 0 \end{array}\right) \Rightarrow$ The general solution is $\begin{pmatrix} 7t - \frac{9}{5} \\ 3t - \frac{11}{5} \\ 5t \end{pmatrix}$.

11 a) $\det(A) = \begin{vmatrix} 3 & -4 & -6 \\ -8 & 5 & 7 \\ -5 & 3 & 4 \end{vmatrix} = 3(20-21) + 4(-32+35) - 6(-24+25) = 3$

b) $$\left(\begin{array}{ccc|ccc} 3 & -4 & -6 & 1 & 0 & 0 \\ -8 & 5 & 7 & 0 & 1 & 0 \\ -5 & 3 & 4 & 0 & 0 & 1 \end{array}\right)\begin{cases} 3R_2+8R_1 \\ 3R_3+5R_1 \end{cases}$$

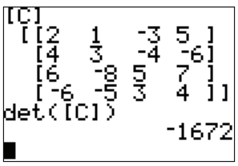
$$\left(\begin{array}{ccc|ccc} 3 & -4 & -6 & 1 & 0 & 0 \\ 0 & -17 & -27 & 8 & 3 & 0 \\ 0 & -11 & -18 & 5 & 0 & 3 \end{array}\right)\left\{17R_3-11R_2\right.$$

$$\left(\begin{array}{ccc|ccc} 3 & -4 & -6 & 1 & 0 & 0 \\ 0 & -17 & -27 & 8 & 3 & 0 \\ 0 & 0 & -9 & -3 & -33 & 51 \end{array}\right)\begin{cases} \frac{1}{3}R_2 \\ \frac{1}{51}R_3 \end{cases}$$

$$\left(\begin{array}{ccc|ccc} 3 & -4 & -6 & 1 & 0 & 0 \\ 0 & -\frac{17}{3} & -9 & \frac{8}{3} & 1 & 0 \\ 0 & 0 & -\frac{3}{17} & -\frac{1}{17} & -\frac{11}{17} & 1 \end{array}\right)$$

c) $\det(B)=3\cdot\left(-\frac{17}{3}\right)\cdot\left(-\frac{3}{17}\right)=3$

d)



e) $$\left(\begin{array}{cccc|cccc} 2 & 1 & -3 & 5 & 1 & 0 & 0 & 0 \\ 4 & 3 & -4 & -6 & 0 & 1 & 0 & 0 \\ 6 & -8 & 5 & 7 & 0 & 0 & 1 & 0 \\ -6 & -5 & 3 & 4 & 0 & 0 & 0 & 1 \end{array}\right)\begin{cases} R_2-2R_1 \\ R_3-3R_1 \\ R_4+3R_1 \end{cases}$$

$$\left(\begin{array}{cccc|cccc} 2 & 1 & -3 & 5 & 1 & 0 & 0 & 0 \\ 0 & 1 & 2 & -16 & -2 & 1 & 0 & 0 \\ 0 & -11 & 14 & -8 & -3 & 0 & 1 & 0 \\ 0 & -2 & -6 & 19 & 3 & 0 & 0 & 1 \end{array}\right)\begin{cases} R_3+11R_2 \\ R_4+2R_2 \end{cases}$$

$$\left(\begin{array}{cccc|cccc} 2 & 1 & -3 & 5 & 1 & 0 & 0 & 0 \\ 0 & 1 & 2 & -16 & -2 & 1 & 0 & 0 \\ 0 & 0 & 36 & -184 & -25 & 11 & 1 & 0 \\ 0 & 0 & -2 & -13 & -1 & 2 & 0 & 1 \end{array}\right)\left\{18R_4+R_3\right.$$

$$\left(\begin{array}{cccc|cccc} 2 & 1 & -3 & 5 & 1 & 0 & 0 & 0 \\ 0 & 1 & 2 & -16 & -2 & 1 & 0 & 0 \\ 0 & 0 & 36 & -184 & -25 & 11 & 1 & 0 \\ 0 & 0 & 0 & -418 & -43 & 47 & 1 & 18 \end{array}\right)\left\{\frac{1}{18}R_4\right.$$

$$\left(\begin{array}{cccc|cccc} 2 & 1 & -3 & 5 & 1 & 0 & 0 & 0 \\ 0 & 1 & 2 & -16 & -2 & 1 & 0 & 0 \\ 0 & 0 & 36 & -184 & -25 & 11 & 1 & 0 \\ 0 & 0 & 0 & -\frac{418}{18} & -\frac{43}{18} & \frac{47}{18} & \frac{1}{18} & 1 \end{array}\right)$$

$$\det(D)=2\cdot 1\cdot 36\cdot\left(-\frac{418}{18}\right)=-1672$$

Chapter 7

Exercise 7.1

1 $60^\circ = 60^\circ\left(\frac{\pi}{180^\circ}\right) = \frac{60^\circ}{180^\circ}\pi = \frac{\pi}{3}$

2 $150^\circ = 150^\circ\left(\frac{\pi}{180^\circ}\right) = \frac{150^\circ}{180^\circ}\pi = \frac{5\pi}{6}$

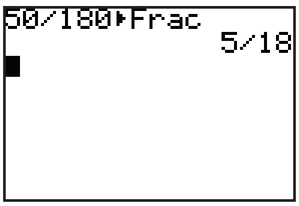
3 $-270^\circ = -270^\circ\left(\frac{\pi}{180^\circ}\right) = -\frac{270^\circ}{180^\circ}\pi = -\frac{3\pi}{2}$

4 $36^\circ = \frac{36^\circ}{180^\circ}\pi = \frac{\pi}{5}$

5 $135^\circ = \frac{135^\circ}{180^\circ}\pi = \frac{3\pi}{4}$

6 $50^\circ = \frac{50^\circ}{180^\circ}\pi = \frac{5\pi}{18}$

(**Note:** A fraction can be simplified by a GDC.)



7 $-45^\circ = -\frac{45^\circ}{180^\circ}\pi = -\frac{\pi}{4}$

8 $400^\circ = \frac{400^\circ}{180^\circ}\pi = \frac{20\pi}{9}$

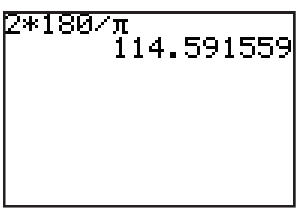
9 $-480^\circ = -\frac{480^\circ}{180^\circ}\pi = -\frac{8\pi}{3}$

10 $\frac{3\pi}{4} = \frac{3}{4}(180^\circ) = 135^\circ$

11 $-\frac{7\pi}{2} = -\frac{7}{2}(180^\circ) = -630^\circ$

12 $2 = 2\left(\frac{180^\circ}{\pi}\right) \approx 115^\circ$

(**Note:** We use: $1 = \frac{180^\circ}{\pi}$)



13 $\frac{7\pi}{6} = \frac{7}{6}(180^\circ) = 210^\circ$

14 $-2.5 = -2.5\left(\frac{180^\circ}{\pi}\right) \approx -143^\circ$

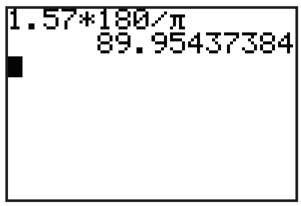
(**Note:** We use: $1 = \frac{180^\circ}{\pi}$)

15 $\frac{5\pi}{3} = \frac{5}{3}(180^\circ) = 300^\circ$

16 $\frac{\pi}{12} = \frac{180^\circ}{12} = 15^\circ$

17 $1.57 = 1.57\left(\frac{180^\circ}{\pi}\right) \approx 90.0^\circ$

(**Note:** We use: $1 = \frac{180^\circ}{\pi}$)



18 $\frac{8\pi}{3} = \frac{8}{3}(180^\circ) = 480^\circ$

19 For example, positive: $30^\circ + 360^\circ = 390^\circ$; negative: $30^\circ - 360^\circ = -330^\circ$.

20 For example, positive: $\frac{3\pi}{2} + 2\pi = \frac{7\pi}{2}$; negative: $\frac{3\pi}{2} - 2\pi = -\frac{\pi}{2}$.

21 For example, positive: $175^\circ + 360^\circ = 535^\circ$; negative: $175^\circ - 360^\circ = -185^\circ$.

22 For example, positive: $-\frac{\pi}{6} + 2\pi = \frac{11\pi}{6}$; negative: $-\frac{\pi}{6} - 2\pi = -\frac{13\pi}{6}$.

23 For example, positive: $\frac{5\pi}{3} + 2\pi = \frac{11\pi}{3}$; negative: $\frac{5\pi}{3} - 2\pi = -\frac{\pi}{3}$.

24 For example, positive: $3.25 + 2\pi \approx 9.53$; negative: $3.25 - 2\pi \approx -3.03$.

25 First convert 120° to radian measure:

$$120^\circ = 120^\circ\left(\frac{\pi}{180^\circ}\right) = \frac{120^\circ}{180^\circ}\pi = \frac{2\pi}{3}$$

Then use the formula $s = r\theta$:

$$s = r\theta \Rightarrow 6\left(\frac{2\pi}{3}\right) = 4\pi \approx 12.6 \text{ cm}$$

26 First convert 70° to radian measure:

$$70^\circ = 70^\circ\left(\frac{\pi}{180^\circ}\right)$$

Then use the formula $s = r\theta$:

$$s = r\theta = 12 \cdot 70\left(\frac{\pi}{180}\right) \approx 14.7 \text{ cm}$$

Note: Do not round off the result for $70^\circ\left(\frac{\pi}{180^\circ}\right) \approx 1.221\,73....$ Rounding off will give you an answer of: $12 \cdot 1.22 = 14.64 \approx 14.6$ cm.

27 We can use the arc length formula to find the angle in radians:

$$s = r\theta \Rightarrow 12 = 8 \cdot \theta \Rightarrow \theta = \frac{3}{2} \text{ radians}$$

Converting 1.5 radians into degrees:

$$1.5 = 1.5\left(\frac{180^\circ}{\pi}\right) \approx 85.9^\circ$$

28 We can use the arc length formula to find the radius:

$$s = r\theta \Rightarrow 15 = r \cdot \frac{2\pi}{3} \Rightarrow r = \frac{45}{2\pi} \approx 7.16 \text{ cm}$$

29 First convert 100° to radian measure:

$$100^\circ = 100^\circ\left(\frac{\pi}{180^\circ}\right)$$

Then use the area of a sector formula,

$$A = \frac{1}{2}r^2\theta:$$

$$A = \frac{1}{2}r^2\theta = \frac{1}{2} \cdot 4^2 \cdot 100 \cdot \left(\frac{\pi}{180}\right) = \frac{40}{9}\pi$$

$$\approx 14.0 \text{ cm}^2$$

30 Use the formula for the area of a sector,

$$A = \frac{1}{2}r^2\theta:$$

$$A = \frac{1}{2}r^2\theta = \frac{1}{2} \cdot 10^2 \cdot \frac{5\pi}{6} = \frac{125}{3}\pi \approx 131 \text{ cm}^2$$

31 We can use the arc length formula to find the angle in radians. Let the angle $= \alpha$:

$$s = r\alpha \Rightarrow 60 = 20 \cdot \theta \Rightarrow \theta = 3 \text{ radians}$$

Converting 3 radians into degrees:

$$3 = 3\left(\frac{180^\circ}{\pi}\right) \approx 172^\circ$$

32 Use the arc length formula: $s = r\theta = 16 \cdot 2$
$= 32$ cm

33 First convert 60° to radian measure:

$$60^\circ = 60^\circ\left(\frac{\pi}{180^\circ}\right) = \frac{\pi}{3}$$

We can use the area of a sector formula to find the radius:

$$A = \frac{1}{2}r^2\theta \Rightarrow 24 = \frac{1}{2} \cdot r^2 \cdot \frac{\pi}{3} \Rightarrow$$

$$\frac{144}{\pi} = r^2 \Rightarrow r = \frac{12}{\sqrt{\pi}} \approx 6.77 \text{ cm}$$

34 **a)** $1.5 \cdot 2\pi = 3\pi$ radians per second

b) Speed is: $1.5 \cdot 70\pi = 105\pi$ cm/s. Now change the units:

$$\frac{105\pi \cdot 60^2}{1000 \cdot 100} \approx 11.9 \text{ km/s}$$

```
1.5*70*π*60²/100
0/100
         11.87522023
```

35 Velocity is:

$$25 \text{ km/hr} = \frac{2\,500\,000}{60^2} \text{ cm/s} = 694.444 \text{ cm/s}$$

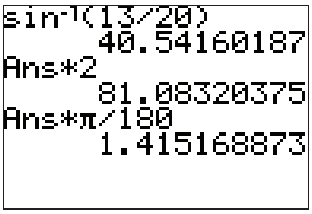
Angular velocity is: $\frac{694.444 \text{ cm/s}}{70\pi \text{ cm}} \approx 3.16$ revolutions per second, or:

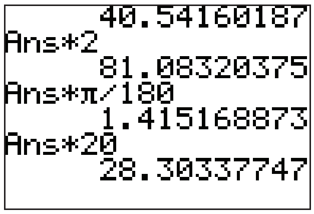
$$\frac{694.444 \text{ cm/s}}{70\pi \text{ cm}} \cdot 2\pi \approx 19.8 \text{ radians per second.}$$

```
25*1000*100/60²
        694.4444444
Ans/(70π)
        3.157836172
Ans*2π
        19.84126984
```

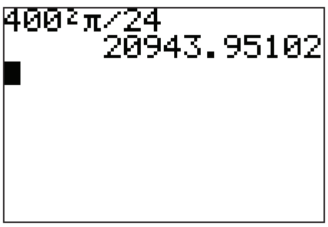
36 $v = w \cdot r$

37 Half the chord and the radius are the leg and hypotenuse of the triangle respectively. To find the angle, we have to use the formula $\sin\frac{\theta}{2} = \frac{13}{20} \Rightarrow \theta \approx 81.1^\circ$. The angle in radians is 1.42. Hence, the length of the arc is: $s = r\theta \approx 28.3$ cm.





38 The area is $400^2\pi$ m²; hence, the area it irrigates each hour is: $\frac{400^2\pi}{24} \approx 20\,944$ m².



39 **a)** An angle in the regular polygon is: $\frac{360}{64} = 5.625$. A right triangle whose hypotenuse is the radius of the circle is formed by half of one side of the polygon and half of the angle. Hence:

$$r = \frac{\frac{3}{2}}{\sin\frac{5.625}{2}} \approx 30.6 \text{ cm.}$$

b) The circumference of the circle is $2r\pi \approx 192.077\,12....$ The perimeter of the polygon is $64 \cdot 3 = 192$. Hence, their difference is approximately 0.0771 cm.

40 The radius of the inscribed circle is $r = \sqrt{50} = 5\sqrt{2}$ cm, and it is related to the height of the equilateral triangle as $\frac{1}{3}h$. Hence, $h = 15\sqrt{2}$. For the height of the equilateral triangle, it holds that:

$$h = \frac{a\sqrt{3}}{2} \Rightarrow 15\sqrt{2} = \frac{a\sqrt{3}}{2} \Rightarrow a = 10\sqrt{6}.$$

The area of the triangle is:

$$A = \frac{\left(10\sqrt{6}\right)^2\sqrt{3}}{4} = 150\sqrt{3} \text{ cm}^2.$$

Note: Alternatively, we could have used the circumscribed circle in which the radius is $\frac{2}{3}h$, and then proceeded in a similar manner.

41 The area of one-quarter of the circle is equal to the area of the right triangle plus the shaded area.

Hence: $\frac{r^2\pi}{4} = \frac{1}{2}r \cdot r + A \Rightarrow$

$$r^2\pi = 2r^2 + 4A \Rightarrow$$

$$r^2(\pi - 2) = 4A \Rightarrow r^2 = \frac{4A}{\pi - 2}$$

The area of the circle is: $r^2\pi = \frac{4\pi A}{\pi - 2}$.

Exercise 7.2

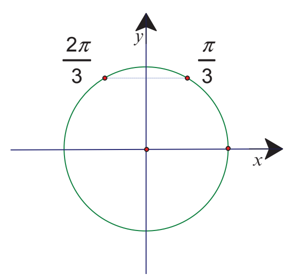
1 **a)** For $\frac{\pi}{6}: \left(\cos\frac{\pi}{6}, \sin\frac{\pi}{6}\right) = \left(\frac{\sqrt{3}x}{2x}, \frac{x}{2x}\right)$

$$= \left(\frac{\sqrt{3}}{2}, \frac{1}{2}\right)$$

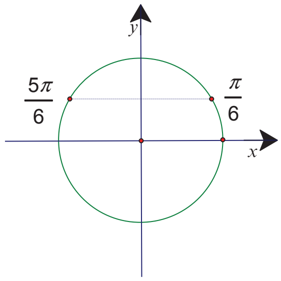
For $\frac{\pi}{3}: \left(\cos\frac{\pi}{3}, \sin\frac{\pi}{3}\right) = \left(\frac{x}{2x}, \frac{\sqrt{3}x}{2x}\right)$

$$= \left(\frac{1}{2}, \frac{\sqrt{3}}{2}\right)$$

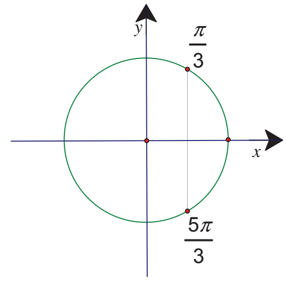
b) For $\frac{2\pi}{3}: \left(-\cos\frac{\pi}{3}, \sin\frac{\pi}{3}\right) = \left(-\frac{1}{2}, \frac{\sqrt{3}}{2}\right)$



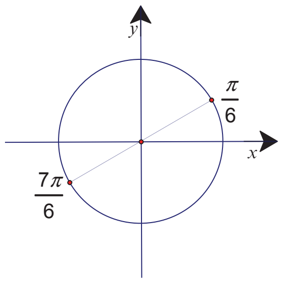
For $\frac{5\pi}{6}$: $\left(-\cos\frac{\pi}{6}, \sin\frac{\pi}{6}\right) = \left(-\frac{\sqrt{3}}{2}, \frac{1}{2}\right)$



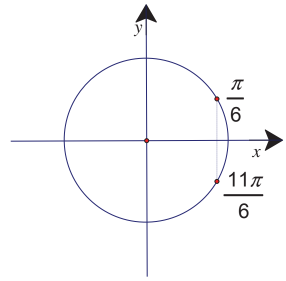
For $\frac{5\pi}{3}$: $\left(\cos\frac{\pi}{3}, -\sin\frac{\pi}{3}\right) = \left(\frac{1}{2}, -\frac{\sqrt{3}}{2}\right)$



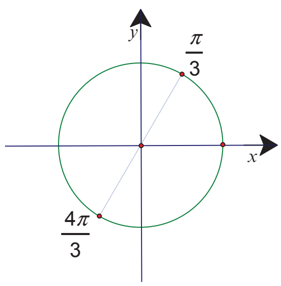
For $\frac{7\pi}{6}$: $\left(-\cos\frac{\pi}{6}, -\sin\frac{\pi}{6}\right) = \left(-\frac{\sqrt{3}}{2}, -\frac{1}{2}\right)$



For $\frac{11\pi}{6}$: $\left(\cos\frac{\pi}{6}, -\sin\frac{\pi}{6}\right) = \left(\frac{\sqrt{3}}{2}, -\frac{1}{2}\right)$



For $\frac{4\pi}{3}$: $\left(-\cos\frac{\pi}{3}, -\sin\frac{\pi}{3}\right) = \left(-\frac{1}{2}, -\frac{\sqrt{3}}{2}\right)$



2 0.6 **3** 1.0 **4** 0.5 **5** 0.5

6 2.7 **7** 0.1 **8** 0.3 **9** 1.6

10 a) I

b) $x = \cos\left(\frac{\pi}{6}\right) = \frac{\sqrt{3}}{2}$, $y = \sin\left(\frac{\pi}{6}\right) = \frac{1}{2}$; therefore, the terminal point is $\left(\frac{\sqrt{3}}{2}, \frac{1}{2}\right)$.

11 a) $\frac{5\pi}{3} = 2\pi - \frac{\pi}{3}$, so the terminal point lies in quadrant IV.

b) $x = \cos\left(\frac{5\pi}{3}\right) = \cos\left(\frac{\pi}{3}\right) = \frac{1}{2}$, $y = \sin\left(\frac{5\pi}{3}\right) = -\sin\left(\frac{\pi}{3}\right) = -\frac{\sqrt{3}}{2}$; therefore, the terminal point is $\left(\frac{1}{2}, -\frac{\sqrt{3}}{2}\right)$.

12 a) $\frac{7\pi}{4} = 2\pi - \frac{\pi}{4}$, so the terminal point lies in quadrant IV.

b) $x = \cos\left(\frac{7\pi}{4}\right) = \cos\left(\frac{\pi}{4}\right) = \frac{\sqrt{2}}{2}$, $y = \sin\left(\frac{7\pi}{4}\right) = -\sin\left(\frac{\pi}{4}\right) = -\frac{\sqrt{2}}{2}$; therefore, the terminal point is $\left(\frac{\sqrt{2}}{2}, -\frac{\sqrt{2}}{2}\right)$.

13 a) $\frac{3\pi}{2}$ terminates at the negative y-axis.

b) The terminal point is $(0, -1)$.

14 a) Since $\frac{\pi}{2} \approx 1.57$ and $\pi \approx 3.14$, then $\frac{\pi}{2} < 2 < \pi$; therefore, $t = 2$ terminates in quadrant II.

b) $x = \cos 2 \approx -0.416$, $y = \sin 2 \approx 0.909$; therefore, the terminal point is $(-0.416, 0.909)$.

15 a) IV

b) $x = \cos\left(-\frac{\pi}{4}\right) = \cos\left(\frac{\pi}{4}\right) = \frac{\sqrt{2}}{2}$, $y = \sin\left(-\frac{\pi}{4}\right) = -\sin\left(\frac{\pi}{4}\right) = -\frac{\sqrt{2}}{2}$; therefore, the terminal point is $\left(\frac{\sqrt{2}}{2}, -\frac{\sqrt{2}}{2}\right)$.

16 a) Since $-\frac{\pi}{2} \approx -1.57$, then $-\frac{\pi}{2} < -1 < 0$; therefore, $t = -1$ terminates in quadrant IV.

b) $x = \cos(-1) \approx 0.540$, $y = \sin(-1) \approx -0.841$; therefore, the terminal point is $(0.540, -0.841)$.

17 a) $-\frac{5\pi}{4} = -\pi - \frac{\pi}{4}$, so the terminal point lies in quadrant II.

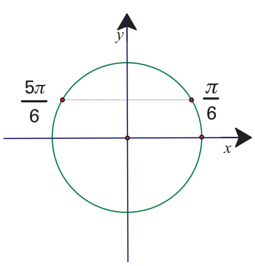
b) $x = \cos\left(-\frac{5\pi}{4}\right) = -\cos\left(\frac{\pi}{4}\right) = -\frac{\sqrt{2}}{2}$, $y = \sin\left(-\frac{5\pi}{4}\right) = \sin\left(\frac{\pi}{4}\right) = \frac{\sqrt{2}}{2}$; therefore, the terminal point is $\left(-\frac{\sqrt{2}}{2}, \frac{\sqrt{2}}{2}\right)$.

18 a) Since $\pi \approx 3.14$ and $\frac{3\pi}{2} \approx 4.71$, then $\pi < 3.52 < \frac{3\pi}{2}$; therefore, $t = 3.52$ terminates in quadrant III.

b) $x = \cos 3.52 \approx -0.929$, $y = \sin 3.52 \approx -0.369$; therefore, the terminal point is $(-0.929, -0.369)$.

19 $\sin\left(\frac{\pi}{3}\right) = \frac{\sqrt{3}}{2}$, $\cos\left(\frac{\pi}{3}\right) = \frac{1}{2}$, $\tan\left(\frac{\pi}{3}\right) = \sqrt{3}$

20

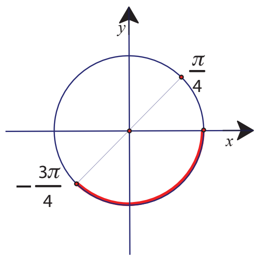


$$\sin\left(\frac{5\pi}{6}\right) = \sin\left(\pi - \frac{\pi}{6}\right) = \sin\left(\frac{\pi}{6}\right) = \frac{1}{2}$$

$$\cos\left(\frac{5\pi}{6}\right) = \cos\left(\pi - \frac{\pi}{6}\right) = -\cos\left(\frac{\pi}{6}\right) = -\frac{\sqrt{3}}{2}$$

$$\tan\left(\frac{5\pi}{6}\right) = \frac{\sin\left(\frac{5\pi}{6}\right)}{\cos\left(\frac{5\pi}{6}\right)} = \frac{\frac{1}{2}}{-\frac{\sqrt{3}}{2}} = -\frac{1}{\sqrt{3}} = -\frac{\sqrt{3}}{3}$$

21



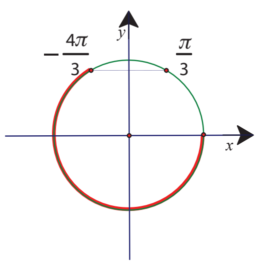
$$\sin\left(-\frac{3\pi}{4}\right) = \sin\left(-\pi + \frac{\pi}{4}\right) = -\sin\left(\frac{\pi}{4}\right) = -\frac{\sqrt{2}}{2}$$

$$\cos\left(-\frac{3\pi}{4}\right) = \cos\left(-\pi + \frac{\pi}{4}\right) = -\cos\left(\frac{\pi}{4}\right) = -\frac{\sqrt{2}}{2}$$

$$\tan\left(-\frac{3\pi}{4}\right) = \frac{\sin\left(-\frac{3\pi}{4}\right)}{\cos\left(-\frac{3\pi}{4}\right)} = \frac{-\frac{\sqrt{2}}{2}}{-\frac{\sqrt{2}}{2}} = 1$$

22 $\sin\left(\frac{\pi}{2}\right) = 1$, $\cos\left(\frac{\pi}{2}\right) = 0$, $\tan\left(\frac{\pi}{2}\right)$ is not defined

23



$$\sin\left(-\frac{4\pi}{3}\right) = \sin\left(-\pi - \frac{\pi}{3}\right) = \sin\left(\frac{\pi}{3}\right) = \frac{\sqrt{3}}{2}$$

$$\cos\left(-\frac{4\pi}{3}\right) = \cos\left(-\pi - \frac{\pi}{3}\right) = -\cos\left(\frac{\pi}{3}\right) = -\frac{1}{2}$$

$$\tan\left(-\frac{4\pi}{3}\right) = \frac{\sin\left(-\frac{4\pi}{3}\right)}{\cos\left(-\frac{4\pi}{3}\right)} = \frac{\frac{\sqrt{3}}{2}}{-\frac{1}{2}} = -\sqrt{3}$$

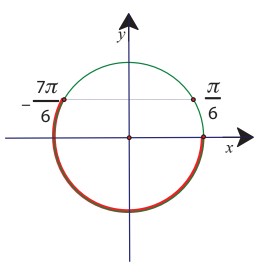
24 $\sin(3\pi) = \sin(2\pi + \pi) = \sin\pi = 0$

$\cos(3\pi) = \cos(2\pi + \pi) = \cos\pi = -1$

$\tan(3\pi) = \frac{\sin(3\pi)}{\cos(3\pi)} = \frac{0}{-1} = 0$

25 $\sin\left(\frac{3\pi}{2}\right) = -1$, $\cos\left(\frac{3\pi}{2}\right) = 0$, $\tan\left(\frac{3\pi}{2}\right)$ is not defined

26

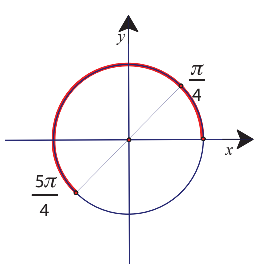


$$\sin\left(-\frac{7\pi}{6}\right) = \sin\left(-\pi - \frac{\pi}{6}\right) = \sin\left(\frac{\pi}{6}\right) = \frac{1}{2}$$

$$\cos\left(-\frac{7\pi}{6}\right) = \cos\left(-\pi - \frac{\pi}{6}\right) = -\cos\left(\frac{\pi}{6}\right) = -\frac{\sqrt{3}}{2}$$

$$\tan\left(-\frac{7\pi}{6}\right) = \frac{\sin\left(-\frac{7\pi}{6}\right)}{\cos\left(-\frac{7\pi}{6}\right)} = \frac{\frac{1}{2}}{-\frac{\sqrt{3}}{2}} = -\frac{1}{\sqrt{3}} = -\frac{\sqrt{3}}{3}$$

27



$$\sin\left(\frac{5\pi}{4}\right) = \sin\left(\pi + \frac{\pi}{4}\right) = -\sin\left(\frac{\pi}{4}\right) = -\frac{\sqrt{2}}{2}$$

$$\cos\left(\frac{5\pi}{4}\right) = \cos\left(\pi + \frac{\pi}{4}\right) = -\cos\left(\frac{\pi}{4}\right) = -\frac{\sqrt{2}}{2}$$

$$\tan\left(\frac{5\pi}{4}\right) = \frac{\sin\left(\frac{5\pi}{4}\right)}{\cos\left(\frac{5\pi}{4}\right)} = \frac{-\frac{\sqrt{2}}{2}}{-\frac{\sqrt{2}}{2}} = 1$$

28 $\frac{13\pi}{6} = \frac{\pi + 12\pi}{6} = \frac{\pi}{6} + 2\pi$, and the period of the sine and cosine function is 2π; therefore:

$$\sin\left(\frac{13\pi}{6}\right) = \sin\left(\frac{\pi}{6} + 2\pi\right) = \sin\left(\frac{\pi}{6}\right) = \frac{1}{2}$$

$$\cos\left(\frac{13\pi}{6}\right) = \cos\left(\frac{\pi}{6} + 2\pi\right) = \cos\left(\frac{\pi}{6}\right) = \frac{\sqrt{3}}{2}$$

29 $\frac{10\pi}{3} = \frac{4\pi + 6\pi}{3} = \frac{4\pi}{3} + 2\pi$, and the period of the sine and cosine function is 2π; therefore:

$$\sin\left(\frac{10\pi}{3}\right) = \sin\left(\frac{4\pi}{3} + 2\pi\right) = \sin\left(\frac{4\pi}{3}\right)$$

$$\cos\left(\frac{10\pi}{3}\right) = \cos\left(\frac{4\pi}{3} + 2\pi\right) = \cos\left(\frac{4\pi}{3}\right)$$

The terminal point for $\frac{4\pi}{3} = \pi + \frac{\pi}{3}$ is in the third quadrant. It is diametrically opposite to the terminal point for $\frac{\pi}{3}$; therefore, the sine and cosine of $\frac{4\pi}{3}$ are opposite in sign to the sine and cosine of $\frac{\pi}{3}$. Hence:

$$\sin\left(\frac{10\pi}{3}\right) = \sin\left(\frac{4\pi}{3}\right) = -\sin\left(\frac{\pi}{3}\right) = -\frac{\sqrt{3}}{2}$$

$$\cos\left(\frac{10\pi}{3}\right) = \cos\left(\frac{4\pi}{3}\right) = -\cos\left(\frac{\pi}{3}\right) = -\frac{1}{2}$$

30 $\frac{15\pi}{4} = \frac{7\pi + 8\pi}{4} = \frac{7\pi}{4} + 2\pi$, and the period of the sine and cosine function is 2π; therefore:

$$\sin\left(\frac{15\pi}{4}\right) = \sin\left(\frac{7\pi}{4} + 2\pi\right) = \sin\left(\frac{7\pi}{4}\right)$$

$$\cos\left(\frac{15\pi}{4}\right) = \cos\left(\frac{7\pi}{4} + 2\pi\right) = \cos\left(\frac{7\pi}{4}\right)$$

The terminal point for $\frac{7\pi}{4} = 2\pi - \frac{\pi}{4}$ is in the fourth quadrant. It is the reflection in the x-axis of the terminal point for $\frac{\pi}{4}$; therefore, the sine is opposite in sign to the sine of $\frac{\pi}{4}$ and the cosine is the same as the cosine of $\frac{\pi}{4}$. Hence:

$$\sin\left(\frac{15\pi}{4}\right) = \sin\left(\frac{7\pi}{4}\right) = -\sin\left(\frac{\pi}{4}\right) = -\frac{\sqrt{2}}{2}$$

$$\cos\left(\frac{15\pi}{4}\right) = \cos\left(\frac{7\pi}{4}\right) = \cos\left(\frac{\pi}{4}\right) = \frac{\sqrt{2}}{2}$$

31 $\frac{17\pi}{6} = \frac{5\pi + 12\pi}{6} = \frac{5\pi}{6} + 2\pi$, and the period of the sine and cosine function is 2π; therefore:

$$\sin\left(\frac{17\pi}{6}\right) = \sin\left(\frac{5\pi}{6} + 2\pi\right) = \sin\left(\frac{5\pi}{6}\right)$$

$$\cos\left(\frac{17\pi}{6}\right) = \cos\left(\frac{5\pi}{6} + 2\pi\right) = \cos\left(\frac{5\pi}{6}\right)$$

The terminal point for $\frac{5\pi}{6} = \pi - \frac{\pi}{6}$ is in the second quadrant. It is the reflection in the y-axis of the terminal point for $\frac{\pi}{6}$; therefore, the sine value is the same as the sine of $\frac{\pi}{6}$ and the cosine is opposite in sign to the cosine of $\frac{\pi}{6}$. Hence:

$$\sin\left(\frac{17\pi}{6}\right) = \sin\left(\frac{5\pi}{6}\right) = \sin\left(\frac{\pi}{6}\right) = \frac{1}{2}$$

$$\cos\left(\frac{17\pi}{6}\right) = \cos\left(\frac{5\pi}{6}\right) = -\cos\left(\frac{\pi}{6}\right) = -\frac{\sqrt{3}}{2}$$

32 a) $\cos\left(\frac{5\pi}{6}\right) = -\cos\left(\frac{\pi}{6}\right) = -\frac{\sqrt{3}}{2}$

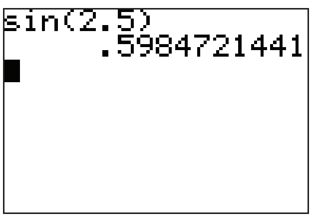
b) $\sin 315^\circ = -\sin 45^\circ = -\frac{\sqrt{2}}{2}$

c) Not defined

d) $\sec\frac{5\pi}{3} = \frac{1}{\cos\frac{5\pi}{3}} = \frac{1}{\cos\frac{\pi}{3}} = \frac{1}{\frac{1}{2}} = 2$

e) $\csc 240^\circ = \frac{1}{\sin 240^\circ} = \frac{1}{-\sin 60^\circ} = -\frac{1}{\frac{\sqrt{3}}{2}} = -\frac{2\sqrt{3}}{3}$

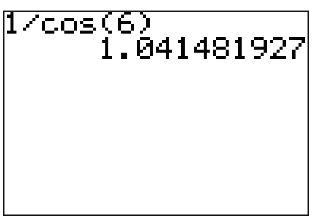
33 a) 0.598



b) $\cot 120^\circ = \frac{1}{\tan 120^\circ} = \frac{1}{-\tan 60^\circ} = -\frac{1}{\sqrt{3}} = -\frac{\sqrt{3}}{3}$

c) $\cos\frac{5\pi}{4} = -\cos\frac{\pi}{4} = -\frac{\sqrt{2}}{2}$

d) 1.04



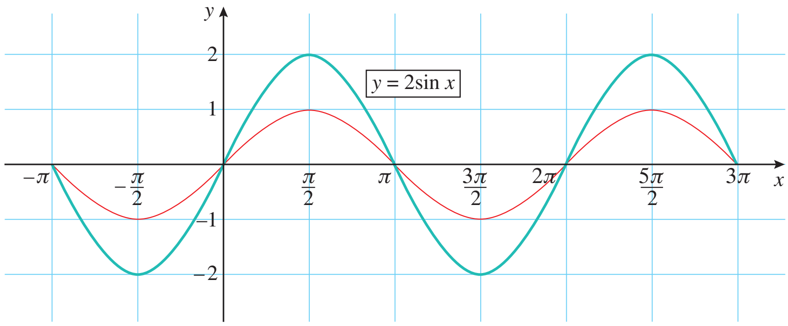
e) 0

34 I, II **35** II **36** III **37** II

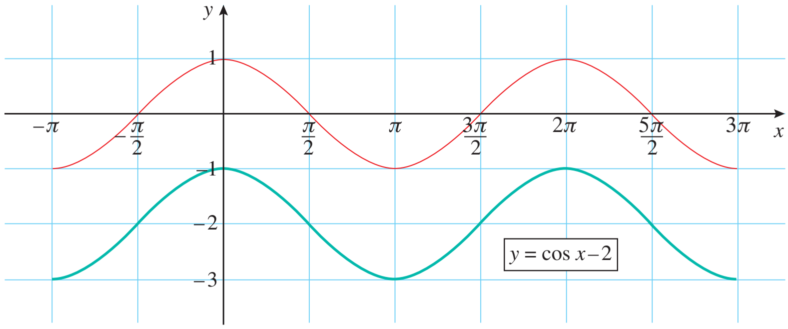
38 I, IV **39** I **40** IV **41** II, IV

Exercise 7.3

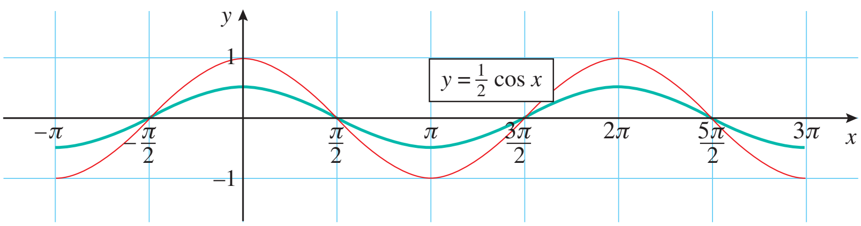
1 The graph is of the form $y = af(x)$. Since $a = 2$, the graph of $y = 2\sin x$ is obtained by vertically stretching the graph of $y = \sin x$ by scale factor 2.



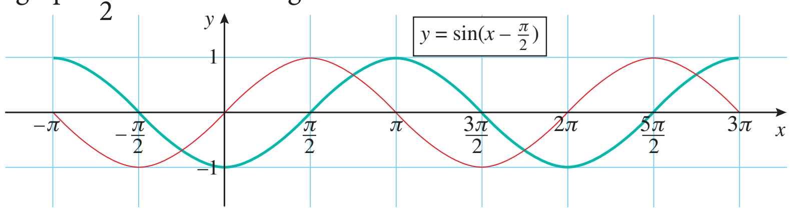
2 The graph is of the form $y = f(x) + d$. Since $d = -2$, the graph of $y = \cos x - 2$ is obtained by translating the graph of $y = \cos x$ down two units.



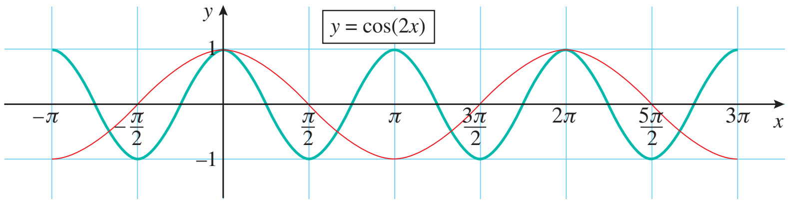
3 The graph is of the form $y = af(x)$. Since $a = \frac{1}{2}$, the graph of $y = \frac{1}{2}\cos x$ is obtained by vertically shrinking the graph of $y = \cos x$ by scale factor $\frac{1}{2}$.



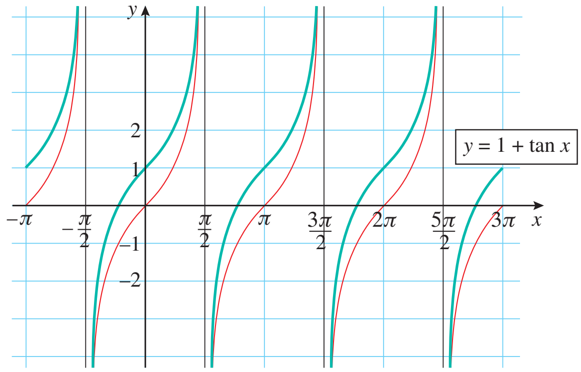
4 The graph is of the form $y = f(x + c)$. Since $c = -\frac{\pi}{2}$, the graph of $y = \sin\left(x - \frac{\pi}{2}\right)$ is obtained by translating the sine graph $\frac{\pi}{2}$ units to the right.



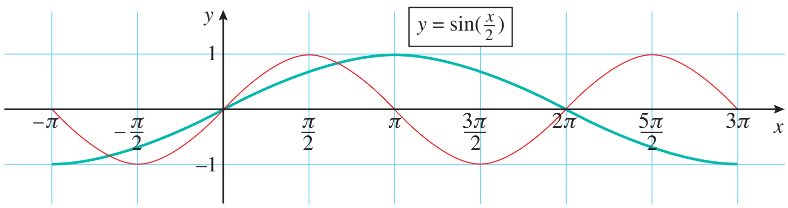
5 The graph is of the form $y = f(bx)$. Since $b = 2$, the graph of $y = \cos(2x)$ is obtained by horizontally shrinking the graph of $y = \cos x$ by scale factor $\frac{1}{2}$.



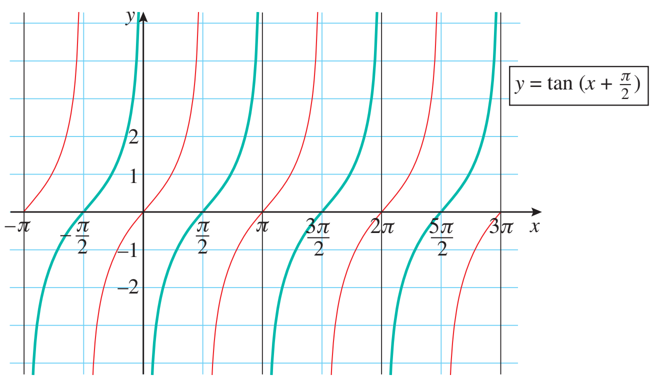
6 The graph is of the form $y = f(x) + d$. Since $d = 1$, the graph of $y = 1 + \tan x$ is obtained by translating the graph of $y = \tan x$ up one unit.



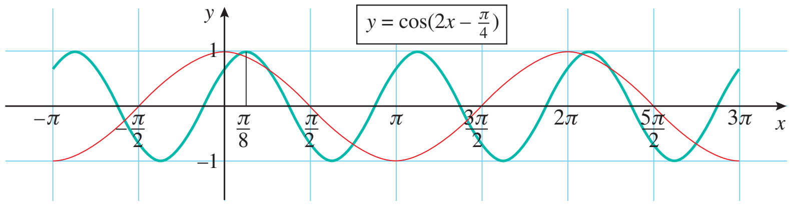
7 The graph is of the form $y = f(bx)$. Since $b = \frac{1}{2}$, the graph of $y = \sin\left(\frac{x}{2}\right)$ is obtained by horizontally stretching the graph of $y = \sin x$ by scale factor 2.



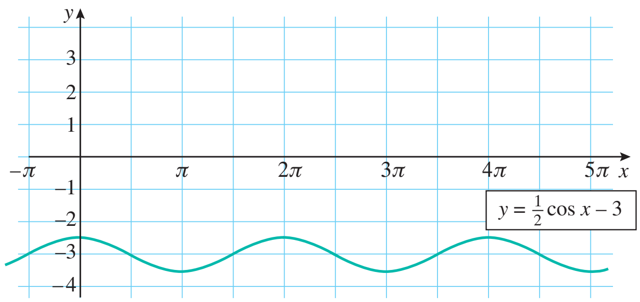
8 The graph is of the form $y = f(x + c)$. Since $c = \frac{\pi}{2}$, the graph of $y = \tan\left(x + \frac{\pi}{2}\right)$ is obtained by translating the tangent graph $\frac{\pi}{2}$ units to the left.



9 To determine the transformation, we need to write the function in the form $y = a\cos[b(x + c)] + d$. Obviously, $a = 1$ and $d = 0$. Since $2x - \frac{\pi}{4} = 2\left(x - \frac{\pi}{8}\right)$, the graph of $y = \cos\left(2x - \frac{\pi}{4}\right)$ is obtained by translating the cosine graph $\frac{\pi}{8}$ units to the right, and then horizontally shrinking the graph of $y = \cos\left(x - \frac{\pi}{8}\right)$ by scale factor $\frac{1}{2}$.



10 a)



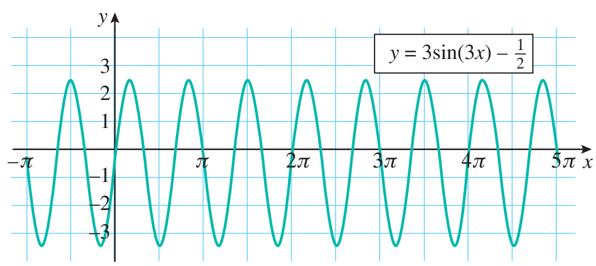
In $f(x) = \frac{1}{2}\cos x - 3$, $a = \frac{1}{2}$, so the amplitude equals $\frac{1}{2}$.

$b = 1$, so the period equals $\frac{2\pi}{1} = 2\pi$.

b) The domain is $x \in \mathbb{R}$.

The graph is obtained by translating the graph of $y = \frac{1}{2}\cos x$ down three units. The range of $\frac{1}{2}\cos x$ is $-0.5 \leqslant y \leqslant 0.5$, so after a vertical translation the range will be $-3 - 0.5 \leqslant y \leqslant -3 + 0.5$. Hence, the range is $-3.5 \leqslant y \leqslant -2.5$.

11 a)



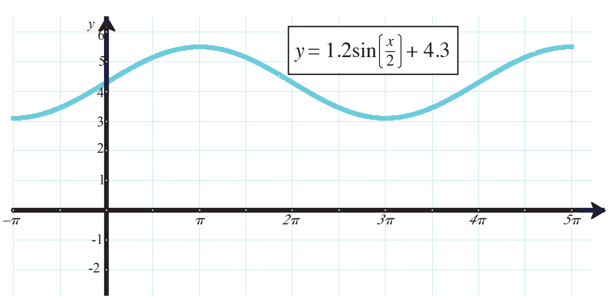
In $g(x) = 3\sin(3x) - \frac{1}{2}$, $a = 3$, so the amplitude equals 3.

$b = 3$, so the period equals $\frac{2\pi}{3}$.

b) The domain is $x \in \mathbb{R}$.

The graph is obtained by translating the graph of $y = 3\sin(3x)$ down 0.5 unit. The range of $3\sin(3x)$ is $-3 \leqslant y \leqslant 3$, so after a vertical translation the range will be $-0.5 - 3 \leqslant y \leqslant -0.5 + 3$. Hence, the range is $-3.5 \leqslant y \leqslant 2.5$.

12 a)



In $g(x) = 1.2\sin\left(\frac{x}{2}\right) + 4.3$, $a = 1.2$, so the amplitude equals 1.2.

$b = \frac{1}{2}$, so the period equals $\frac{2\pi}{\frac{1}{2}} = 4\pi$.

b) The domain is $x \in \mathbb{R}$.

The graph is obtained by translating the graph of $y = 1.2\sin\left(\frac{x}{2}\right)$ up 4.3 units. The range of $1.2\sin\left(\frac{x}{2}\right)$ is $-1.2 \leqslant y \leqslant 1.2$, so after the vertical translation the range will be $4.3 - 1.2 \leqslant y \leqslant 4.3 + 1.2$. Hence, the range is $3.1 \leqslant y \leqslant 5.5$.

13 To determine B we have to determine the mid-line; so, we have to find the average of the function's maximum and minimum value:

$$B = \frac{10 + 4}{2} = 7$$

To determine A, we have to determine the amplitude. The amplitude is the difference between the function's maximum value and the mid-line: $|A| = 10 - 7 = 3$. So, $A = \pm 3$. We see that the graph, starting from initial position at $x = 0$, first reaches the maximum value, and then the minimum value, so A is positive; hence, $A = 3$.

14 To determine B we have to determine the mid-line; so, we have to find the average of the function's maximum and minimum value:

$$B = \frac{8.6 + 3.2}{2} = 5.9$$

To determine A, we have to determine the amplitude. The amplitude is the difference between the function's maximum value and the mid-line: $|A| = 8.6 - 5.9 = 2.7$. So, $A = \pm 2.7$. We see that the graph, starting from initial position at $x = 0$, first reaches the maximum value, and then the minimum value, so A is positive; hence, $A = 2.7$.

15 To determine B we have to determine the mid-line; so, we have to find the average of the function's maximum and minimum value:

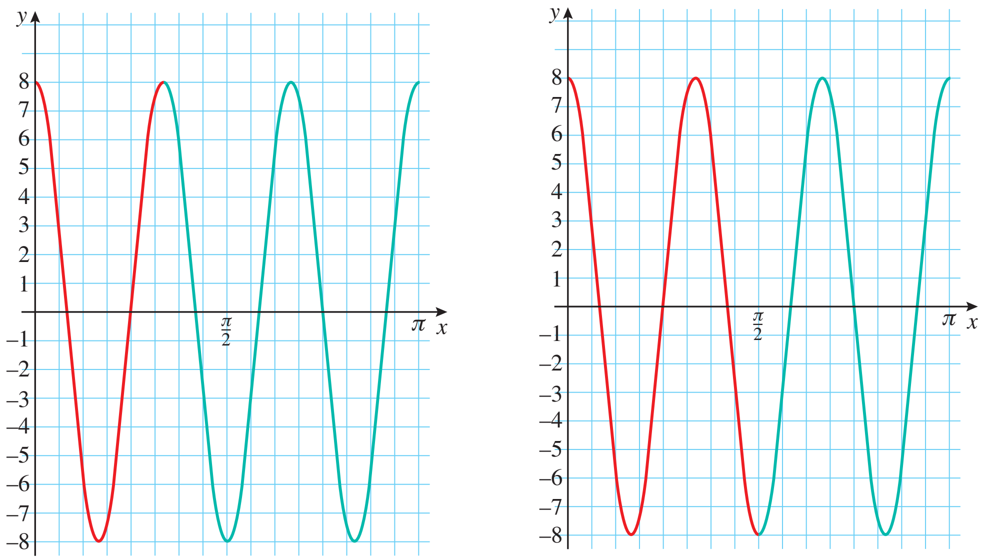
$$B = \frac{6.2 + 2.4}{2} = 4.3$$

To determine A, we have to determine the amplitude. The amplitude is the difference between the function's maximum value and the mid-line: $|A| = 6.2 - 4.3 = 1.9$. So, $A = \pm 1.9$. To determine which value of A is correct, we need to look at the graph. We see that the graph at $x = 0$ reaches maximum position; hence, the graph is either cosine, or sine with horizontal shift. If we choose a cosine graph, we use

$A = 1.9$ and the function is $y = 1.9\cos\left(\frac{\pi}{4}x\right) + 4.3$. If we choose a sine graph, we use $A = 1.9$ and the function is $y = 1.9\sin\left(\frac{\pi}{4}(x - 6)\right) + 4.3$ (since $x = 6$ is in initial position and the graph moves towards the maximum); or we can use $A = -1.9$ and then the function is $y = -1.9\sin\left(\frac{\pi}{4}(x - 2)\right) + 4.3$ (since $x = 2$ is in initial position and the graph moves towards the minimum).

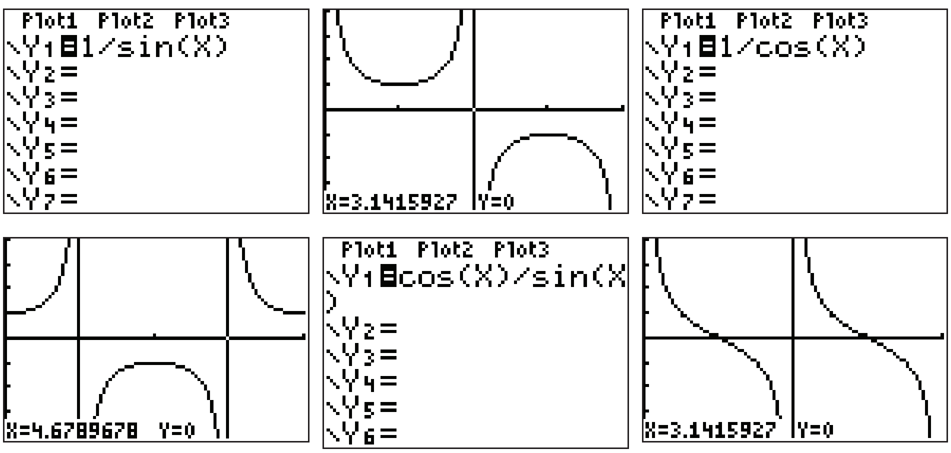
16 **a)** The constant p is equivalent to the constant a in the general form of a cosine function. The amplitude of the graph is 8, and the graph moves from its maximum to its minimum value, so a is positive; hence, $p = 8$.

b) The constant q is equivalent to the constant b in the general form of a cosine function. By inspecting the graph we can see the period.



Observing the graph, we notice that it is quite complicated to determine the value of the period, but we can easily see that one-and-a-half periods equals $\frac{\pi}{2}$. So, from $\frac{3}{2}$ period $= \frac{\pi}{2}$, we can conclude that the period is $\frac{\pi}{3}$. Hence: period $= \frac{2\pi}{q}$, so $\frac{\pi}{3} = \frac{2\pi}{q} \Rightarrow q = 6$.

17 a)



b) $y = \csc x$: range is $y \leqslant -1,\ y \geqslant 1$

$y = \sec x$: range is $y \leqslant -1,\ y \geqslant 1$

$y = \cot x$: range is $\mathbb{R}$

18 a) $a = \frac{1-(-3)}{2} = 2$

$c = \frac{1+(-3)}{2} = -1$

Fundamental period is $\frac{2\pi}{3}$, so: $\frac{2\pi}{3} = \frac{2\pi}{b} \Rightarrow b = 3$

b) We have to solve the equation $2\sin(3x) - 1 = 0 \Rightarrow \sin(3x) = \frac{1}{2}$.

The first solution of the equation $\sin(3x) = \frac{1}{2} \Rightarrow 3x = \frac{\pi}{6}$ is the first zero on the graph. The second solution of the equation $\sin(3x) = \frac{1}{2}$ is the x-coordinate of point P. Hence:

$$\sin(3x) = \frac{1}{2} \Rightarrow 3x = \frac{5\pi}{6} \Rightarrow x = \frac{5\pi}{18}$$

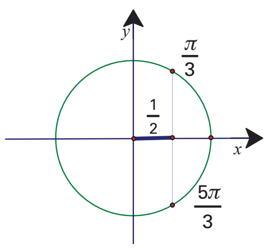
19 $a = \dfrac{2-(-4)}{2} = 3$

$c = \dfrac{2+(-4)}{2} = -1$

$b = \dfrac{\pi}{2} - \dfrac{3\pi}{4} = -\dfrac{\pi}{4}$

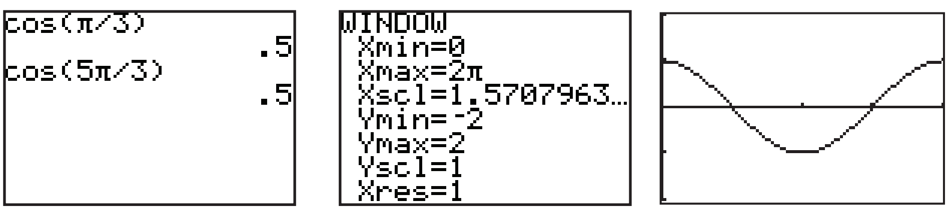
Exercise 7.4

1

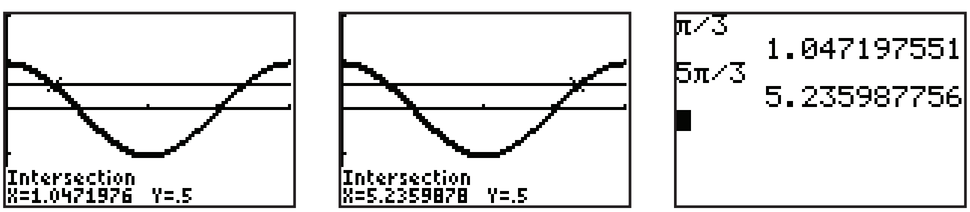


Solutions are: $\frac{\pi}{3}, \frac{5\pi}{3}$.

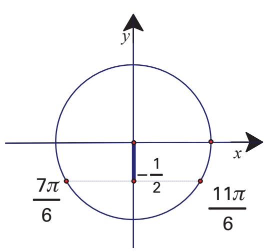
GDC verification:



Graphical solution:



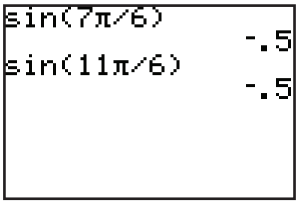
2



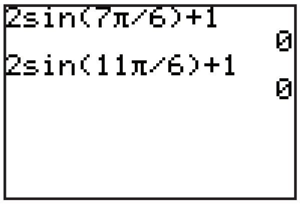
$2\sin x + 1 = 0 \Rightarrow 2\sin x = -1 \Rightarrow \sin x = -\frac{1}{2}$

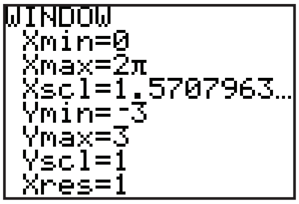
Solutions are: $\frac{7\pi}{6}, \frac{11\pi}{6}$.

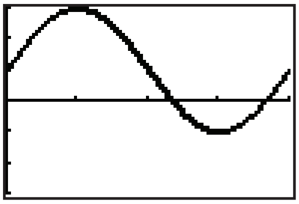
GDC verification:



or

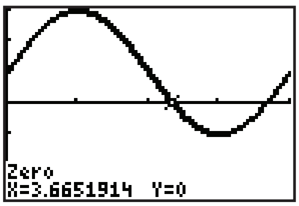


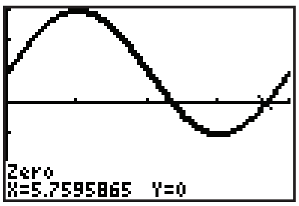


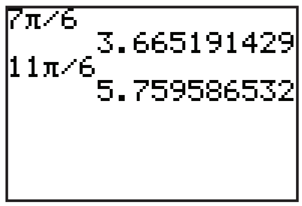


From the graph above we can see that there are only two solutions on the interval, and we have already checked that both of our solutions are correct!

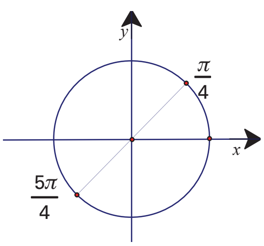
Graphical solution:







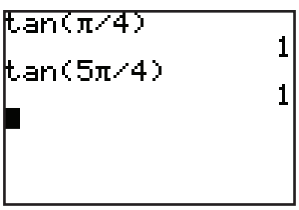
3



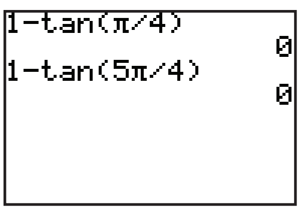
$1 - \tan x = 0 \Rightarrow \tan x = 1$

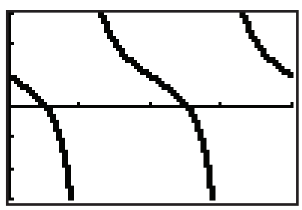
Solutions are: $\frac{\pi}{4}, \frac{5\pi}{4}$.

GDC verification:



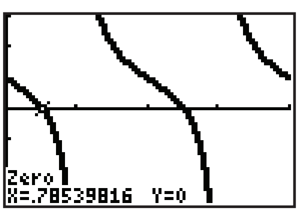
or

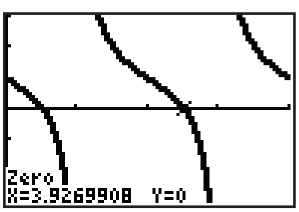


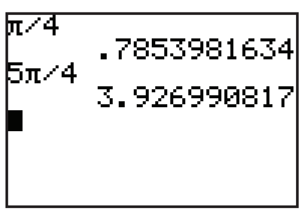


From the graph above we can see that there are only two solutions on the interval, and we have already checked that both of our solutions are correct!

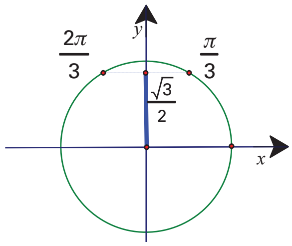
Graphical solution:







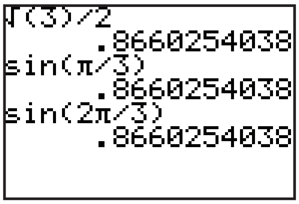
4



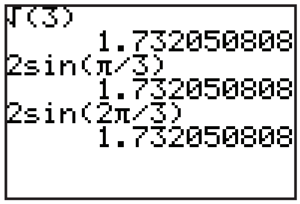
$$\sqrt{3} = 2\sin x \Rightarrow \frac{\sqrt{3}}{2} = \sin x$$

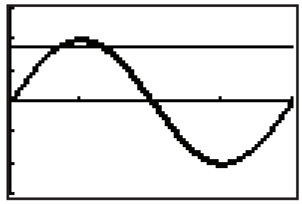
Solutions are: $\frac{\pi}{3}, \frac{2\pi}{3}$.

GDC verification:



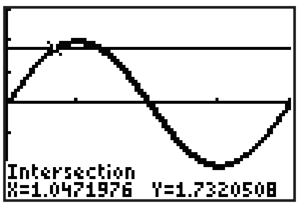
or

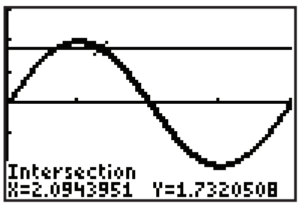


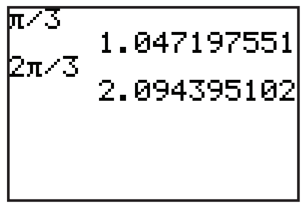


From the graph above we can see that there are only two solutions on the interval, and we have already checked that both of our solutions are correct!

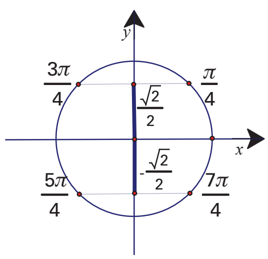
Graphical solution:







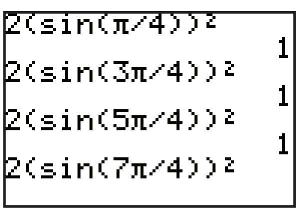
5

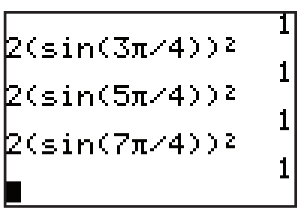


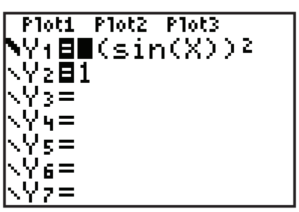
$$2\sin^2 x = 1 \Rightarrow (\sin x)^2 = \frac{1}{2} \Rightarrow \sin x = \pm\sqrt{\frac{1}{2}} \Rightarrow \sin x = \frac{\sqrt{2}}{2}, \text{ or } \sin x = -\frac{\sqrt{2}}{2}$$

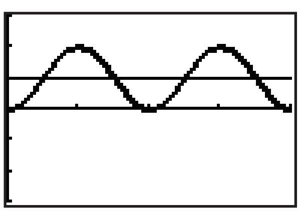
Solutions are: $\frac{\pi}{4}, \frac{3\pi}{4}, \frac{5\pi}{4}, \frac{7\pi}{4}$.

GDC verification:



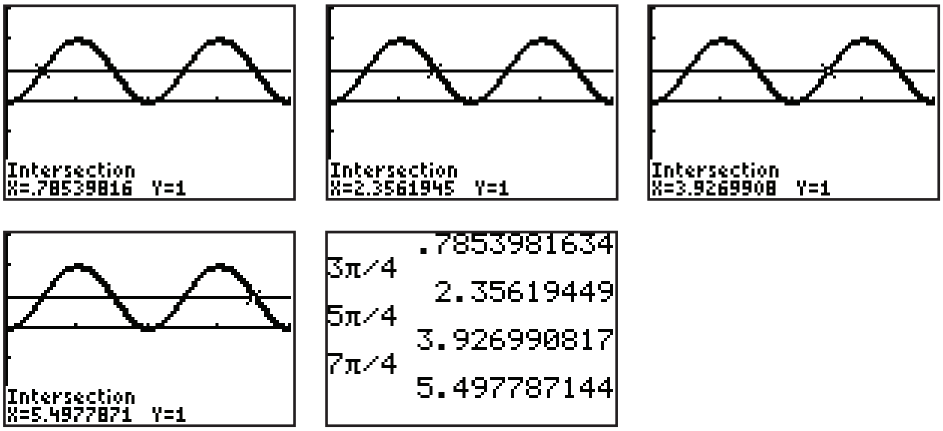




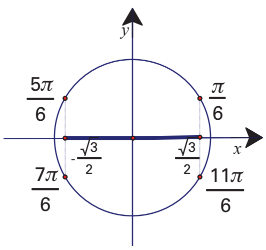


From the previous graph we can see that there are four solutions on the interval, and we have already checked that all our solutions are correct!

Graphical solution:



6

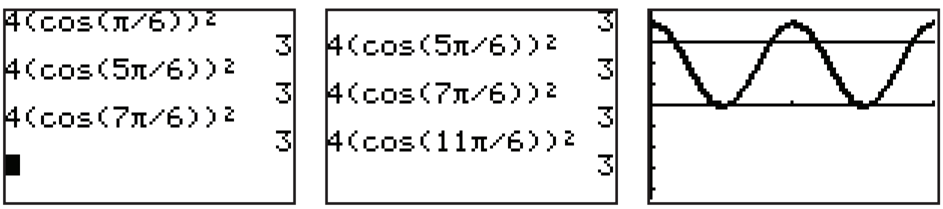


$$4\cos^2 x = 3 \Rightarrow (\cos x)^2 = \frac{3}{4} \Rightarrow \cos x = \pm\sqrt{\frac{3}{4}} \Rightarrow$$

$$\cos x = \frac{\sqrt{3}}{2}, \text{ or } \cos x = -\frac{\sqrt{3}}{2}$$

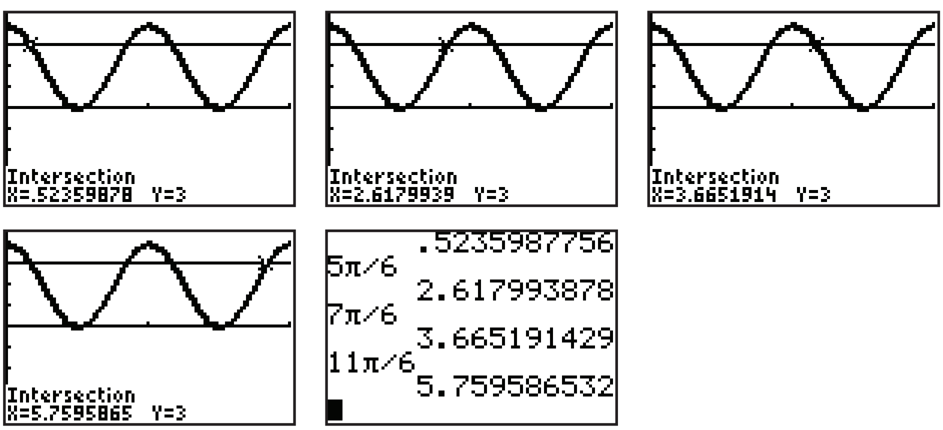
Solutions are: $\frac{\pi}{6}, \frac{5\pi}{6}, \frac{7\pi}{6}, \frac{11\pi}{6}$.

GDC verification:

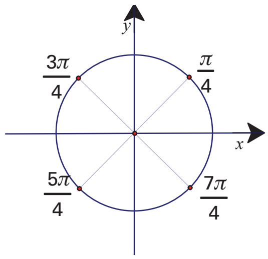


From the graph above we can see that there are four solutions on the interval, and we have already checked that all our solutions are correct!

Graphical solution:



7

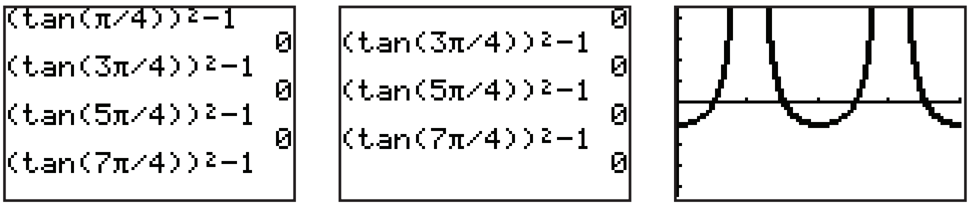


$\tan^2 x - 1 = 0 \Rightarrow (\tan x)^2 = 1 \Rightarrow \tan x = \pm 1 \Rightarrow$

$\tan x = 1$, or $\tan x = -1$

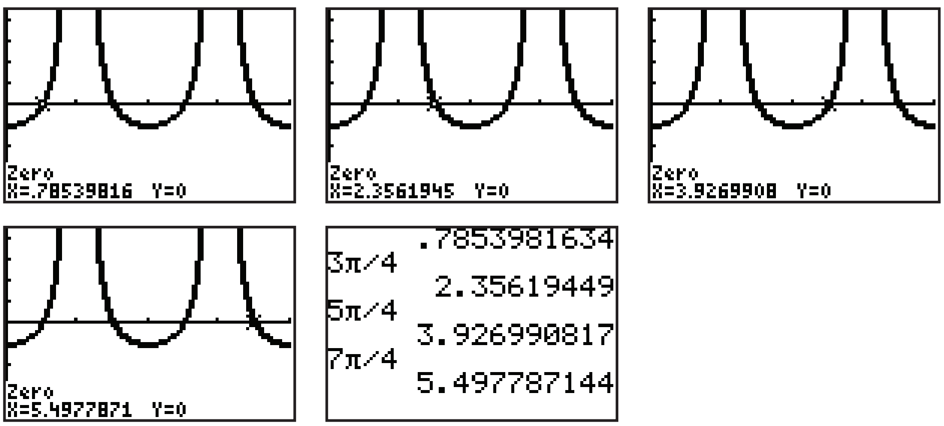
Solutions are: $\frac{\pi}{4}, \frac{3\pi}{4}, \frac{5\pi}{4}, \frac{7\pi}{4}$.

GDC verification:

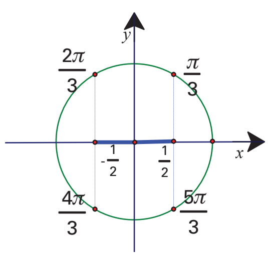


From the graph above we can see that there are four solutions on the interval, and we have already checked that all our solutions are correct!

Graphical solution:



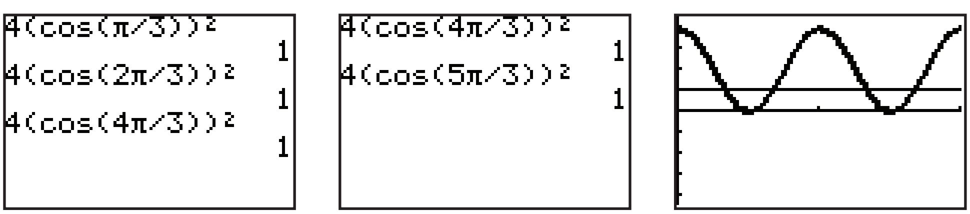
8



$4\cos^2 x = 1 \Rightarrow (\cos x)^2 = \frac{1}{4} \Rightarrow \cos x = \pm\sqrt{\frac{1}{4}} \Rightarrow \cos x = \frac{1}{2}$, or $\cos x = -\frac{1}{2}$

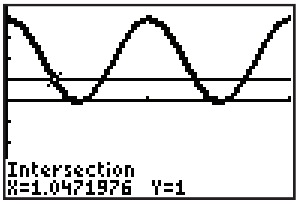
Solutions are: $\frac{\pi}{3}, \frac{2\pi}{3}, \frac{4\pi}{3}, \frac{5\pi}{3}$.

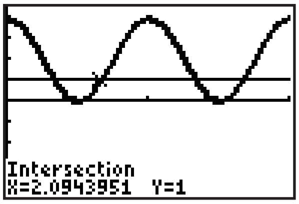
GDC verification:

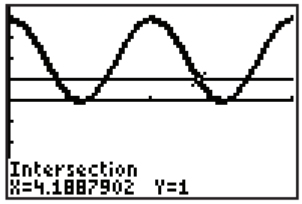


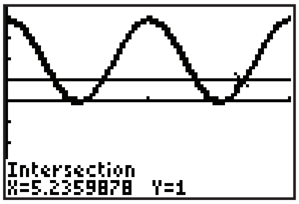
From the previous graph we can see that there are four solutions on the interval, and we have already checked that all our solutions are correct!

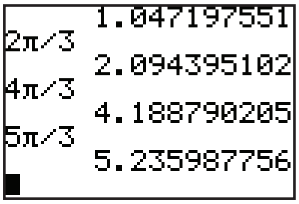
Graphical solution:







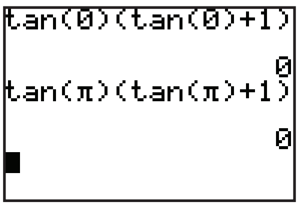


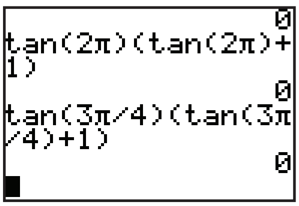


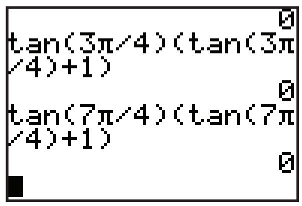
9 $\tan x(\tan x + 1) = 0 \Rightarrow \begin{cases} \tan x = 0 \Rightarrow x = 0, \pi, 2\pi \\ \tan x + 1 = 0 \Rightarrow \tan x = -1 \Rightarrow x = \dfrac{3\pi}{4}, \dfrac{7\pi}{4} \end{cases}$

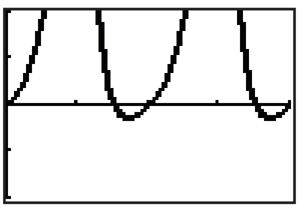
Solutions are: $0, \dfrac{3\pi}{4}, \pi, \dfrac{7\pi}{4}, 2\pi$.

GDC verification:





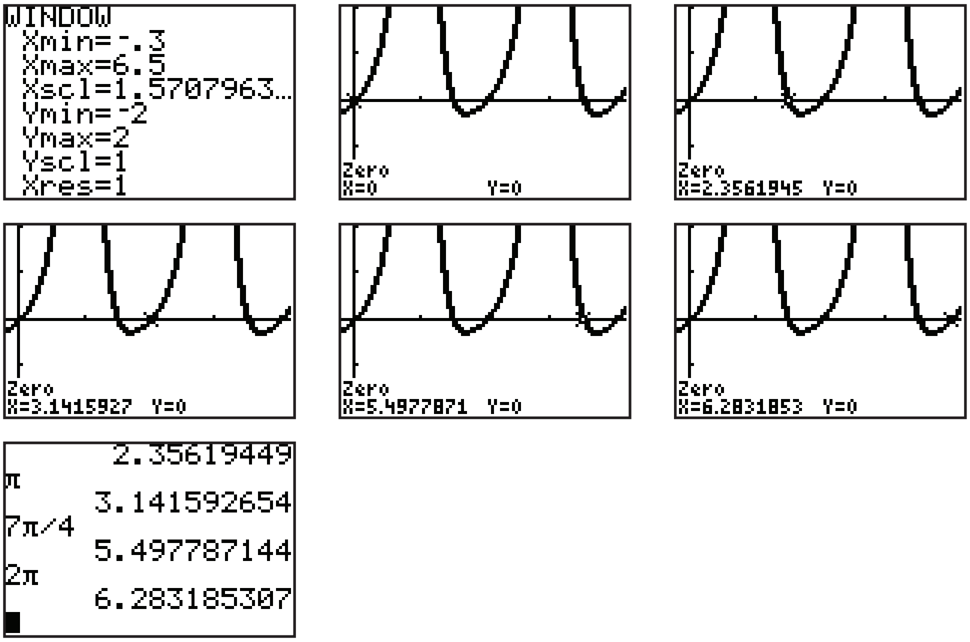




From the graph above we can see that there are five solutions on the interval, and we have already checked that all our solutions are correct!

Graphical solution:

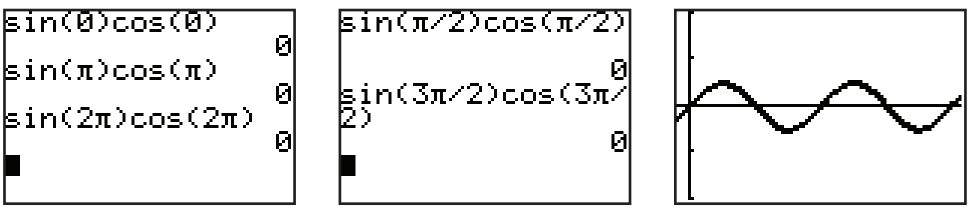
We have to enlarge a window to find the solutions at the endpoint of the interval.



10 $\sin x \cos x = 0 \Rightarrow \begin{cases} \sin x = 0 \Rightarrow x = 0, \pi, 2\pi \\ \cos x = 0 \Rightarrow x = \dfrac{\pi}{2}, \dfrac{3\pi}{2} \end{cases}$

Solutions are: $0, \dfrac{\pi}{2}, \pi, \dfrac{3\pi}{2}, 2\pi.$

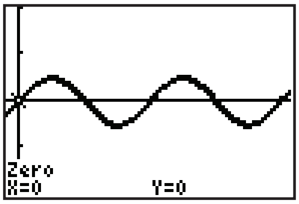
GDC verification:

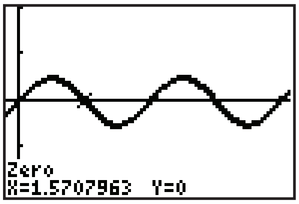


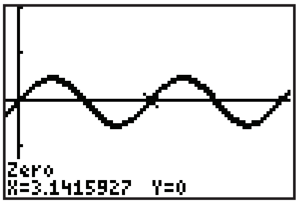
From the graph above we can see that there are five solutions on the interval, and we have already checked that all our solutions are correct!

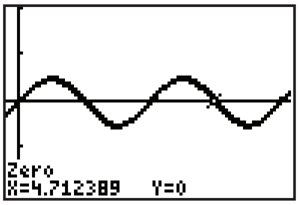
Graphical solution:

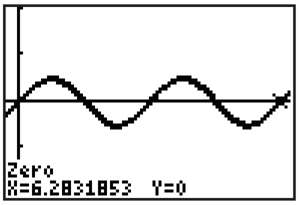
We have to enlarge a window to find the solutions at the endpoint of the interval.

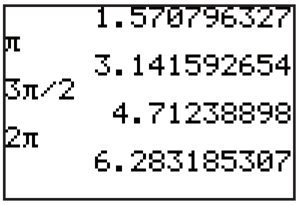




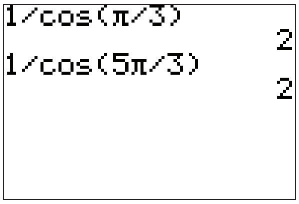


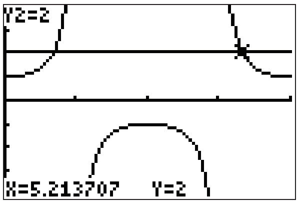






11 $\sec x = 2 \Rightarrow \dfrac{1}{\cos x} = 2 \Rightarrow x = \dfrac{\pi}{3}, \dfrac{5\pi}{3}$

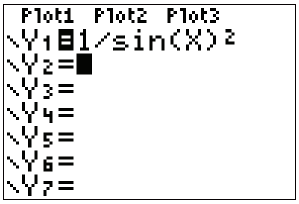


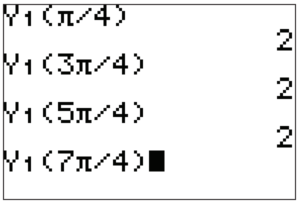


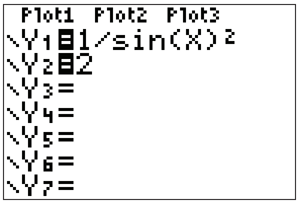
12 $\csc^2 x = 2 \Rightarrow \left(\dfrac{1}{\sin x}\right)^2$. Hence:

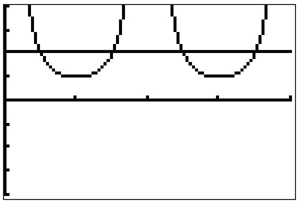
$$\frac{1}{\sin x} = \sqrt{2} \Rightarrow \sin x = \frac{\sqrt{2}}{2} \Rightarrow x = \frac{\pi}{4}, \frac{3\pi}{4}, \text{ or}$$

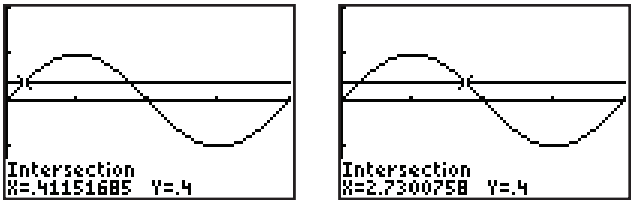
$$\frac{1}{\sin x} = -\sqrt{2} \Rightarrow \sin x = -\frac{\sqrt{2}}{2} \Rightarrow x = \frac{5\pi}{4}, \frac{7\pi}{4}$$



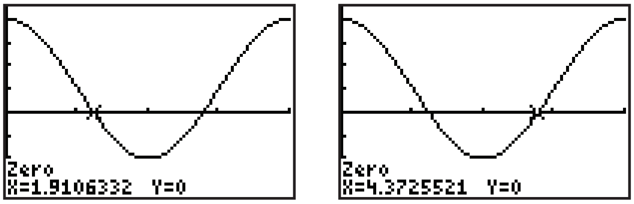




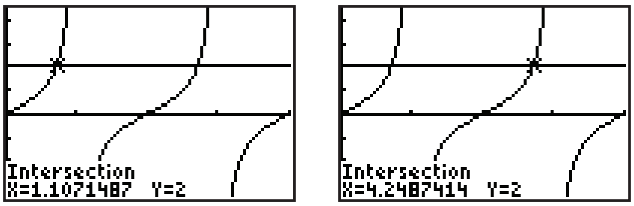


13 

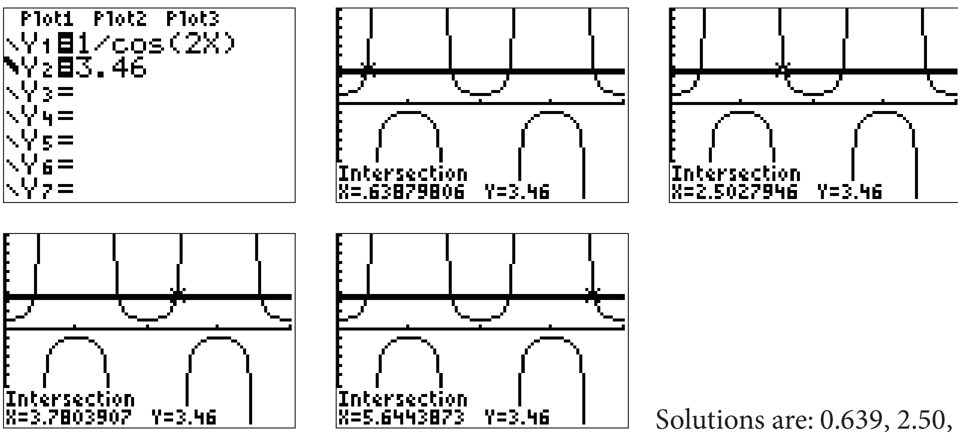
Solutions are: 0.412, 2.73.

14 

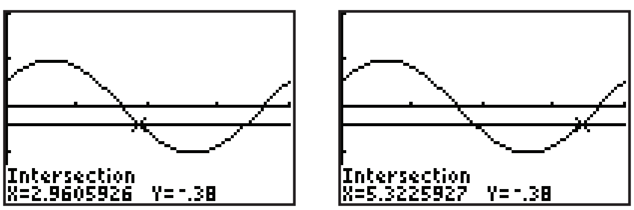
Solutions are: 1.91, 4.37.

15 

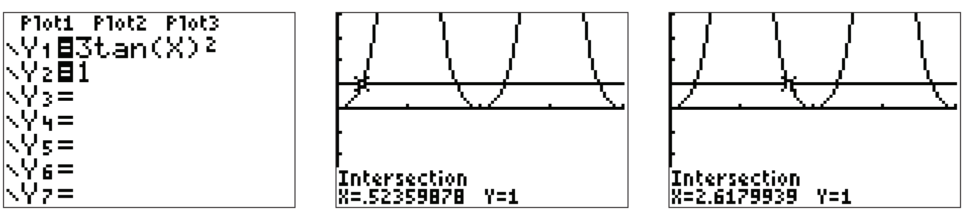
Solutions are: 1.11, 4.25.

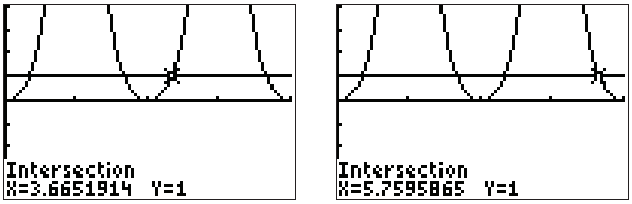
16 

Solutions are: 0.639, 2.50, 3.78, 5.64.

17 

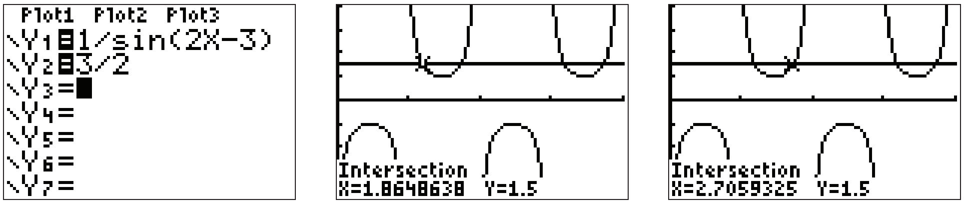
Solutions are: 2.96, 5.32.

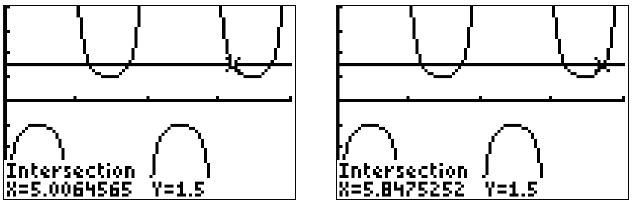
18 



Solutions are: 0.523, 2.62, 3.67, 5.76.

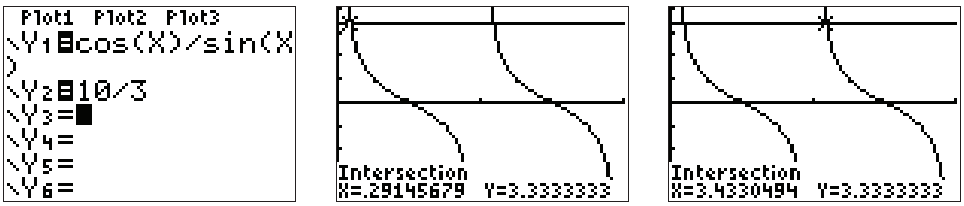
19





Solutions are: 1.86, 2.71, 5.01, 5.85.

20



Solutions are: 0.291, 3.43.

21 $x = \frac{\pi}{2} + k \cdot \pi$

k	0	1	2	3
x	$\frac{\pi}{2}$	$\frac{\pi}{2} + \pi = \frac{3\pi}{2}$	$\frac{\pi}{2} + 2\pi = \frac{5\pi}{2}$	~~$\frac{\pi}{2} + 3\pi > 3\pi$~~

k	−1	−2	−3	−4
x	$\frac{\pi}{2} - \pi = -\frac{\pi}{2}$	$\frac{\pi}{2} - 2\pi = -\frac{3\pi}{2}$	$\frac{\pi}{2} - 3\pi = -\frac{5\pi}{2}$	~~$\frac{\pi}{2} - 4\pi < -3\pi$~~

$$x = -\frac{5\pi}{2}, -\frac{3\pi}{2}, -\frac{\pi}{2}, \frac{\pi}{2}, \frac{3\pi}{2}, \frac{5\pi}{2}$$

22 $x = \frac{\pi}{6} + k \cdot 2\pi$

k	0	1	−1	−2
x	$\frac{\pi}{6}$	~~$\frac{\pi}{6} + 2\pi > 2\pi$~~	$\frac{\pi}{6} - 2\pi = -\frac{11\pi}{6}$	~~$\frac{\pi}{6} - 4\pi < -2\pi$~~

$$x = -\frac{11\pi}{6}, \frac{\pi}{6}$$

23 $x = \frac{7\pi}{12} + k \cdot \pi$

k	0	1	−1
x	$\frac{7\pi}{12}$	$\frac{7\pi}{12} + \pi = \frac{19\pi}{12}$	~~$\frac{7\pi}{12} + 2\pi > 2\pi$~~

$x = \frac{7\pi}{12}, \frac{19\pi}{12}$

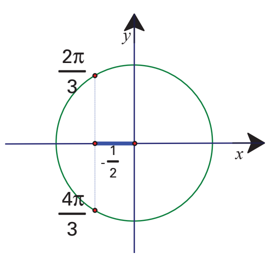
24 $x = \frac{\pi}{4} + k \cdot \frac{\pi}{4}$

k	0	1	2	3	4	5
x	$\frac{\pi}{4}$	$\frac{\pi}{4} + \frac{\pi}{4} = \frac{\pi}{2}$	$\frac{\pi}{4} + 2\frac{\pi}{4} = \frac{3\pi}{4}$	$\frac{\pi}{4} + 3\frac{\pi}{4} = \pi$	$\frac{\pi}{4} + 4\frac{\pi}{4} = \frac{5\pi}{4}$	$\frac{\pi}{4} + 5\frac{\pi}{4} = \frac{3\pi}{2}$

k	6	7	8	−1	−2
x	$\frac{\pi}{4} + 6\frac{\pi}{4} = \frac{7\pi}{4}$	$\frac{\pi}{4} + 7\frac{\pi}{4} = 2\pi$	~~$\frac{\pi}{4} + 8\frac{\pi}{4} > 2\pi$~~	$\frac{\pi}{4} - \frac{\pi}{4} = 0$	~~$\frac{\pi}{4} - 2\frac{\pi}{4} < 0$~~

$x = 0, \frac{\pi}{4}, \frac{\pi}{2}, \frac{3\pi}{4}, \pi, \frac{5\pi}{4}, \frac{3\pi}{2}, \frac{7\pi}{4}, 2\pi$

25



$\left\{..., -\frac{4\pi}{3}, -\frac{2\pi}{3}, \frac{2\pi}{3}, \frac{4\pi}{3}, ...\right\}$ have cosines equal to $-\frac{1}{2}$, that is,

$x - \frac{\pi}{6} = \pm\frac{2\pi}{3} + k \cdot 2\pi.$

So, either $x - \frac{\pi}{6} = \frac{2\pi}{3} + k \cdot 2\pi \Rightarrow x = \frac{\pi}{6} + \frac{2\pi}{3} + k \cdot 2\pi = \frac{5\pi}{6} + k \cdot 2\pi$, or

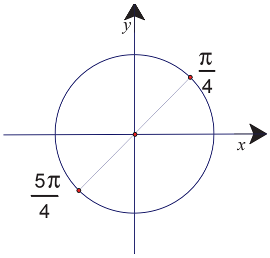
$x - \frac{\pi}{6} = -\frac{2\pi}{3} + k \cdot 2\pi \Rightarrow x = \frac{\pi}{6} - \frac{2\pi}{3} + k \cdot 2\pi = -\frac{\pi}{2} + k \cdot 2\pi.$

k	0	1
$x = \frac{5\pi}{6} + k \cdot 2\pi$	$\frac{5\pi}{6}$	~~$\frac{5\pi}{6} + 2\pi$~~

k	0	1	2
$x = -\frac{\pi}{2} + k \cdot 2\pi$	~~$-\frac{\pi}{2}$~~	$-\frac{\pi}{2} + 2\pi = \frac{3\pi}{2}$	~~$-\frac{\pi}{2} + 2 \cdot 2\pi$~~

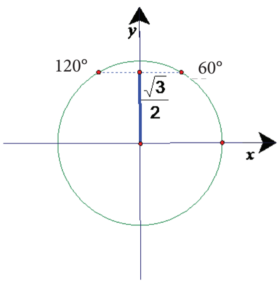
Solutions are: $\frac{5\pi}{6}, \frac{3\pi}{2}$.

26 Since the period of the tangent function is π, it holds that $\tan(\theta + \pi) = \tan\theta$. So, we have to solve the equation $\tan\theta = 1$.



$\left\{\ldots, -\frac{7\pi}{4}, -\frac{3\pi}{4}, \frac{\pi}{4}, \frac{5\pi}{4}, \frac{9\pi}{4}, \ldots\right\}$ have tangents equal to 1. So, from the set of listed values, we can see that the solutions are: $-\frac{3\pi}{4}, \frac{\pi}{4}$.

27



$\{\ldots, -300^\circ, -240^\circ, 60^\circ, 120^\circ, \ldots\}$ have sines equal to $\frac{\sqrt{3}}{2}$, that is,

$2x = 60^\circ + k\cdot 360^\circ \Rightarrow x = 30^\circ + k\cdot 180^\circ$, or

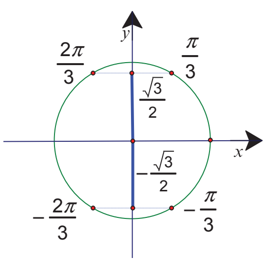
$2x = 120^\circ + k\cdot 360^\circ \Rightarrow x = 60^\circ + k\cdot 180^\circ$.

k	0	1	2
$x = 30^\circ + k\cdot 180^\circ$	30°	$30^\circ + 180^\circ = 210^\circ$	~~$30^\circ + 2\cdot 180^\circ$~~

k	0	1	2
$x = 60^\circ + k\cdot 180^\circ$	60°	$60^\circ + 180^\circ = 240^\circ$	~~$60^\circ + 2\cdot 180^\circ$~~

Solutions are: $30^\circ, 60^\circ, 210^\circ, 240^\circ$.

28 $\sin^2\left(\alpha + \frac{\pi}{2}\right) = \frac{3}{4} \Rightarrow \left(\sin\left(\alpha + \frac{\pi}{2}\right)\right)^2 = \frac{3}{4} \Rightarrow \sin\left(\alpha + \frac{\pi}{2}\right) = \pm\sqrt{\frac{3}{4}}$. So, we have to solve the following equations: $\sin\left(\alpha + \frac{\pi}{2}\right) = \frac{\sqrt{3}}{2}$ or $\sin\left(\alpha + \frac{\pi}{2}\right) = -\frac{\sqrt{3}}{2}$.



$\left\{\ldots, -\frac{2\pi}{3}, -\frac{\pi}{3}, \frac{\pi}{3}, \frac{2\pi}{3}, \ldots\right\}$ have sines equal to either $\frac{\sqrt{3}}{2}$ or $-\frac{\sqrt{3}}{2}$, that is,

$\alpha + \frac{\pi}{2} = \frac{\pi}{3} + k\cdot\pi \Rightarrow \alpha = -\frac{\pi}{6} + k\cdot\pi$, or

$\alpha + \frac{\pi}{2} = \frac{2\pi}{3} + k\cdot\pi \Rightarrow \alpha = \frac{\pi}{6} + k\cdot\pi$.

k	–1	0	1
$\alpha = -\frac{\pi}{6} + k\cdot\pi$	~~$-\frac{\pi}{6} - \pi = -\frac{7\pi}{6}$~~	$-\frac{\pi}{6}$	~~$-\frac{\pi}{6} + \pi = \frac{5\pi}{6}$~~

k	0	1
$\alpha = \frac{\pi}{6} + k\cdot\pi$	$\frac{\pi}{6}$	~~$\frac{\pi}{6} + \pi = \frac{7\pi}{6}$~~

Solutions are: $-\frac{\pi}{6}, \frac{\pi}{6}$.

29 We will use the substitution $\cos\theta = t \Rightarrow 2t^2 - 5t - 3 = 0 \Rightarrow t_1 = -\frac{1}{2}, t_2 = 3$. Since $\cos\theta = 3$ has no solutions, we have to solve: $\cos\theta = -\frac{1}{2}$. On the interval from 0 to 2π, the solutions are: $\frac{2\pi}{3}, \frac{4\pi}{3}$.

30 First write the tangent using sine and cosine, and then transform the equation:

$$3\frac{\sin x}{\cos x} = 2\cos x \Rightarrow 3\sin x = 2\cos^2 x$$

Write the equation in sine:

$$3\sin x = 2\left(1 - \sin^2 x\right) \Rightarrow 2\sin^2 x + 3\sin x - 2 = 0$$

The quadratic equation in sine has the solutions: $\sin x = -2$ (which is impossible and hence is not a solution) and $\sin x = \frac{1}{2}$, whose solutions on the interval from 0 to 2π are: $\frac{\pi}{6}, \frac{5\pi}{6}$.

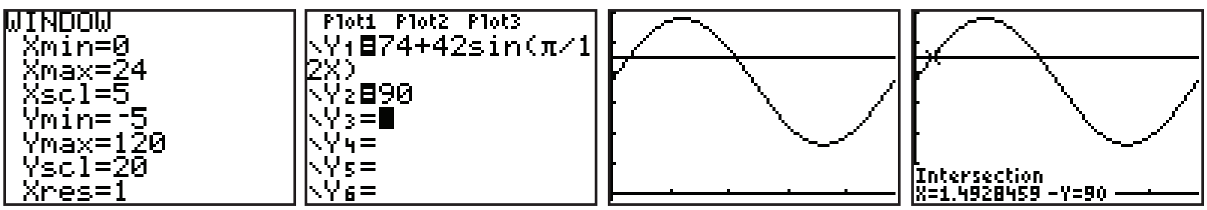
31 $\cos(x + 90^\circ) = \frac{\sqrt{2}}{2}$. Hence, $x + 90^\circ = 45^\circ + k\cdot 360^\circ \Rightarrow x = -45^\circ + k\cdot 360^\circ$, or $x + 90^\circ = -45^\circ + k\cdot 360^\circ \Rightarrow x = -135^\circ + k\cdot 360^\circ$. On the interval from 0 to 360°, the solutions are:

$225^\circ, 315^\circ$.

32 $\frac{9}{\cos^2\theta} = 12 \Rightarrow \cos^2\theta = \frac{3}{4}$. Hence, $\cos\theta = \frac{\sqrt{3}}{2}$, or $\cos\theta = -\frac{\sqrt{3}}{2}$. On the interval from 0 to π, the solutions are: $\frac{\pi}{6}, \frac{5\pi}{6}$.

33 We have been asked to approximate the answer to one decimal place, so we will use a graphical method for this task.

We have to choose an appropriate window:
Number of hours after midnight are from 1 to 24 – this will determine the x-values.
Values of the function range from $74 - 42 = 32$ to $74 + 42 = 116$ – this will determine the y-values.



The number of empty nests first equals 90 approximately 1.5 hours after midnight.

Solution Paper 1 type

34 a We have to solve the equation $H = 12$ on the interval from 0 to 365.

$$12 + 7.26\sin\left[\frac{2\pi}{365}(D - 80)\right] = 12 \Rightarrow 7.26\sin\left[\frac{2\pi}{365}(D - 80)\right] = 0 \Rightarrow \sin\left[\frac{2\pi}{365}(D - 80)\right] = 0$$

$\{\dots, -\pi, 0, \pi, 2\pi, \dots\}$ have sines equal to 0; therefore,

$$\frac{2\pi}{365}(D - 80) = k\cdot\pi \Rightarrow D - 80 = \frac{k\cdot 365}{2} \Rightarrow D = 80 + \frac{k\cdot 365}{2} \Rightarrow D = 80, 262.5$$

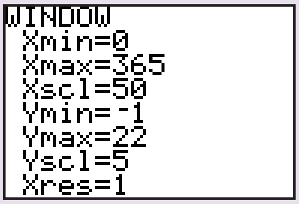
So, for the interval $[0, 365]$, the solutions are: $D = 80, 262.5$.

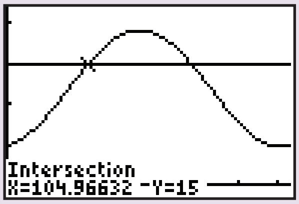
The 80th day of the year ($80 = 31 + 28 + 21$, March 21) has 12 hours of daylight.

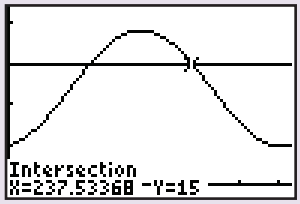
And, approximately the 263rd day ($263 = 31 + 28 + 31 + 30 + 31 + 30 + 31 + 31 + 20$, September 20) has 12 hours of daylight.

Solution Paper 2 type

34 b We have to solve the equation $H = 15$ on the interval from 0 to 365.



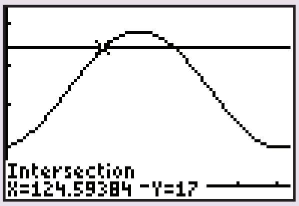


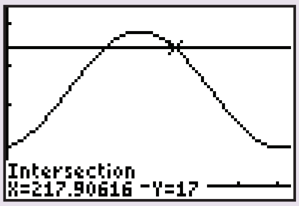


The 105th day (105 = 31 + 28 + 31 + 15, April 15) has 15 hours of daylight.

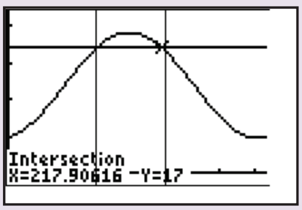
Also, the 238th day (238 = 31+ 28 + 31+ 30 + 31+ 30 + 31+ 26, August 26) has 15 hours of daylight.

c We have to solve the inequality $H > 17$ on the interval from 0 to 365.





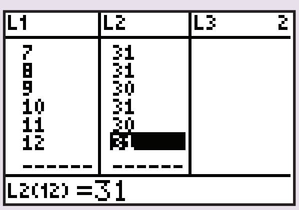
Solutions of the equation $H = 17$ are approximately 124.60 and 217.91.

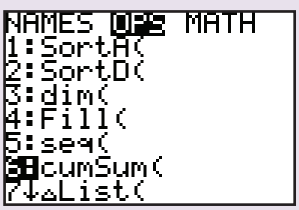


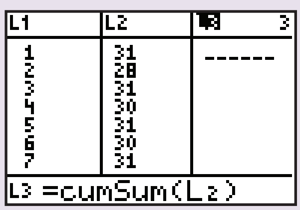
We can see that on the interval $[124.60, 217.91]$ the values of the function are greater than 17. So, from the 125th day to the 217th day, there will be more than 17 hours of daylight. This is a total of 93 days.

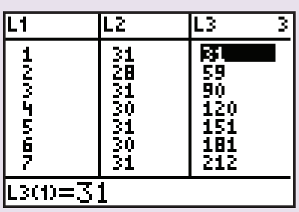
Note: To determine which date is on which day after December 31, we can use lists on a GDC.

In L1 we input a number for a month (e.g. 7 for July), in L2 we input the number of days in that month (e.g. 31 for July), and in L3 we input the cumulative sum of L2.









So, when we have to find out which day is, for example, the 105th, we can see that it is 15 days after the 90th day, so it is the 15th April. Similarly, the 125th day is 5 days after the 120th day, so it is the 5th May.

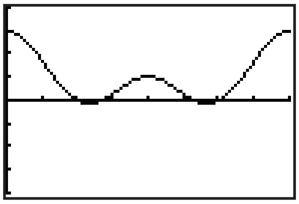
35 After factorizing we have:

$$2\cos^2 x + \cos x = 0 \Rightarrow \cos x\,(2\cos x + 1) = 0$$

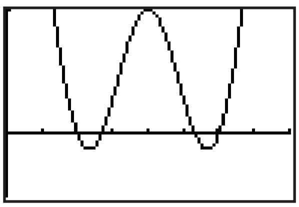
$$\Rightarrow \begin{cases} \cos x = 0 \Rightarrow x = \dfrac{\pi}{2}, \dfrac{3\pi}{2} \\ 2\cos x + 1 = 0 \Rightarrow \cos x = -\dfrac{1}{2} \Rightarrow x = \dfrac{2\pi}{3}, \dfrac{4\pi}{3} \end{cases}$$

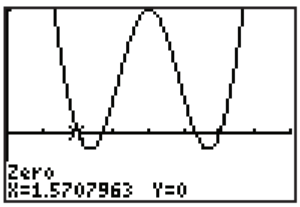
Solutions are: $\dfrac{\pi}{2}, \dfrac{2\pi}{3}, \dfrac{4\pi}{3}, \dfrac{3\pi}{2}$.

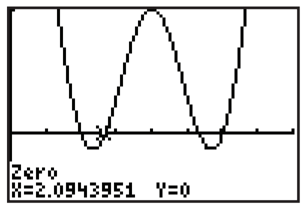
GDC verification:

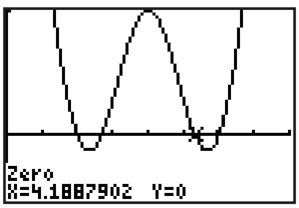


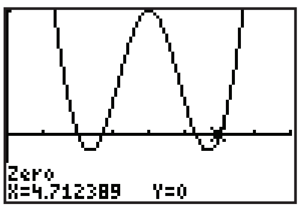
Change the window to distinguish the zero points:

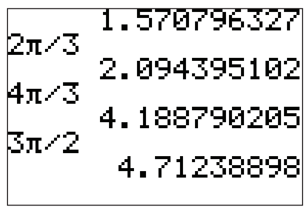












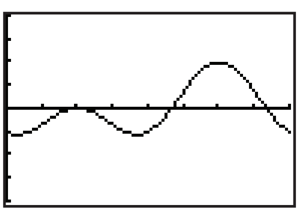
36 The equation $2\sin^2\theta - \sin\theta - 1 = 0$ is a quadratic equation in $\sin\theta$, so we can solve it by using the substitution $\sin\theta = t$.

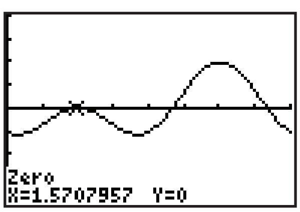
$$2t^2 - t - 1 = 0 \Rightarrow t_{1,2} = \frac{1 \pm \sqrt{1+8}}{4} \Rightarrow t_1 = 1 \text{ or } t_2 = -\frac{1}{2}$$

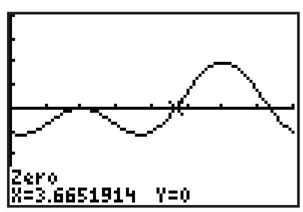
Hence, $\sin\theta = 1 \Rightarrow \theta = \dfrac{\pi}{2}$, or $\sin\theta = -\dfrac{1}{2} \Rightarrow \theta = \dfrac{7\pi}{6}, \dfrac{11\pi}{6}$.

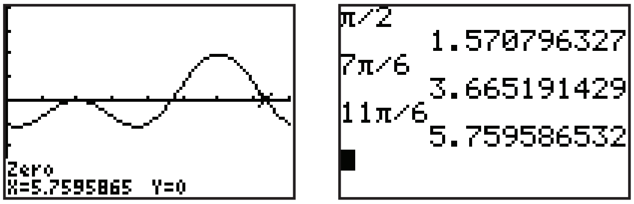
Solutions are: $\dfrac{\pi}{2}, \dfrac{7\pi}{6}, \dfrac{11\pi}{6}$.

GDC verification:







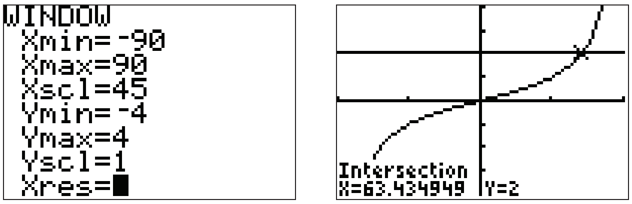


37 $\tan^2 x - \tan x = 2$

$t^2 - t - 2 = 0 \Rightarrow t_1 = -1, t_2 = 2$ Substitute: $\tan x = t$.

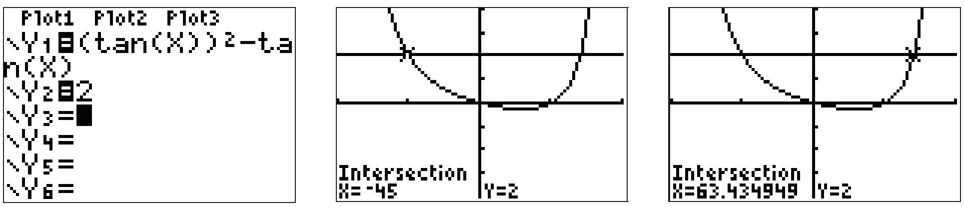
$\tan x = -1 \Rightarrow x = -45^\circ$

We will solve the equation $\tan x = 2$ graphically.



Solutions are: $-45^\circ, 63.4^\circ$.

GDC verification:

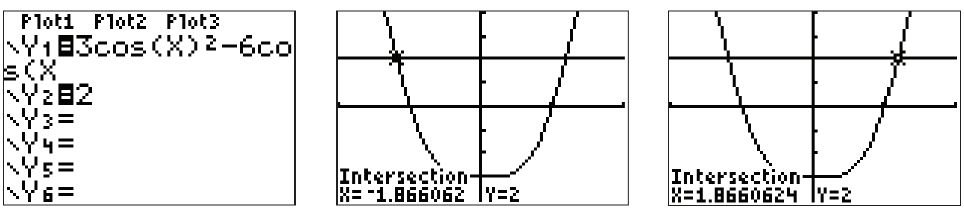


38 Using the substitution $\cos x = t$, and solving the quadratic equation:

$3t^2 - 6t - 2 = 0 \Rightarrow t_1 = -0.290\,99..., t_2 = 2.290\,99...$

$3t^2 - 6t - 2 = 0 \Rightarrow t_1 \approx -2.91, t_2 \approx 2.29$

We see that we cannot find exact solutions, so we will solve using a GDC.



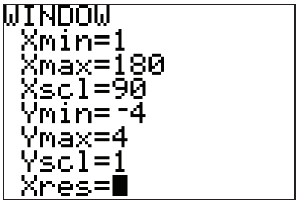
Solutions are: $-1.87, 1.87$.

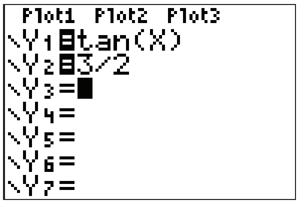
Note: An alternative method would be to continue with the quadratic equation and use a GDC to solve $\cos x = -2.91$ (has no solution) and $\cos x = -0.290\,99 \Rightarrow x = \pm 1.87$.

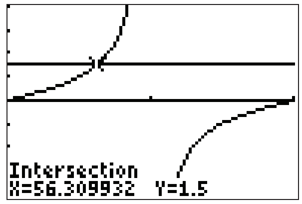
39 $2 \sin \beta = 3 \cos \beta$

We can divide the equation by $2 \cos \beta$: $\dfrac{\sin \beta}{\cos \beta} = \dfrac{3}{2} \Rightarrow \tan \beta = \dfrac{3}{2}$

We will solve the equation graphically.

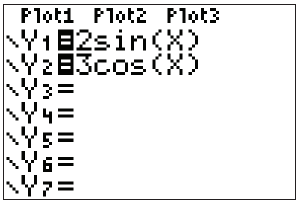


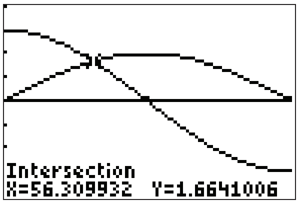




Solution is: 56.3°.

GDC verification:





40 $\sin^2 x = \cos^2 x$

$$\frac{\sin^2 x}{\cos^2 x} = 1$$

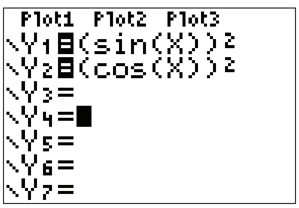
Divide both sides by $\cos^2 x$.

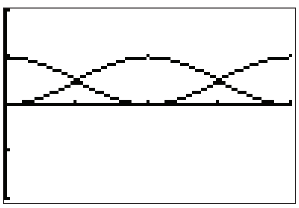
$\tan^2 x = 1 \Rightarrow \tan x = \pm 1$

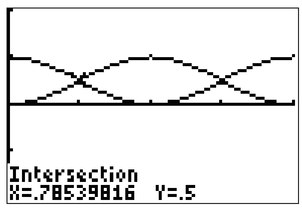
$\tan x = 1 \Rightarrow x = \dfrac{\pi}{4}$

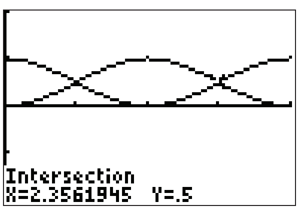
$\tan x = -1 \Rightarrow x = -\dfrac{\pi}{4} + \pi = \dfrac{3\pi}{4}$

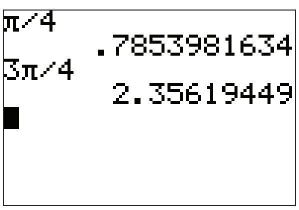
Solutions are: $\dfrac{\pi}{4}, \dfrac{3\pi}{4}$.







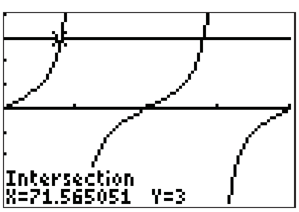


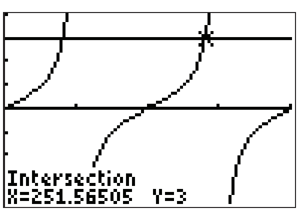


41 Using the substitution $\sec^2 x = t$, we have the quadratic equation $t^2 + 2t + 4 = 0$, which has no real solutions; hence, the equation has no solution.

42 Factorizing the equation we have: $\sin x \tan x - 3\sin x = 0 \Rightarrow \sin x\,(\tan x - 3) = 0$.

Hence, either $\sin x = 0 \Rightarrow 0°, 180°$, or $\tan x = 3$. We will use a graphical method to find the solutions.





Solutions are: 0°, 71.6°, 180°, 251.6°.

Exercise 7.5

1 $\cos\left(\frac{7\pi}{12}\right) = \cos\left(\frac{\pi}{3} + \frac{\pi}{4}\right) = \cos\frac{\pi}{3}\cos\frac{\pi}{4} - \sin\frac{\pi}{3}\sin\frac{\pi}{4} = \frac{1}{2}\frac{\sqrt{2}}{2} - \frac{\sqrt{3}}{2}\frac{\sqrt{2}}{2} = \frac{\sqrt{2}-\sqrt{6}}{4}$

2 $\sin 165^\circ = \sin(120^\circ + 45^\circ) = \sin 120^\circ \cos 45^\circ + \cos 120^\circ \sin 45^\circ$

$$= \frac{\sqrt{3}}{2}\frac{\sqrt{2}}{2} + \left(-\frac{1}{2}\right)\frac{\sqrt{2}}{2} = \frac{\sqrt{6}-\sqrt{2}}{4}$$

3 $\tan\left(\frac{\pi}{12}\right) = \tan\left(\frac{\pi}{3} - \frac{\pi}{4}\right) = \dfrac{\tan\frac{\pi}{3} - \tan\frac{\pi}{4}}{1 + \tan\frac{\pi}{3}\cdot\tan\frac{\pi}{4}} = \frac{\sqrt{3}-1}{1+\sqrt{3}}$

Rationalizing the denominator and simplifying the expression:

$$\frac{\sqrt{3}-1}{1+\sqrt{3}} = \frac{4-2\sqrt{2}}{2} = 2-\sqrt{3}$$

4 $\sin\left(-\frac{5\pi}{12}\right) = \sin\left(\frac{\pi}{4} - \frac{2\pi}{3}\right) = \sin\frac{\pi}{4}\cos\frac{2\pi}{3} - \cos\frac{\pi}{4}\sin\frac{2\pi}{3}$

$$= \frac{\sqrt{2}}{2}\left(-\frac{1}{2}\right) - \frac{\sqrt{2}}{2}\frac{\sqrt{3}}{2} = \frac{-\sqrt{2}-\sqrt{6}}{4}$$

5 $\cos 255^\circ = \cos(225^\circ + 30^\circ) = \cos 225^\circ \cos 30^\circ - \sin 225^\circ \sin 30^\circ$

$$= -\frac{\sqrt{2}}{2}\cdot\frac{\sqrt{3}}{2} - \left(-\frac{\sqrt{2}}{2}\cdot\frac{1}{2}\right) = \frac{-\sqrt{6}+\sqrt{2}}{4}$$

6 $\cot 75^\circ = \dfrac{1}{\tan 75^\circ} = \dfrac{1}{\tan(30^\circ + 45^\circ)} = \dfrac{1 - \tan 30^\circ \tan 45^\circ}{\tan 30^\circ + \tan 45^\circ}$

$$\frac{1 - \frac{1}{\sqrt{3}}\cdot 1}{\frac{1}{\sqrt{3}} + 1} = \frac{\frac{\sqrt{3}-1}{\sqrt{3}}}{\frac{1+\sqrt{3}}{\sqrt{3}}} = \frac{4-2\sqrt{3}}{2} = 2-\sqrt{3}$$

7 **a)** $\cos\left(\frac{\pi}{12}\right) = \cos\left(\frac{\pi}{3} - \frac{\pi}{4}\right) = \cos\frac{\pi}{3}\cos\frac{\pi}{4} + \sin\frac{\pi}{3}\sin\frac{\pi}{4} = \frac{1}{2}\frac{\sqrt{2}}{2} + \frac{\sqrt{3}}{2}\frac{\sqrt{2}}{2} = \frac{\sqrt{2}+\sqrt{6}}{4}$

b) $\frac{\sqrt{2}+\sqrt{6}}{4} = \cos\left(\frac{\pi}{12}\right) = \cos\left(2\frac{\pi}{24}\right) = 2\cos^2\frac{\pi}{24} - 1$. Hence:

$2\cos^2\frac{\pi}{24} - 1 = \frac{\sqrt{2}+\sqrt{6}}{4} \Rightarrow \cos^2\frac{\pi}{24} = \frac{\sqrt{2}+\sqrt{6}+4}{8}$. Since $\frac{\pi}{24}$ is in the first quadrant, $\cos\frac{\pi}{24}$ is positive, and thus: $\cos\frac{\pi}{24} = \sqrt{\frac{\sqrt{2}+\sqrt{6}+4}{8}}$.

8 $\tan\left(\frac{\pi}{2} - \theta\right) = \dfrac{\sin\left(\frac{\pi}{2} - \theta\right)}{\cos\left(\frac{\pi}{2} - \theta\right)} = \dfrac{\sin\frac{\pi}{2}\cos\theta - \cos\frac{\pi}{2}\sin\theta}{\cos\frac{\pi}{2}\cos\theta + \sin\frac{\pi}{2}\sin\theta} = \dfrac{1\cdot\cos\theta - 0}{0 + 1\cdot\sin\theta} = \dfrac{\cos\theta}{\sin\theta} = \cot\theta$

9 $\sin\left(\frac{\pi}{2}-\theta\right)=\sin\frac{\pi}{2}\cos\theta-\cos\frac{\pi}{2}\sin\theta=1\cdot\cos\theta-0=\cos\theta$

10 $\csc\left(\frac{\pi}{2}-\theta\right)=\frac{1}{\sin\left(\frac{\pi}{2}-\theta\right)}=\frac{1}{\sin\frac{\pi}{2}\cos\theta-\cos\frac{\pi}{2}\sin\theta}=\frac{1}{1\cdot\cos\theta-0}=\frac{1}{\cos\theta}=\sec\theta$

11 Given $0 < x < \frac{\pi}{2}$, it follows that $\cos x > 0$.

a) Using the Pythagorean identity for cosine:

$$\cos^2 x = 1-\sin^2 x \Rightarrow \cos x = +\sqrt{1-\sin^2 x}=\sqrt{1-\left(\frac{3}{5}\right)^2}=\sqrt{\frac{16}{25}}=\frac{4}{5}$$

b) Using the double angle identity for cosine:

$$\cos 2x = 1-2\sin^2 x = 1-2\left(\frac{3}{5}\right)^2=1-\frac{18}{25}=\frac{7}{25}$$

c) Using the double angle formula for sine and the result from **a**:

$$\sin 2x = 2\sin x\cos x = 2\cdot\frac{3}{5}\cdot\frac{4}{5}=\frac{24}{25}$$

12 Given $\frac{\pi}{2} < x < \pi$, it follows that $\sin x > 0$.

a) Using the Pythagorean identity for sine:

$$\sin^2 x = 1-\cos^2 x \Rightarrow \sin x = +\sqrt{1-\cos^2 x}=\sqrt{1-\left(-\frac{2}{3}\right)^2}=\sqrt{\frac{5}{9}}=\frac{\sqrt{5}}{3}$$

b) Using the double angle identity for sine and the result from **a**:

$$\sin 2x = 2\sin x\cos x = 2\cdot\frac{\sqrt{5}}{3}\cdot\frac{-2}{3}=-\frac{4\sqrt{5}}{9}$$

c) Using the double angle formula for cosine:

$$\cos 2x = 2\cos^2 x-1=2\left(-\frac{2}{3}\right)^2-1=\frac{8}{9}-1=-\frac{1}{9}$$

13 Firstly, we will find $\cos\theta$:

$\cos\theta=-\sqrt{1-\sin^2\theta}=-\sqrt{1-\frac{4}{9}}=-\frac{\sqrt{5}}{3}$. Now we can find the other values:

$$\sin 2\theta = 2\sin\theta\cos\theta = 2\cdot\frac{2}{3}\left(-\frac{\sqrt{5}}{3}\right)=-\frac{4\sqrt{5}}{9}$$

$$\cos 2\theta = 1-2\sin^2\theta = 1-2\frac{4}{9}=\frac{1}{9}$$

$$\tan 2\theta = \frac{\sin 2\theta}{\cos\theta}=\frac{-\frac{4\sqrt{5}}{9}}{\frac{1}{9}}=-4\sqrt{5}$$

Note: To find the value of $\cos 2\theta$ there is no need to find $\cos\theta$ first.

14 Firstly, we will find $\sin\theta$:

$\sin\theta=-\sqrt{1-\cos^2\theta}=-\sqrt{1-\frac{16}{25}}=-\frac{3}{5}$. Now we can find the other values:

$$\sin 2\theta = 2\sin\theta\cos\theta = 2\cdot\left(-\frac{3}{5}\right)\left(-\frac{4}{5}\right)=\frac{24}{25}$$

$$\cos 2\theta = 2\cos^2\theta - 1 = 2\left(\frac{16}{25}\right) - 1 = \frac{7}{25}$$

$$\tan 2\theta = \frac{\sin 2\theta}{\cos\theta} = \frac{\frac{24}{25}}{\frac{7}{25}} = \frac{24}{7}$$

Note: To find the value of $\cos 2\theta$ there is no need to find $\sin\theta$ first.

15 From $\tan\theta = 2 \Rightarrow \frac{\sin\theta}{\cos\theta} = 2 \Rightarrow \sin\theta = 2\cos\theta$, and using the Pythagorean identity for sine and cosine, we have: $\sin^2\theta + \cos^2\theta = 1 \Rightarrow (2\cos\theta)^2 + \cos^2\theta = 1 \Rightarrow \cos^2\theta = \frac{1}{5}$. For angles in the first quadrant, all values are positive; hence: $\cos\theta = \frac{\sqrt{5}}{5}$, $\sin\theta = \frac{2\sqrt{5}}{5}$. Now we can find the double angle values:

$$\sin 2\theta = 2\sin\theta\cos\theta = 2\cdot\left(\frac{2\sqrt{5}}{5}\right)\left(\frac{\sqrt{5}}{5}\right) = \frac{4}{5}$$

$$\cos 2\theta = 2\cos^2\theta - 1 = 2\left(\frac{1}{5}\right) - 1 = -\frac{3}{5}$$

$$\tan 2\theta = \frac{\sin 2\theta}{\cos\theta} = \frac{\frac{4}{5}}{-\frac{3}{5}} = -\frac{4}{3}$$

Note: To find the value of $\tan 2\theta$ there is no need to find $\cos\theta$ first. In this case we have to use the double angle formula for tangent: $\tan 2\theta = \frac{2\tan\theta}{1-\tan^2\theta} = \frac{4}{1-4} = -\frac{4}{3}$.

16 From $\sec\theta = \frac{1}{\cos\theta} = -4 \Rightarrow \cos\theta = -\frac{1}{4}$, we can find $\sin\theta$ ($\csc\theta$ is positive which means that $\sin\theta$ is positive): $\sin\theta = \sqrt{1-\left(-\frac{1}{4}\right)^2} = \frac{\sqrt{15}}{4}$. Now we can find the other values:

$$\sin 2\theta = 2\sin\theta\cos\theta = 2\cdot\frac{\sqrt{15}}{4}\left(-\frac{1}{4}\right) = -\frac{\sqrt{15}}{8}$$

$$\cos 2\theta = 2\cos^2\theta - 1 = 2\left(\frac{1}{16}\right) - 1 = -\frac{7}{8}$$

$$\tan 2\theta = \frac{\sin 2\theta}{\cos\theta} = \frac{-\frac{\sqrt{15}}{8}}{-\frac{7}{8}} = \frac{\sqrt{15}}{7}$$

17 $\cos(x-\pi) = \cos x\cos\pi + \sin x\sin\pi = \cos x\cdot(-1) + \sin x\cdot 0 = -\cos x$

18 $\sin\left(x-\frac{\pi}{2}\right) = \sin x\cos\frac{\pi}{2} - \cos x\sin\frac{\pi}{2} = \sin x\cdot 0 - \cos x\cdot 1 = -\cos x$

19 $\tan(x+\pi) = \tan(x+\pi) = \frac{\tan x + \tan\pi}{1-\tan x\cdot\tan\pi} = \frac{\tan x + 0}{1-\tan x\cdot 0} = \tan x$

20 $\cos\left(x+\frac{\pi}{2}\right) = \cos x\cos\frac{\pi}{2} - \sin x\sin\frac{\pi}{2} = \cos x\cdot 0 - \sin x\cdot 1 = -\sin x$

21 $\sec\theta + \sin\theta = \dfrac{1}{\cos\theta} + \sin\theta = \dfrac{1 + \sin\theta\cos\theta}{\cos\theta}$

22 $\dfrac{\sec\theta\csc\theta}{\tan\theta\sin\theta} = \dfrac{\dfrac{1}{\cancel{\cos\theta}\,\sin\theta}}{\dfrac{\sin\theta}{\cancel{\cos\theta}}\sin\theta} = \dfrac{\dfrac{1}{\sin\theta}}{\sin^2\theta} = \dfrac{1}{\sin^3\theta}$

23 $\dfrac{\sec\theta + \csc\theta}{2} = \dfrac{\dfrac{1}{\cos\theta} + \dfrac{1}{\sin\theta}}{2} = \dfrac{\dfrac{\sin\theta + \cos\theta}{\cos\theta\sin\theta}}{2} = \dfrac{\sin\theta + \cos\theta}{2\sin\theta\cos\theta} = \dfrac{\sin\theta + \cos\theta}{\sin 2\theta}$

24 $\dfrac{1}{\cos^2\theta} + \dfrac{1}{\cot^2\theta} = \dfrac{1}{\cos^2\theta} + \dfrac{1}{\dfrac{\cos^2\theta}{\sin^2\theta}} = \dfrac{1}{\cos^2\theta} + \dfrac{\sin^2\theta}{\cos^2\theta} = \dfrac{1 + \sin^2\theta}{\cos^2\theta}$

25 $\cos\theta - \cos\theta\sin^2\theta = \cos\theta\left(1 - \sin^2\theta\right) = \cos\theta\cos^2\theta = \cos^3\theta$

26 $\dfrac{1 - \cos^2\theta}{\sin^2\theta} = \dfrac{\sin^2\theta}{\sin^2\theta} = 1$

27 $\cos 2\theta + \sin^2\theta = \cos^2\theta - \sin^2\theta + \sin^2\theta = \cos^2\theta$

28 $\dfrac{\sin^2\theta}{\cos^2\theta} + \dfrac{1}{\cot^2\theta} = \dfrac{\sin^2\theta}{\cos^2\theta} + \dfrac{1}{\dfrac{\cos^2\theta}{\sin^2\theta}} = \dfrac{\sin^2\theta}{\cos^2\theta} + \dfrac{\sin^2\theta}{\cos^2\theta} = \dfrac{2\sin^2\theta}{\cos^2\theta} = 2\tan^2\theta$

29 $\sin(\alpha + \beta) + \sin(\alpha - \beta) = \sin\alpha\cos\beta + \cos\alpha\sin\beta + \sin\alpha\cos\beta - \cos\alpha\sin\beta = 2\sin\alpha\cos\beta$

30 $\dfrac{1 + \cos 2A}{2} = \dfrac{1 + 2\cos^2 A - 1}{2} = \dfrac{2\cos^2 A}{2} = \cos^2 A$

31 $\cos(\alpha + \beta) + \cos(\alpha - \beta) = \cos\alpha\cos\beta - \sin\alpha\sin\beta + \cos\alpha\cos\beta + \sin\alpha\sin\beta = 2\cos\alpha\cos\beta$

32 $2\cos^2\theta - \cos 2\theta = 2\cos^2\theta - \left(2\cos^2\theta - 1\right) = 1$

33 Using the double angle formula for cosine and the formula for the difference of squares, we have:

$$\frac{\cos 2\theta}{\cos\theta + \sin\theta} = \frac{\cos^2\theta - \sin^2\theta}{\cos\theta + \sin\theta} = \frac{\cancel{(\cos\theta + \sin\theta)}(\cos\theta - \sin\theta)}{\cancel{\cos\theta + \sin\theta}} = \cos\theta - \sin\theta$$

34 $(1 - \cos\alpha)(1 + \sec\alpha) = (1 - \cos\alpha)\left(1 + \dfrac{1}{\cos\alpha}\right) = (1 - \cos\alpha)\left(\dfrac{\cos\alpha + 1}{\cos\alpha}\right) = \dfrac{(1 - \cos\alpha)(\cos\alpha + 1)}{\cos\alpha}$

$$= \frac{1 - \cos^2\alpha}{\cos\alpha} = \frac{\sin^2\alpha}{\cos\alpha} = \frac{\sin\alpha}{\cos\alpha}\sin\alpha = \tan\alpha\sin\alpha$$

35 $\dfrac{1 - \tan^2 x}{1 + \tan^2 x} = \dfrac{1 - \dfrac{\sin^2 x}{\cos^2 x}}{1 + \dfrac{\sin^2 x}{\cos^2 x}} = \dfrac{\dfrac{\cos^2 x - \sin^2 x}{\cos^2 x}}{\dfrac{\cos^2 x + \sin^2 x}{\cos^2 x}} = \dfrac{\cos^2 x - \sin^2 x}{\underbrace{\cos^2 x + \sin^2 x}_{=1}} = \cos^2 x - \sin^2 x = \cos 2x$

36 Using the formula $(a^4 - b^4) = (a^2 - b^2)(a^2 + b^2)$, and then the Pythagorean identity for sine and cosine, and the double angle formula for cosine, we have:

$$\cos^4\theta - \sin^4\theta = (\cos^2\theta - \sin^2\theta)\underbrace{(\cos^2\theta + \sin^2\theta)}_{=1} = \cos^2\theta - \sin^2\theta = \cos 2\theta$$

37 $$\cot\theta - \tan\theta = \frac{1}{\tan\theta} - \tan\theta = \frac{1-\tan^2\theta}{\tan\theta} = \frac{1}{\dfrac{\tan\theta}{1-\tan^2\theta}}$$

We will multiply the numerator and denominator by 2 so that we can apply the double angle formula for tangent: $\dfrac{2}{2\dfrac{\tan\theta}{1-\tan^2\theta}} = \dfrac{2}{\tan 2\theta} = 2\cot 2\theta$

38 We will multiply the numerator and denominator by $\cos\beta + \sin\beta$:

$$\frac{(\cos\beta - \sin\beta)(\cos\beta + \sin\beta)}{(\cos\beta + \sin\beta)(\cos\beta + \sin\beta)} = \frac{\cos^2\beta - \sin^2\beta}{\cos^2\beta + 2\cos\beta\sin\beta + \sin^2\beta} = \frac{\cos 2\beta}{1 + 2\cos\beta\sin\beta} = \frac{\cos 2\beta}{1+\sin 2\beta}$$

39 $$\frac{1}{\sec\theta(1-\sin\theta)} = \frac{1}{\dfrac{1}{\cos\theta}(1-\sin\theta)}$$

Now, we will multiply the numerator and denominator by $1+\sin\theta$:

$$\frac{1+\sin\theta}{\dfrac{1}{\cos\theta}(1-\sin\theta)(1+\sin\theta)} = \frac{1+\sin\theta}{\dfrac{1-\sin^2\theta}{\cos\theta}} = \frac{1+\sin\theta}{\dfrac{\cos^2\theta}{\cos\theta}} = \frac{1+\sin\theta}{\cos\theta} = \frac{1}{\cos\theta} + \frac{\sin\theta}{\cos\theta} = \sec\theta + \tan\theta$$

40 $$(\tan A - \sec A)^2 = \left(\frac{\sin A}{\cos A} - \frac{1}{\cos A}\right)^2 = \frac{(\sin A - 1)^2}{\cos^2 A}$$

We will use the fact that the squares of opposite numbers are the same, and apply the Pythagorean identity for cosine:

$$= \frac{(1-\sin A)^2}{1-\sin^2 A} = \frac{(1-\sin A)^2}{(1-\sin A)(1+\sin A)} = \frac{1-\sin A}{1+\sin A}$$

41 $$\frac{\tan 2x\tan x}{\tan 2x - \tan x} = \frac{\dfrac{2\tan x}{1-\tan^2 x}\tan x}{\dfrac{2\tan x}{1-\tan^2 x} - \tan x} = \frac{\dfrac{2\tan^2 x}{1-\tan^2 x}}{\dfrac{2\tan x - \tan x + \tan^3 x}{1-\tan^2 x}} = \frac{2\tan^2 x}{\tan x + \tan^3 x}$$

$$= \frac{2\tan^2 x}{\tan x(1+\tan^2 x)} = \frac{2\tan x}{1+\tan^2 x} = \frac{2\dfrac{\sin x}{\cos x}}{1+\dfrac{\sin^2 x}{\cos^2 x}} = \frac{2\dfrac{\sin x}{\cos x}}{\dfrac{\cos^2 x + \sin^2 x}{\cos^2 x}} = \frac{2\sin x}{\dfrac{1}{\cos x}} = 2\sin x\cos x = \sin 2x$$

42 $$\frac{\sin 2\theta - \cos 2\theta + 1}{\sin 2\theta + \cos 2\theta + 1} = \frac{2\sin\theta\cos\theta - \cos^2\theta + \sin^2\theta + \cos^2\theta + \sin^2\theta}{2\sin\theta\cos\theta + \cos^2\theta - \sin^2\theta + \cos^2\theta + \sin^2\theta}$$

$$= \frac{2\sin\theta\cos\theta + 2\sin^2\theta}{2\sin\theta\cos\theta + 2\cos^2\theta} = \frac{2\sin\theta(\cos\theta + \sin\theta)}{2\cos\theta(\sin\theta + \cos\theta)} = \frac{\sin\theta}{\cos\theta} = \tan\theta$$

43 We will transform the right side of the identity:

$$2\csc\alpha - \frac{\sin\alpha}{1+\cos\alpha} = \frac{2}{\sin\alpha} - \frac{\sin\alpha}{1+\cos\alpha} = \frac{2(1+\cos\alpha)-\sin^2\alpha}{\sin\alpha(1+\cos\alpha)} = \frac{1+2\cos\alpha+1-\sin^2\alpha}{\sin\alpha(1+\cos\alpha)}$$

$$\frac{1+2\cos\alpha+\cos^2\alpha}{\sin\alpha(1+\cos\alpha)} = \frac{(1+\cos\alpha)^2}{\sin\alpha(1+\cos\alpha)} = \frac{\cos\alpha+1}{\sin\alpha}$$

44 $$\frac{1+\cos\beta}{\sin\beta} + \frac{\sin\beta}{1+\cos\beta} = \frac{(1+\cos\beta)(1+\cos\beta)+\sin^2\beta}{\sin\beta(1+\cos\beta)} = \frac{1+2\cos\beta+\cos^2\beta+\sin^2\beta}{\sin\beta(1+\cos\beta)}$$

$$= \frac{2+2\cos\beta}{\sin\beta(1+\cos\beta)} = \frac{2}{\sin\beta} = 2\csc\beta$$

45 $$\frac{\cot x-1}{1-\tan x} = \frac{\frac{\cos x}{\sin x}-1}{1-\frac{\sin x}{\cos x}} = \frac{\frac{\cos x-\sin x}{\sin x}}{\frac{\cos x-\sin x}{\cos x}} = \frac{\cos x}{\sin x} = \frac{1}{\sin x}\frac{1}{\frac{1}{\cos x}} = \csc x\frac{1}{\sec x} = \frac{\csc x}{\sec x}$$

46 Using the double angle formula for cosine, we have:

$$\cos 2x = 1-2\sin^2 x \Rightarrow 2\sin^2 x = 1-\cos 2x \Rightarrow \sin^2 x = \frac{1-\cos 2x}{2}$$

Using the substitution $x = \frac{\theta}{2}$, we have:

$$\sin^2\left(\frac{\theta}{2}\right) = \frac{1-\cos 2\left(\frac{\theta}{2}\right)}{2} = \frac{1-\cos\theta}{2} \Rightarrow \sin\left(\frac{\theta}{2}\right) = \pm\sqrt{\frac{1-\cos\theta}{2}}$$

47 Denote the angle opposite the leg of 2 in the smaller right triangle by α. From the figure, we can see that:

$\tan\alpha = \frac{2}{x}$ and $\tan(\alpha+\theta) = \frac{5+2}{x}$.

Using the compound formula for tangent, we can reduce the last expression.

$$\tan(\alpha+\theta) = \frac{\tan\alpha+\tan\theta}{1-\tan\alpha\cdot\tan\theta} = \frac{\frac{2}{x}+\tan\theta}{1-\frac{2}{x}\tan\theta} = \frac{7}{x}$$

Now, we will solve for $\tan\theta$:

$$\frac{\frac{2+x\tan\theta}{x}}{\frac{x-2\tan\theta}{x}} = \frac{7}{x} \Rightarrow \frac{2+x\tan\theta}{x-2\tan\theta} = \frac{7}{x} \Rightarrow x(2+x\tan\theta) = 7(x-2\tan\theta)$$

$$2x+x^2\tan\theta = 7x-14\tan\theta \Rightarrow x^2\tan\theta+14\tan\theta = 7x-2x$$

$$\Rightarrow (x^2+14)\tan\theta = 5x \Rightarrow \tan\theta = \frac{5x}{x^2+14}$$

48 We will rewrite the equation using cosine only:

$$2\left(1-\cos^2 x\right)-\cos x = 1 \Rightarrow -2\cos^2 x-\cos x+1 = 0$$

Using the substitution $\cos x = t$, we will have: $-2t^2-t+1=0 \Rightarrow t_1 = -1, t_2 = \frac{1}{2}$. Hence, either $\cos x = -1 \Rightarrow x = \pi$, or $\cos x = \frac{1}{2} \Rightarrow x = \frac{\pi}{3}, \frac{5\pi}{3}$. Solutions are: $\frac{\pi}{3}, \pi, \frac{5\pi}{3}$.

49 $\dfrac{1}{\cos^2 x} = 8\cos x \Rightarrow \cos^3 x = \dfrac{1}{8} \Rightarrow \cos x = \dfrac{1}{2} \Rightarrow x = -\dfrac{\pi}{3}, \dfrac{\pi}{3}$

50 $2\cos x + \sin 2x = 0$

After using the double angle identity for sine, we have: $2\cos x + 2\sin x\cos x = 0$.

Factorizing:

$2\cos x\,(1 + \sin x) = 0$

$$\Rightarrow \begin{array}{l} \cos x = 0 \Rightarrow x = 90^\circ, -90^\circ \\ 1 + \sin x = 0 \Rightarrow \sin x = -1 \Rightarrow x = -90^\circ \end{array}$$

Solutions are: $90^\circ, -90^\circ$.

51 $2\sin x = \cos 2x$

After using the double angle identity for cosine, we have:

$2\sin x = 1 - 2\sin^2 x$

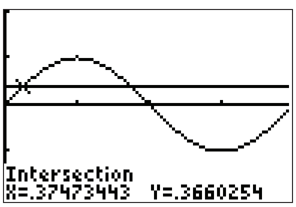
$2\sin^2 x + 2\sin x - 1 = 0$

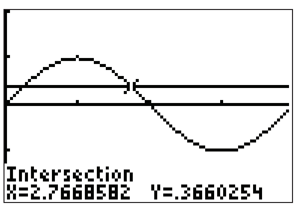
Using the substitution $\sin x = t$:

$2t^2 + 2t - 1 = 0 \Rightarrow t_{1,2} = \dfrac{-2 \pm \sqrt{4+8}}{4} = \dfrac{-1 \pm \sqrt{3}}{2}$

$\sin x = \dfrac{-1-\sqrt{3}}{2}$ has no solution because $\dfrac{-1-\sqrt{3}}{2} < -1$.

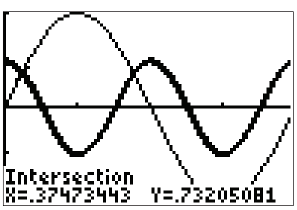
$\sin x = \dfrac{-1+\sqrt{3}}{2}$ cannot be solved exactly, so we will use a GDC.

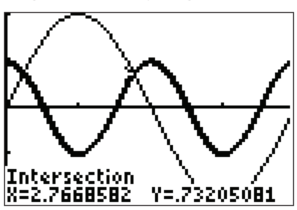




Solutions are: 0.375, 2.77.

Note: We can solve the task graphically right from the start.

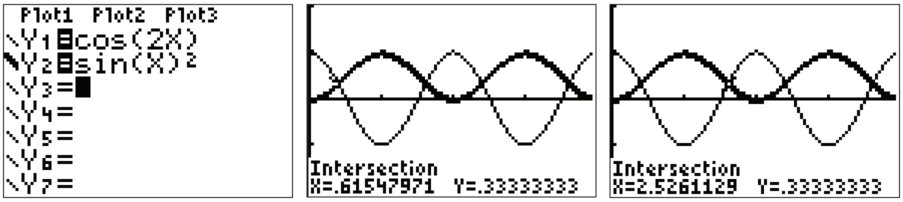


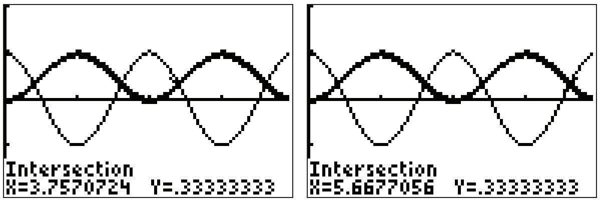


52 We will rewrite the equation using cosine only:

$1 - 2\sin^2 x = \sin^2 x \Rightarrow 3\sin^2 x = 1 \Rightarrow \sin x = \pm\dfrac{\sqrt{3}}{3}$

We cannot solve the equations exactly, so we will use a GDC.





Solutions are: 0.615, 2.53, 3.76, 5.67.

53 Using the double angle formula for sine:

$$\sin 2x = -1 \Rightarrow 2x = \frac{3\pi}{2} + 2k\pi \Rightarrow x = \frac{3\pi}{4} + k\pi$$

Hence, the solutions are: $\frac{3\pi}{4}, \frac{7\pi}{4}$.

54 Using the double angle formula for cosine:

$$\cos 2x = -\frac{1}{2} \Rightarrow 2x = \pm\frac{2\pi}{3} + 2k\pi \Rightarrow x = \pm\frac{\pi}{3} + k\pi$$

Hence, the solutions are: $\frac{\pi}{3}, \frac{2\pi}{3}$.

55 Using the definition of the secant function and writing 1 as: $\cos^2 x + \sin^2 x$.

$$\frac{1}{\cos^2 x} - \tan x - 1 = 0 \Rightarrow \frac{\cos^2 x + \sin^2 x}{\cos^2 x} - \tan x - 1 = 0 \Rightarrow$$

$$1 + \tan^2 x - \tan x - 1 = 0 \Rightarrow \tan^2 x - \tan x = 0$$

Factorizing the equation:

$$\tan^2 x - \tan x = 0 \Rightarrow \tan x(\tan x - 1) = 0$$

Hence, either: $\tan x = 0 \Rightarrow 0, \pi$, or $\tan x = 1 \Rightarrow \frac{\pi}{4}, \frac{5\pi}{4}$

Solutions are: $0, \frac{\pi}{4}, \pi, \frac{5\pi}{4}$.

56 Using the double angle formula for tangent:

$$\frac{2\tan x}{1-\tan^2 x} + \tan x = 0 \Rightarrow \tan x\left(\frac{2}{1-\tan^2 x} + 1\right) = 0 \Rightarrow \tan x\left(\frac{2+1-\tan^2 x}{1-\tan^2 x}\right) = 0$$

Hence, either: $\tan x = 0 \Rightarrow x = 0, \pi$, or $\frac{3-\tan^2 x}{1-\tan^2 x} = 0 \Rightarrow \tan x = \pm\sqrt{3} \Rightarrow \frac{\pi}{3}, \frac{2\pi}{3}, \frac{4\pi}{3}, \frac{5\pi}{3}$

Solutions are: $0, \frac{\pi}{3}, \pi, \frac{2\pi}{3}, \frac{4\pi}{3}, \frac{5\pi}{3}$.

57 Factorize the equation: $\cos 3x\,(2\sin 2x + 1) = 0$

Hence, either: $\cos 3x = 0 \Rightarrow 3x = \pm 90^\circ + k360^\circ \Rightarrow x = \pm 30^\circ + k120^\circ \Rightarrow x = 30^\circ, 90^\circ, 150^\circ$, or

$$\sin 2x = -\frac{1}{2} \Rightarrow \begin{cases} 2x = 210^\circ + k360^\circ \Rightarrow x = 105^\circ + k180^\circ \\ 2x = -30^\circ + k360^\circ \Rightarrow x = -15^\circ + k180^\circ \end{cases} \Rightarrow x = 105^\circ, 165^\circ$$

Solutions are: $30^\circ, 90^\circ, 105^\circ, 150^\circ, 165^\circ$.

58 Use the compound formula for $3x = 2x + x$, and the double angle formula for sine and cosine:
$\sin(2x + x) = \sin 2x \cos x + \cos 2x \sin x = 2 \sin x \cos^2 x + (1 - 2\sin^2 x) \sin x$

Now simplify: $2 \sin x(1 - \sin^2 x) + \sin x - 2\sin^3 x = 3 \sin x - 4 \sin^3 x$

59 **a)** $(\sin^2 x + \cos^2 x)^2 = 1 \Rightarrow \sin^4 x + 2\sin^2 x \cos^2 x + \cos^4 x = 1$

$$\Rightarrow \sin^4 x + \cos^4 x + \frac{1}{2} 4 \sin^2 x \cos^2 x = 1$$

$$\Rightarrow \sin^4 x + \cos^4 x + \frac{1}{2}(\sin 2x)^2 = 1$$

$$\Rightarrow \sin^4 x + \cos^4 x = 1 - \frac{1}{2}\sin^2 2x$$

We have to transform $\sin^2 2x$ in such a way that we obtain $\cos 8x$:

$\sin^2 2x = -\frac{1}{2}(1 - 2\sin^2 2x - 1) = -\frac{1}{2}(\cos 4x - 1)$

Now, we will return to the formula:

$\sin^4 x + \cos^4 x = 1 - \frac{1}{2}\sin^2 2x \Rightarrow \sin^4 x + \cos^4 x = 1 - \frac{1}{2}\left(-\frac{1}{2}(\cos 4x - 1)\right)$

$$\Rightarrow \sin^4 x + \cos^4 x = 1 + \frac{1}{4}\cos 4x - \frac{1}{4}$$

Hence, $\sin^4 x + \cos^4 x = \frac{1}{4}(\cos 4x + 3)$.

b) Using the formula from **a**, we have: $\frac{1}{4}(\cos 4x + 3) = \frac{1}{2} \Rightarrow \cos 4x = -1 \Rightarrow 4x = \pi + 2k\pi$

Hence: $x = \frac{\pi}{4} + k\frac{\pi}{2}$

Solutions are: $\frac{\pi}{4}, \frac{3\pi}{4}, \frac{5\pi}{4}, \frac{7\pi}{4}$.

Exercise 7.6

1 Since $\sin\frac{\pi}{2} = 1$, then $\arcsin 1 = \frac{\pi}{2}$.

2 Since $\cos\frac{\pi}{4} = \frac{1}{\sqrt{2}}$, then $\arccos\frac{1}{\sqrt{2}} = \frac{\pi}{4}$.

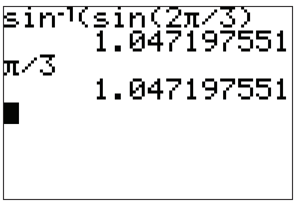
3 Since $\tan\left(-\frac{\pi}{3}\right) = -\sqrt{3}$, then $\arctan -\sqrt{3} = -\frac{\pi}{3}$.

4 Since $\cos\frac{2\pi}{3} = -\frac{1}{2}$, then $\arccos\left(-\frac{1}{2}\right) = \frac{2\pi}{3}$.

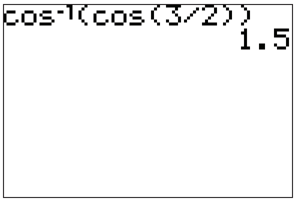
5 Since $\tan 0 = 0$, then $\arctan 0 = 0$.

6 Since $\sin\left(-\frac{\pi}{3}\right) = -\frac{\sqrt{3}}{2}$, then $\arcsin\left(-\frac{\sqrt{3}}{2}\right) = -\frac{\pi}{3}$.

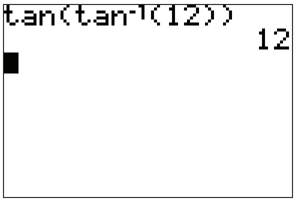
7 Since $\sin\frac{2\pi}{3} = \sin\frac{\pi}{3}$, then $\sin^{-1}\left(\sin\frac{2\pi}{3}\right) = \frac{\pi}{3}$.



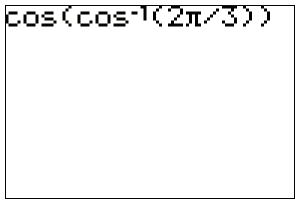
8 $\dfrac{3}{2}$

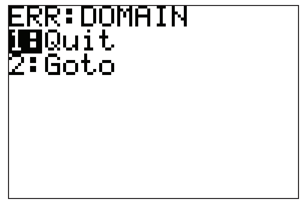


9 12

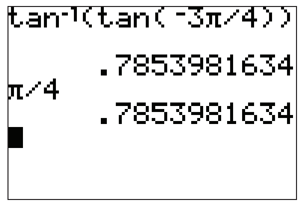


10 $\arccos\left(\dfrac{2\pi}{3}\right)$ is not defined and hence it is not possible.

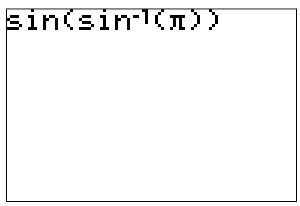


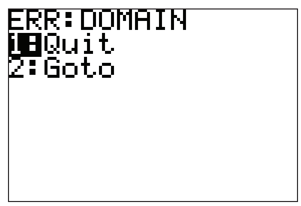


11 Since $\tan\left(-\dfrac{3\pi}{4}\right) = \tan\dfrac{\pi}{4}$, then $\arctan\left(\tan\left(-\dfrac{3\pi}{4}\right)\right) = \dfrac{\pi}{4}$.

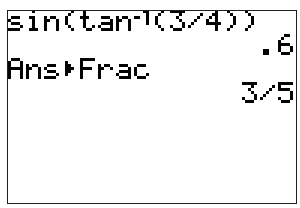


12 $\arcsin \pi$ is not defined and hence it is not possible.

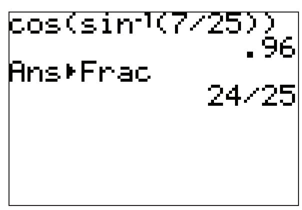




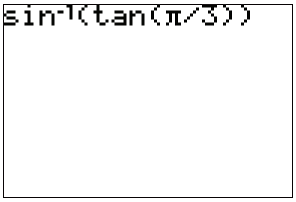
13 If $\tan\theta = \dfrac{3}{4}$, then $\sin\theta = \dfrac{3}{\sqrt{3^2+4^2}} = \dfrac{3}{5}$; hence, $\sin\left(\arctan\dfrac{3}{4}\right) = \dfrac{3}{5}$.

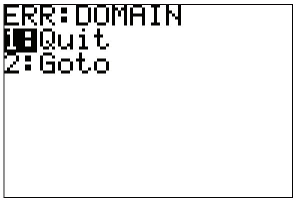


14 If $\sin\theta = \dfrac{7}{25}$, then $\cos\theta = \dfrac{\sqrt{25^2-7^2}}{25} = \dfrac{24}{25}$; hence, $\cos\left(\arcsin\left(\dfrac{7}{25}\right)\right) = \dfrac{24}{25}$.

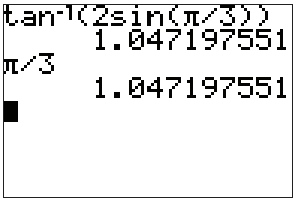


15 Since $\tan\frac{\pi}{3} = \sqrt{3} > 1$, it is not possible.

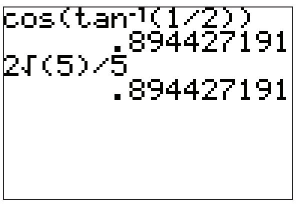




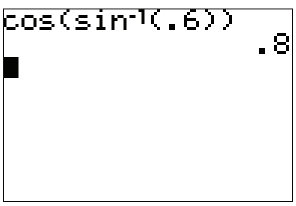
16 $\tan^{-1}\left(2\cdot\frac{\sqrt{3}}{2}\right) = \tan^{-1}\sqrt{3} = \frac{\pi}{3}$



17 If $\tan\theta = \frac{1}{2}$, then $\cos\theta = \frac{2}{\sqrt{1^2+2^2}} = \frac{2}{\sqrt{5}} = \frac{2\sqrt{5}}{5}$; hence, $\cos\left(\arctan\frac{1}{2}\right) = \frac{2\sqrt{5}}{5}$.



18 If $\sin\theta = 0.6 = \frac{3}{5}$, then $\cos\theta = \frac{\sqrt{5^2-3^2}}{5} = \frac{4}{5}$; hence, $\cos\left(\sin^{-1}(0.6)\right) = 0.8$.



19 We need to use the compound formula for sine:

$$\sin\left(\arccos\left(\frac{3}{5}\right)+\arctan\left(\frac{5}{12}\right)\right) = \sin\left(\arccos\left(\frac{3}{5}\right)\right)\cos\left(\arctan\left(\frac{5}{12}\right)\right) + \cos\left(\arccos\left(\frac{3}{5}\right)\right)\sin\left(\arctan\left(\frac{5}{12}\right)\right)$$

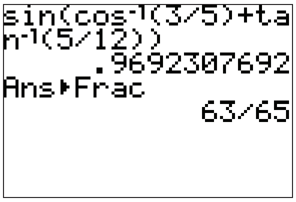
Since: $\sin\left(\arccos\left(\frac{3}{5}\right)\right) = \frac{\sqrt{5^2-3^2}}{5} = \frac{4}{5}$

$\cos\left(\arccos\left(\frac{3}{5}\right)\right) = \frac{3}{5}$

$\cos\left(\arctan\left(\frac{5}{12}\right)\right) = \frac{12}{\sqrt{12^2+5^2}} = \frac{12}{13}$

$\sin\left(\arctan\left(\frac{5}{12}\right)\right) = \frac{5}{\sqrt{12^2+5^2}} = \frac{5}{13}$

The value is: $\frac{4}{5}\cdot\frac{12}{13}+\frac{3}{5}\cdot\frac{5}{13}=\frac{63}{65}$.



20 We need to use the compound formula for cosine:

$$\cos\left(\tan^{-1}3+\sin^{-1}\left(\frac{1}{3}\right)\right)=\cos\left(\tan^{-1}3\right)\cos\left(\sin^{-1}\left(\frac{1}{3}\right)\right)-\sin\left(\tan^{-1}3\right)\sin\left(\sin^{-1}\left(\frac{1}{3}\right)\right)$$

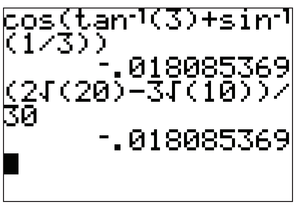
Since: $\cos\left(\tan^{-1}3\right)=\frac{1}{\sqrt{1^2+3^2}}=\frac{1}{\sqrt{10}}=\frac{\sqrt{10}}{10}$

$\sin\left(\tan^{-1}3\right)=\frac{3}{\sqrt{1^2+3^2}}=\frac{3}{\sqrt{10}}=\frac{3\sqrt{10}}{10}$

$\cos\left(\sin^{-1}\left(\frac{1}{3}\right)\right)=\frac{\sqrt{3^2-1^2}}{3}=\frac{\sqrt{8}}{3}=\frac{2\sqrt{2}}{3}$

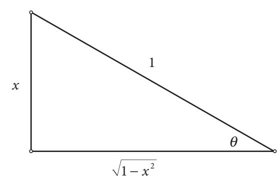
$\sin\left(\sin^{-1}\left(\frac{1}{3}\right)\right)=\frac{1}{3}$

The value is: $\frac{\sqrt{10}}{10}\cdot\frac{2\sqrt{2}}{3}-\frac{3\sqrt{10}}{10}\cdot\frac{1}{3}=\frac{2\sqrt{20}-3\sqrt{10}}{30}$.



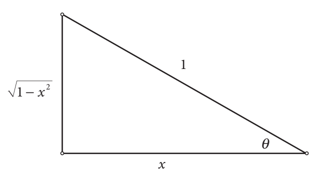
21 If $\sin\theta=x=\frac{x}{1}$, then $\cos\theta=\frac{\sqrt{1^2-x^2}}{1}=\sqrt{1-x^2}$;

hence, $\cos\left(\sin^{-1}(x)\right)=\sqrt{1-x^2}$.

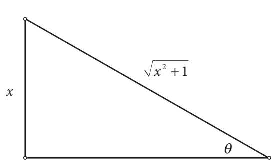


22 If $\cos\theta=x=\frac{x}{1}$, then $\tan\theta=\frac{\sqrt{1^2-x^2}}{x}=\frac{\sqrt{1-x^2}}{x}$;

hence, $\tan(\arccos(x))=\frac{\sqrt{1-x^2}}{x}$.



23 If $\tan\theta=x=\frac{x}{1}$, then $\cos\theta=\frac{1}{\sqrt{1^2+x^2}}=\frac{1}{\sqrt{x^2+1}}$; hence, $\cos\left(\tan^{-1}x\right)=\frac{1}{\sqrt{x^2+1}}$.

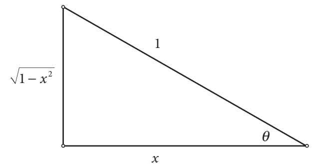


24 Using the double angle formula for sine, we have: $2\sin\left(\cos^{-1}x\right)\cos\left(\cos^{-1}x\right)$. Since:

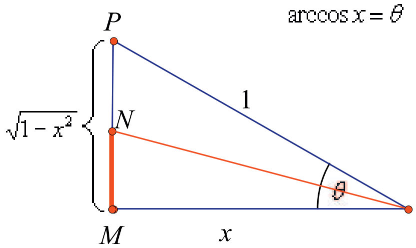
$\sin\left(\cos^{-1}x\right)=\sqrt{1-x^2}$

$\cos\left(\cos^{-1}x\right)=x$, then:

$\sin\left(2\cos^{-1}x\right)=2x\sqrt{1-x^2}$



25 From the triangle below, we can see that $\tan\left(\dfrac{1}{2}\arccos x\right)=\dfrac{NM}{x}$.



Using the fact that the line bisecting an angle divides the side opposite the angle in the ratio of the other two sides, we have:

$$\frac{PN}{NM}=\frac{1}{x}\Rightarrow PN=\frac{NM}{x}$$

Since: $\sqrt{1-x^2}=PN+NM=\dfrac{NM}{x}+NM=NM\left(\dfrac{1+x}{x}\right)\Rightarrow\dfrac{NM}{x}=\dfrac{\sqrt{1-x^2}}{1+x}=\dfrac{\sqrt{(1-x)(1+x)}}{\sqrt{(1+x)^2}}=\sqrt{\dfrac{1-x}{1+x}}$

Therefore: $\tan\left(\dfrac{1}{2}\arccos x\right)=\sqrt{\dfrac{1-x}{1+x}}$.

26 Using the compound angle formula for sine, we have:

$\sin(\arcsin x)\cos(2\arctan x)+\cos(\arcsin x)\sin(2\arctan x)$

Since: $\sin(\arcsin x)=x$

$\cos(\arcsin x)=\sqrt{1-x^2}$

Using the double angle formula for cosine and the result from question 23, we have:

$$\cos(2\arctan x)=2\cos^2(\arctan x)-1=2\left(\frac{1}{\sqrt{x^2+1}}\right)^2-1=\frac{2}{x^2+1}-1=\frac{1-x^2}{x^2+1}$$

Using the double angle formula for sine and the result for $\sin(\arctan x)=\dfrac{x}{\sqrt{x^2+1}}$, we have:

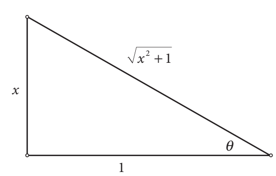
$$\sin(2\arctan x)=2\sin(\arctan x)\cos(\arctan x)=2\frac{x}{\sqrt{x^2+1}}\frac{1}{\sqrt{x^2+1}}=\frac{2x}{x^2+1}$$

Hence: $\sin(\arcsin x)\cos(2\arctan x)+\cos(\arcsin x)\sin(2\arctan x)$

$$=x\frac{1-x^2}{x^2+1}+\sqrt{1-x^2}\frac{2x}{x^2+1}$$

$$=\sin(\arcsin x+2\arctan x)=\frac{x-x^3+2x\sqrt{1-x^2}}{x^2+1}$$

We can use a diagram to verify the result for: $\sin(\arctan x)=\dfrac{x}{\sqrt{x^2+1}}$.



27 We will find the cosine of both sides of the equation. The cosine of the right side is $\frac{16}{65}$. So, now we will find the cosine of the left side (applying the compound angle formula for cosine):

$\cos\left(\arcsin\frac{4}{5}+\arcsin\frac{5}{13}\right)=\cos\left(\arcsin\frac{4}{5}\right)\cos\left(\arcsin\frac{5}{13}\right)-\sin\left(\arcsin\frac{4}{5}\right)\sin\left(\arcsin\frac{5}{13}\right)$. For the first part of the calculation, we will use the formula from question 21:

$$\sqrt{1-\left(\frac{4}{5}\right)^2}\sqrt{1-\left(\frac{5}{13}\right)^2}-\frac{4}{5}\cdot\frac{5}{13}=\frac{3}{5}\cdot\frac{12}{13}-\frac{4}{5}\cdot\frac{5}{13}=\frac{16}{65}$$

Hence: $\arcsin\frac{4}{5}+\arcsin\frac{5}{13}=\arccos\frac{16}{65}$.

28 We will find the tangent of both sides of the equation. The tangent of the right side is 1. So, now we will find the tangent of the left side (applying the compound angle formula for tangent):

$$\tan\left(\arctan\frac{1}{2}+\arctan\frac{1}{3}\right)=\frac{\tan\left(\arctan\frac{1}{2}\right)+\tan\left(\arctan\frac{1}{3}\right)}{1-\tan\left(\arctan\frac{1}{2}\right)\cdot\tan\left(\arctan\frac{1}{3}\right)}=\frac{\frac{1}{2}+\frac{1}{3}}{1-\frac{1}{2}\cdot\frac{1}{3}}=\frac{\frac{5}{6}}{1-\frac{1}{6}}=1$$

Hence: $\arctan\frac{1}{2}+\arctan\frac{1}{3}=\frac{\pi}{4}$.

29 We will find the tangent of both sides of the equation (applying the compound angle formula for tangent):

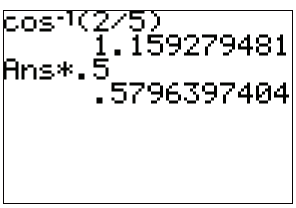
$$\tan\left(\tan^{-1}x+\tan^{-1}(1-x)\right)=\tan\tan^{-1}\left(\frac{4}{3}\right)$$

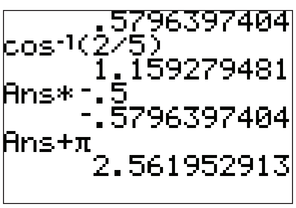
$$\frac{\tan\left(\tan^{-1}x\right)+\tan\left(\tan^{-1}(1-x)\right)}{1-\tan\left(\tan^{-1}x\right)\cdot\tan\left(\tan^{-1}(1-x)\right)}=\frac{4}{3}$$

$$\frac{x+(1-x)}{1-x(1-x)}=\frac{4}{3}\Rightarrow\frac{1}{1-x+x^2}=\frac{4}{3}\Rightarrow 3=4-4x+4x^2\Rightarrow 4x^2-4x+1=0\Rightarrow x_{1,2}=\frac{1}{2}$$

30 $\cos 2x=\frac{2}{5}\Rightarrow 2x=\pm\arccos\left(\frac{2}{5}\right)+2k\pi\Rightarrow x=\pm\frac{1}{2}\arccos\left(\frac{2}{5}\right)+k\pi$

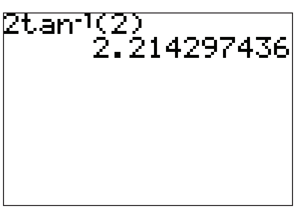
Hence, the solutions are: $\frac{1}{2}\arccos\left(\frac{2}{5}\right)\approx 0.580,\ -\frac{1}{2}\arccos\left(\frac{2}{5}\right)+\pi\approx 2.56.$





31 $\frac{x}{2}=\arctan 2+k\pi\Rightarrow x=2\arctan 2+2k\pi$

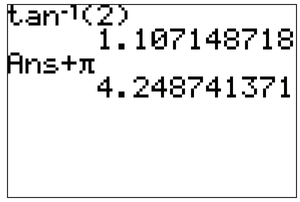
Hence, the solution is: $2\arctan 2\approx 2.21$.



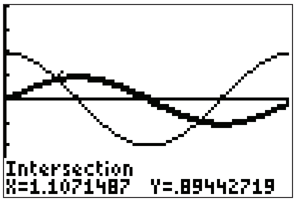
32 We can rewrite the equation in tangent only:

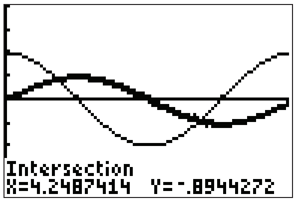
$$2\cos x = \sin x \Rightarrow 2 = \frac{\sin x}{\cos x} \Rightarrow \tan x = 2 \Rightarrow x = \arctan 2 + k\pi$$

Hence, the solutions are: $\arctan 2 \approx 1.11, \arctan 2 + \pi \approx 4.25.$



Note: We can solve the equation directly, reading the results from a GDC.



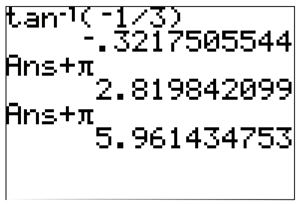


33 Since $\sec^2 x = \frac{1}{\cos^2 x} = \frac{\cos^2 x + \sin^2 x}{\cos^2 x} = 1 + \frac{\sin^2 x}{\cos^2 x} = 1 + \tan^2 x$, the equation contains tangent only; hence, we can substitute $\tan x = t$:

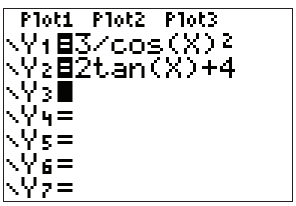
$$3\left(1+t^2\right) = 2t + 4 \Rightarrow 3t^2 - 2t - 1 = 0 \Rightarrow t_1 = 1, t_2 = -\frac{1}{3}$$

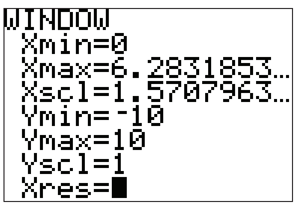
$$\tan x = 1 \Rightarrow x = \frac{\pi}{4}, \frac{5\pi}{4}$$

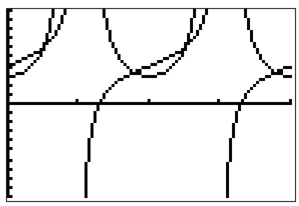
$$\tan x = -\frac{1}{3} \Rightarrow x \approx 2.82, 5.96$$

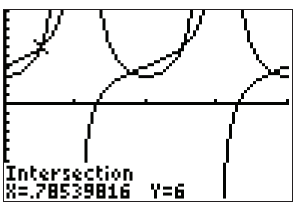


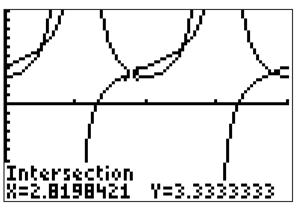
Note: We can solve directly using a GDC.

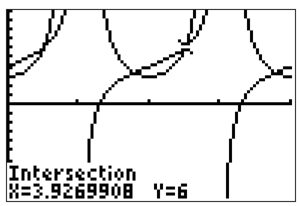


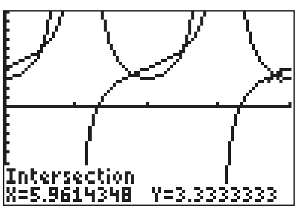


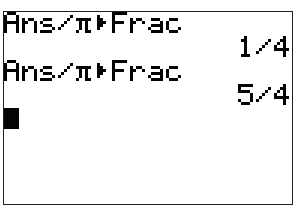










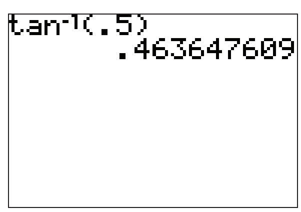


34 Substituting $\tan x = t$ will give us a quadratic equation:

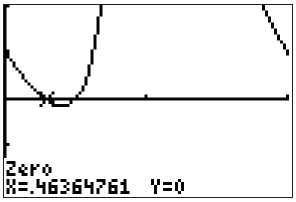
$$2t^2 - 3\tan x + 1 = 0 \Rightarrow t_1 = \frac{1}{2}, t_2 = 1$$

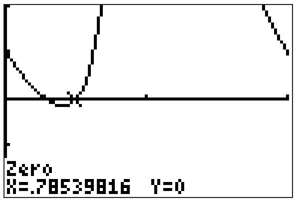
So, either: $\tan x = \frac{1}{2} \Rightarrow x = \arctan\frac{1}{2} + k\pi$, or $\tan x = 1 \Rightarrow x = \arctan 1 + k\pi$

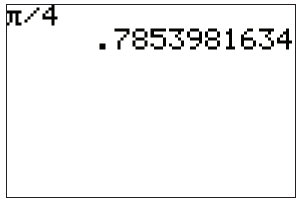
Hence, the solutions are: $\arctan\frac{1}{2} \approx 0.464, \frac{\pi}{4}$.



Note: We can solve the equation directly, reading the results from a GDC.

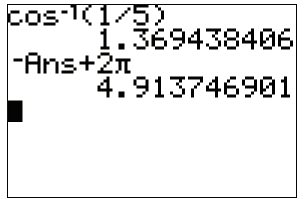




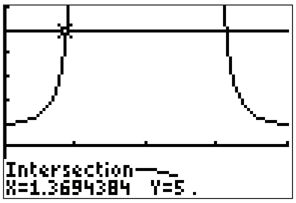


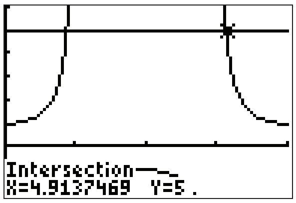
35 $\frac{\sin x}{\cos x} \cdot \frac{1}{\sin x} = 5 \Rightarrow \cos x = \frac{1}{5} \Rightarrow x = \pm \arccos \frac{1}{5} + 2k\pi$

Hence, the solutions are: $\arccos \frac{1}{5} \approx 1.37, -\arccos \frac{1}{5} + 2\pi \approx 4.91.$



Note: We can solve the equation directly, reading the results from a GDC.





36 Using the double angle formula for tangent:

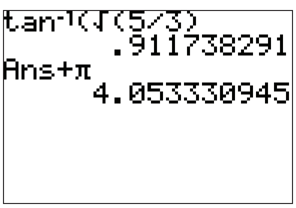
$$\frac{2\tan x}{1-\tan^2 x} + 3\tan x = 0 \Rightarrow \tan x\left(\frac{2+3(1-\tan^2 x)}{1-\tan^2 x}\right) = 0 \Rightarrow \tan x\left(\frac{5-3\tan^2 x}{1-\tan^2 x}\right) = 0$$

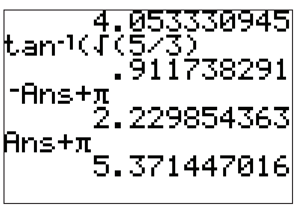
So, either: $\tan x = 0 \Rightarrow x = k\pi$, or

$$\frac{5-3\tan^2 x}{1-\tan^2 x} = 0 \Rightarrow 5-3\tan^2 x = 0 \Rightarrow \tan^2 x = \frac{5}{3} \Rightarrow \tan x = \pm\sqrt{\frac{5}{3}} \Rightarrow x = \pm \arctan\sqrt{\frac{5}{3}} + k\pi$$

Hence, the solutions are:

$$\pi, 2\pi, \arctan\sqrt{\frac{5}{3}} \approx 0.912, \arctan\sqrt{\frac{5}{3}} + \pi \approx 4.05, -\arctan\sqrt{\frac{5}{3}} + \pi \approx 2.23, -\arctan\sqrt{\frac{5}{3}} + 2\pi \approx 5.37.$$





37 Simplifying the equation:

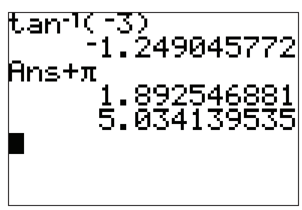
$$2\cos^2 x - 3 \cdot 2\sin x \cos x = 2 \Rightarrow 2\cos^2 x - 6\sin x\cos x = 2\cos^2 x + 2\sin^2 x$$

$$\Rightarrow 2\sin^2 x + 6\sin x\cos x = 0 \Rightarrow 2\sin x(\sin x + 3\cos x) = 0$$

So, either: $\sin x = 0 \Rightarrow x = k\pi$, or

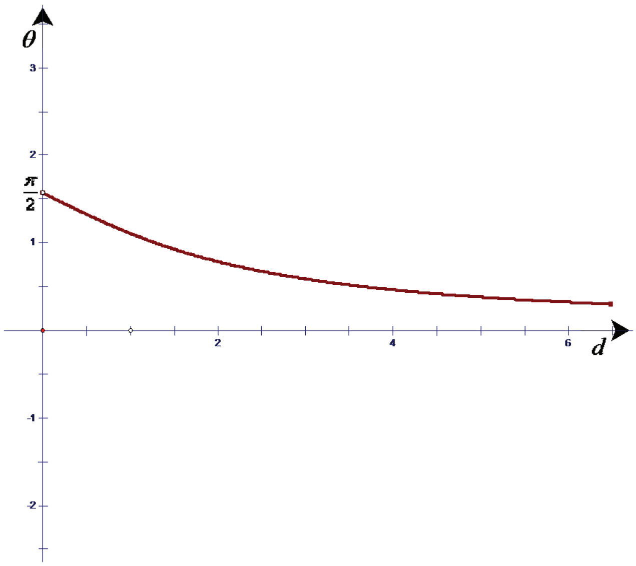
$$\sin x + 3\cos x = 0 \Rightarrow \sin x = -3\cos x \Rightarrow \tan x = -3 \Rightarrow x = \arctan(-3) + k\pi$$

Hence, the solutions are: $0, \pi, \arctan(-3) + \pi \approx 1.89, \arctan(-3) + 2\pi \approx 5.03.$

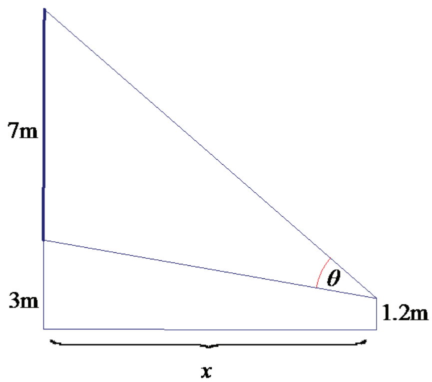


38 From the diagram, we can see that: $\tan\theta = \frac{2}{d} \Rightarrow \theta = \arctan\left(\frac{2}{d}\right)$.

For positive d, the graph is as below:



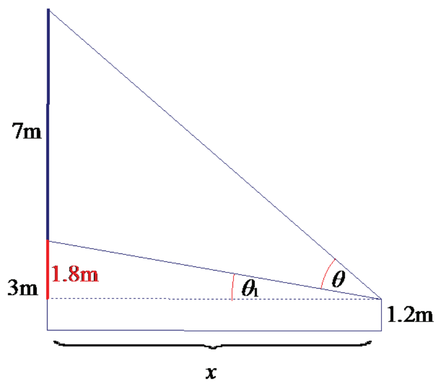
39 a) i)

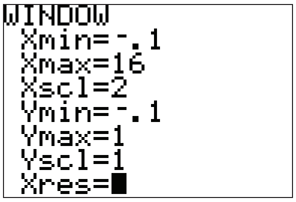


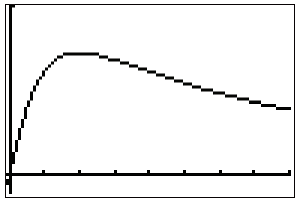
ii) From the diagram below, we can see that: $\tan(\theta + \theta_1) = \frac{7 + 1.8}{x}$, and $\tan(\theta_1) = \frac{1.8}{x}$. Hence,

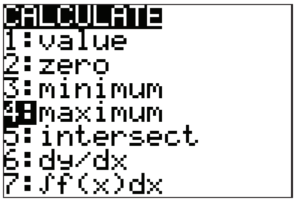
$$\tan\theta = \tan((\theta + \theta_1) - \theta_1) = \frac{\tan(\theta + \theta_1) - \tan(\theta_1)}{1 + \tan(\theta + \theta_1)\cdot\tan(\theta_1)} = \frac{\frac{8.8}{x} - \frac{1.8}{x}}{1 + \frac{8.8}{x}\cdot\frac{1.8}{x}} = \frac{\frac{7}{x}}{\frac{x^2 + 15.84}{x^2}} = \frac{7x}{x^2 + 15.84}.$$

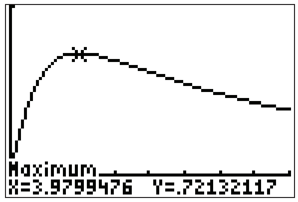
So, $\theta = \arctan\frac{7x}{x^2 + 15.84}$.



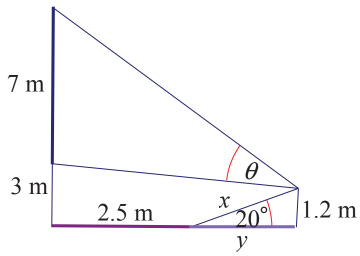
iii) 



iv) 

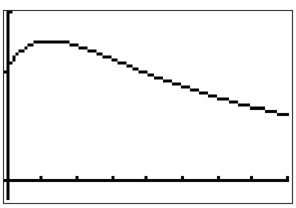


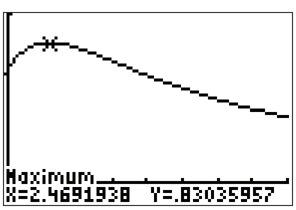
Therefore, $x \approx 3.98$, which means you should sit in the second row.

b) i) 

ii) We can use the result from **a ii**. The differences are the distance from the screen to the seat and the distance up the wall, which is reduced by $x \sin \frac{\pi}{9}$. So, we have to replace x by $2.5 + y$, where $2.5 + y = 2.5 + x \cos \frac{\pi}{9}$. Hence,

$$\theta = \arctan \frac{7\left(x \cos \frac{\pi}{9} + 2.5\right)}{\left(x \cos \frac{\pi}{9} + 2.5\right)^2 + \left(8.8 - x \sin \frac{\pi}{9}\right)\left(1.8 - x \sin \frac{\pi}{9}\right)}$$

iii) 

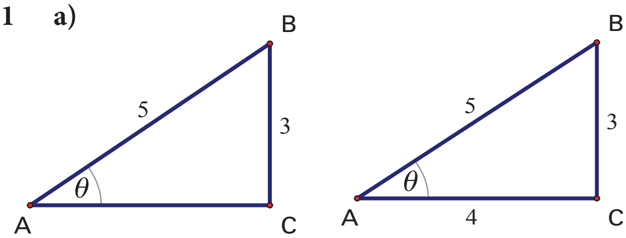
iv) 

Therefore, $x \approx 2.47$ and you should sit in the third row.

Chapter 8

Exercise 8.1

1 a)



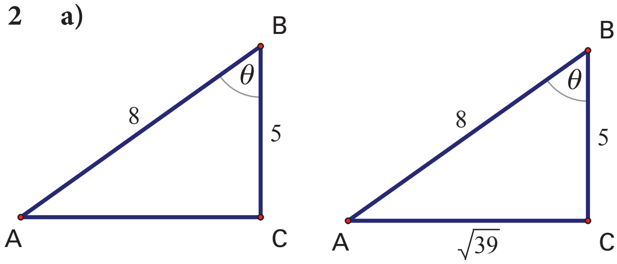
b) $b = \sqrt{5^2 - 3^2} = 4$

$\cos\theta = \frac{4}{5}$, $\tan\theta = \frac{3}{4}$, $\cot\theta = \frac{4}{3}$,

$\sec\theta = \frac{5}{4}$, $\csc\theta = \frac{5}{3}$

c) $\theta \approx 36.9^\circ$, $90^\circ - \theta \approx 53.1^\circ$

2 a)



b) $b = \sqrt{8^2 - 5^2} = \sqrt{39}$

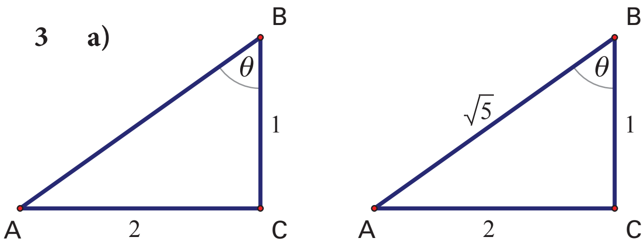
$\sin\theta = \frac{\sqrt{39}}{8}$, $\tan\theta = \frac{\sqrt{39}}{5}$,

$\cot\theta = \frac{5}{\sqrt{39}} = \frac{5\sqrt{39}}{39}$, $\sec\theta = \frac{8}{5}$,

$\csc\theta = \frac{8}{\sqrt{39}} = \frac{8\sqrt{39}}{39}$

c) $\theta \approx 51.3^\circ$, $90^\circ - \theta \approx 38.7^\circ$

3 a)



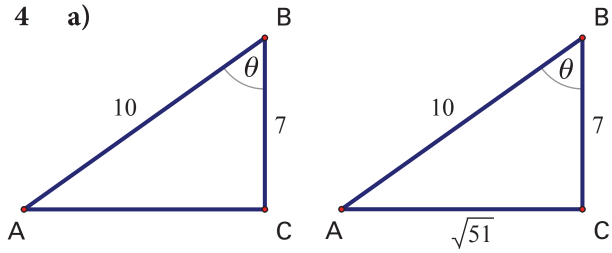
b) $c = \sqrt{2^2 + 1^2} = \sqrt{5}$

$\sin\theta = \frac{2}{\sqrt{5}} = \frac{2\sqrt{5}}{5}$, $\cos\theta = \frac{1}{\sqrt{5}} = \frac{\sqrt{5}}{5}$,

$\cot\theta = \frac{1}{2}$, $\sec\theta = \sqrt{5}$, $\csc\theta = \frac{\sqrt{5}}{2}$

c) $\theta \approx 63.4^\circ$, $90^\circ - \theta \approx 26.6^\circ$

4 a)



b) $b = \sqrt{10^2 - 7^2} = \sqrt{51}$

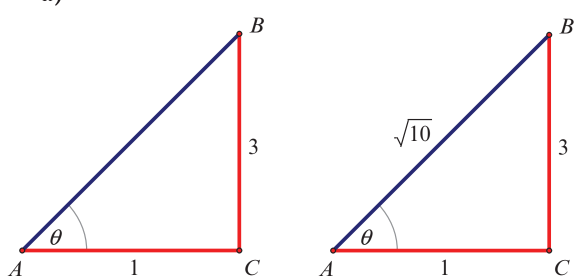
$\sin\theta = \frac{\sqrt{51}}{10}$, $\tan\theta = \frac{\sqrt{51}}{7}$,

$\cot\theta = \frac{7}{\sqrt{51}} = \frac{7\sqrt{51}}{51}$, $\sec\theta = \frac{10}{7}$,

$\csc\theta = \frac{10}{\sqrt{51}} = \frac{10\sqrt{51}}{51}$

c) $\theta \approx 45.6^\circ$, $90^\circ - \theta \approx 44.4^\circ$

5 a)



b) $c = \sqrt{3^2 + 1^2} = \sqrt{10}$

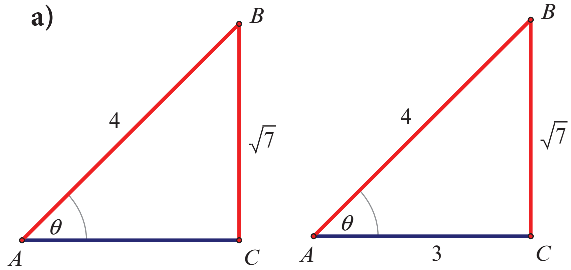
$\sin\theta = \frac{3}{\sqrt{10}} = \frac{3\sqrt{10}}{10}$, $\cos\theta = \frac{1}{\sqrt{10}} = \frac{\sqrt{10}}{10}$,

$\tan\theta = \frac{3}{1} = 3$, $\sec\theta = \frac{\sqrt{10}}{1} = \sqrt{10}$,

$\csc\theta = \frac{\sqrt{10}}{3}$

c) $\theta \approx 71.6^\circ$, $90^\circ - \theta \approx 18.4^\circ$

6 **a)**



b) $b = \sqrt{4^2 - \sqrt{7}^2} = 3$

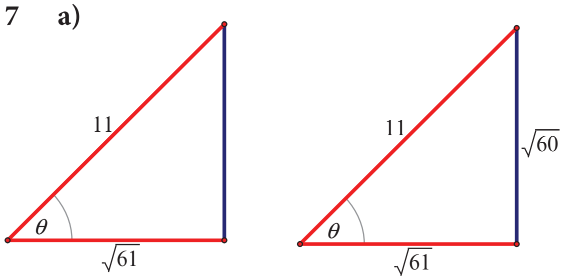
$\cos\theta = \dfrac{3}{4}$, $\tan\theta = \dfrac{\sqrt{7}}{3}$,

$\cot\theta = \dfrac{3}{\sqrt{7}} = \dfrac{3\sqrt{7}}{7}$, $\sec\theta = \dfrac{4}{3}$,

$\csc\theta = \dfrac{4}{\sqrt{7}} = \dfrac{4\sqrt{7}}{7}$

c) $\theta \approx 41.4^\circ$, $90^\circ - \theta \approx 48.6^\circ$

7 **a)**



b) $\sin\theta = \dfrac{\sqrt{60}}{11}$, $\cos\theta = \dfrac{\sqrt{61}}{11}$,

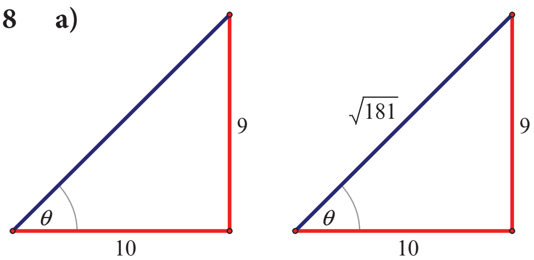
$\tan\theta = \dfrac{\sqrt{60}}{\sqrt{61}} = \dfrac{2\sqrt{915}}{61}$,

$\cot\theta = \dfrac{\sqrt{61}}{\sqrt{60}} = \dfrac{\sqrt{915}}{30}$,

$\csc\theta = \dfrac{11}{\sqrt{60}} = \dfrac{11\sqrt{15}}{30}$

c) $\theta \approx 44.8^\circ$, $90^\circ - \theta \approx 45.2^\circ$

8 **a)**



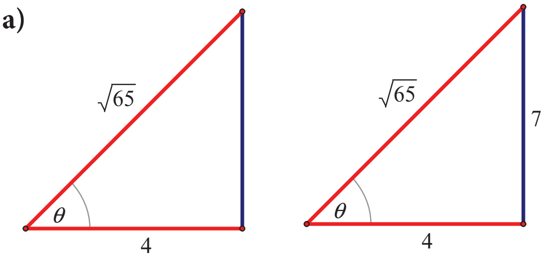
b) $\sin\theta = \dfrac{9}{\sqrt{181}} = \dfrac{9\sqrt{181}}{181}$,

$\cos\theta = \dfrac{10}{\sqrt{181}} = \dfrac{10\sqrt{181}}{181}$, $\cot\theta = \dfrac{10}{9}$,

$\sec\theta = \dfrac{\sqrt{181}}{10}$, $\csc\theta = \dfrac{\sqrt{181}}{9}$

c) $\theta \approx 42.0^\circ$, $90^\circ - \theta \approx 48.0^\circ$

9 **a)**

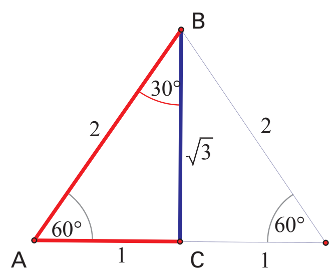


b) $\sin\theta = \dfrac{7}{\sqrt{65}} = \dfrac{7\sqrt{65}}{65}$, $\tan\theta = \dfrac{7}{4}$,

$\cot\theta = \dfrac{4}{7}$, $\sec\theta = \dfrac{\sqrt{65}}{4}$, $\csc\theta = \dfrac{\sqrt{65}}{7}$

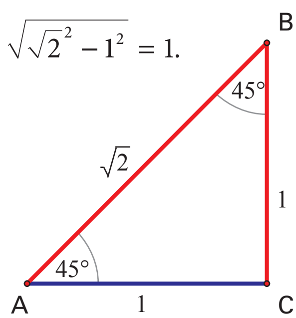
c) $\theta \approx 60.3^\circ$, $90^\circ - \theta \approx 29.7^\circ$

10 $\cos\theta = \dfrac{1}{2}$ in a triangle with adjacent leg 1, hypotenuse 2, and opposite leg $\sqrt{2^2 - 1^2} = \sqrt{3}$.



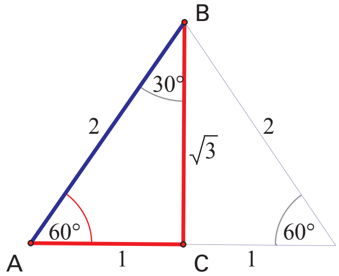
$\theta = 60^\circ = \dfrac{\pi}{3}$

11 $\sin\theta = \dfrac{\sqrt{2}}{2} = \dfrac{1}{\sqrt{2}}$ in a triangle with opposite leg 1, hypotenuse $\sqrt{2}$, and adjacent leg $\sqrt{\sqrt{2}^2 - 1^2} = 1$.



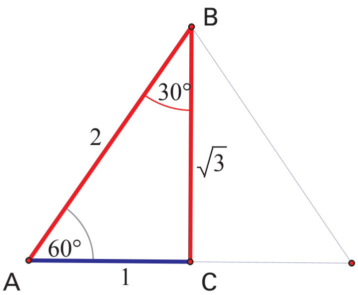
$\theta = 45^\circ = \dfrac{\pi}{4}$

12 $\tan\theta = \sqrt{3}$ in a triangle with opposite leg $\sqrt{3}$, adjacent leg 1, and hypotenuse $\sqrt{\sqrt{3}^2 + 1^2} = 2$.



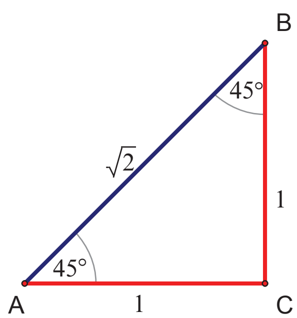
$\theta = 60^\circ = \dfrac{\pi}{3}$

13 $\csc\theta = \dfrac{2\sqrt{3}}{3} = \dfrac{2}{\sqrt{3}} \Rightarrow \sin\theta = \dfrac{\sqrt{3}}{2}$ in a triangle with opposite leg $\sqrt{3}$, hypotenuse 2, and adjacent leg $\sqrt{2^2 - \sqrt{3}^2} = 1$.



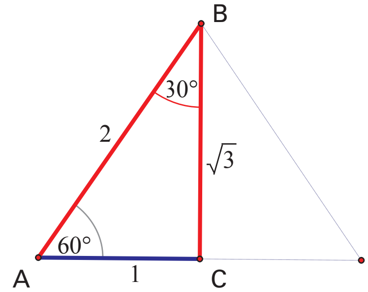
$\theta = 60^\circ = \dfrac{\pi}{3}$

14 $\cot\theta = 1$ in a triangle with both legs 1, and hypotenuse $\sqrt{1^2 + 1^2} = \sqrt{2}$.



$\theta = 45^\circ = \dfrac{\pi}{4}$

15 $\cos\theta = \dfrac{\sqrt{3}}{2}$ in a triangle with adjacent leg $\sqrt{3}$, hypotenuse 2, and opposite leg $\sqrt{2^2 - \sqrt{3}^2} = 1$.



$\theta = 30^\circ = \dfrac{\pi}{6}$

16 From the definition of tangent, we have:

$\tan 60^\circ = \dfrac{x}{50} \Rightarrow x = 50\tan 60^\circ = 50\sqrt{3} \approx 86.6$

From the definition of cosine, we have:

$\cos 60^\circ = \dfrac{50}{y} \Rightarrow y = \dfrac{50}{\cos 60^\circ} = \dfrac{50}{\frac{1}{2}} = 100$

```
50tan(60)
            86.60254038
50/cos(60)
                    100
```

17 From the definition of cosine, we have:

$\cos 55^\circ = \dfrac{x}{15} \Rightarrow x = 15\cos 55^\circ \approx 8.60$

From the definition of sine, we have:

$\sin 55^\circ = \dfrac{y}{15} \Rightarrow y = 15\sin 55^\circ \approx 12.3$

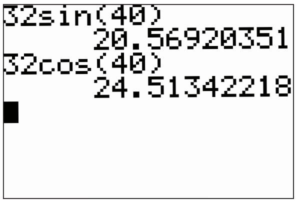
```
15cos(55)
            8.603646545
15sin(55)
            12.28728066
█
```

18 From the definition of sine, we have:

$$\sin 40^\circ = \frac{x}{32} \Rightarrow x = 32\sin 40^\circ \approx 20.6$$

From the definition of cosine, we have:

$$\cos 40^\circ = \frac{y}{32} \Rightarrow y = 32\cos 40^\circ \approx 24.5$$

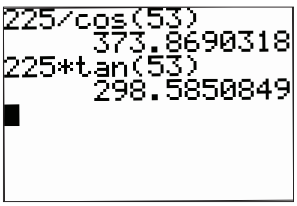


19 From the definition of cosine, we have:

$$\cos 53^\circ = \frac{225}{x} \Rightarrow x = \frac{225}{\cos 53^\circ} \approx 374$$

From the definition of tangent, we have:

$$\tan 53^\circ = \frac{y}{225} \Rightarrow y = 225\tan 53^\circ \approx 299$$



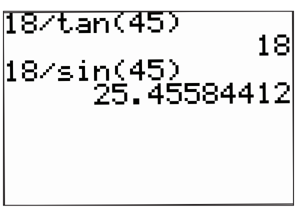
20 From the definition of tangent, we have:

$$\tan 45^\circ = \frac{18}{x} \Rightarrow x = \frac{18}{\tan 45^\circ} = 18$$

From the definition of sine, we have:

$$\sin 45^\circ = \frac{18}{y} \Rightarrow y = \frac{18}{\sin 45^\circ} = \frac{18}{\frac{1}{\sqrt{2}}}$$

$$= 18\sqrt{2} \approx 25.5$$



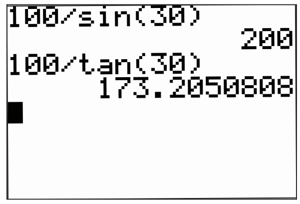
21 From the definition of sine, we have:

$$\sin 30^\circ = \frac{100}{x} \Rightarrow x = \frac{100}{\sin 30^\circ} = 200$$

From the definition of tangent, we have:

$$\tan 30^\circ = \frac{100}{y} \Rightarrow y = \frac{100}{\tan 30^\circ} = \frac{100}{\frac{1}{\sqrt{3}}}$$

$$= 100\sqrt{3} \approx 173$$



22 $\tan\alpha = \frac{10}{\sqrt{300}} = \frac{10}{10\sqrt{3}} = \frac{\sqrt{3}}{3} \Rightarrow \alpha = 30^\circ$; hence, $\beta = 60^\circ$.

Note: We can find β first:

$\tan\beta = \frac{\sqrt{300}}{10} = \frac{10\sqrt{3}}{10} = \sqrt{3} \Rightarrow \beta = 60^\circ$, and then α.

23 $\cos\alpha = \frac{15}{39} \Rightarrow \alpha = \cos^{-1}\left(\frac{5}{13}\right) \approx 67.4^\circ \Rightarrow$

$$\beta = 90^\circ - \alpha \approx 22.6^\circ$$

Note: We can start with $\sin\beta = \frac{15}{39}$ and then find β and α.

24 $\sin\alpha = \frac{\sqrt{7}}{\sqrt{28}} = \frac{1}{2} \Rightarrow \alpha = 30^\circ \Rightarrow$

$$\beta = 90^\circ - \alpha = 60^\circ$$

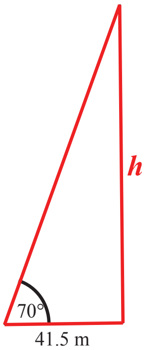
Note: We can start with $\cos\beta = \frac{\sqrt{7}}{\sqrt{28}}$ and then find β and α.

25 $\tan\alpha = \frac{44}{121} = \frac{4}{11} \Rightarrow \alpha = \cos^{-1}\left(\frac{4}{11}\right) \approx 20.0^\circ \Rightarrow$

$$\beta = 90^\circ - \alpha \approx 70^\circ$$

Note: We can start with $\tan\beta = \frac{121}{44} = \frac{11}{4}$ and then find β and α.

26

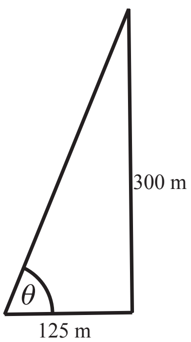


We have to find the other leg of the triangle (h), so we use the definition of the tangent function:

$$\tan 70^\circ = \frac{h}{41.5} \Rightarrow h = 41.5 \tan 70^\circ \approx 114$$

So, the tree is approximately 114 m tall.

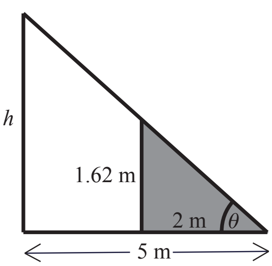
27 We have to find angle θ:



Using the definition of the tangent function, we have:

$$\tan \theta = \frac{300}{125} = 2.4 \Rightarrow \theta = \tan^{-1} 2.4 = 67.4^\circ$$

28 Firstly, we find angle θ, and then the height h.



Shaded triangle:

Using the definition of the tangent function:

$$\tan \theta = \frac{1.62}{2}$$

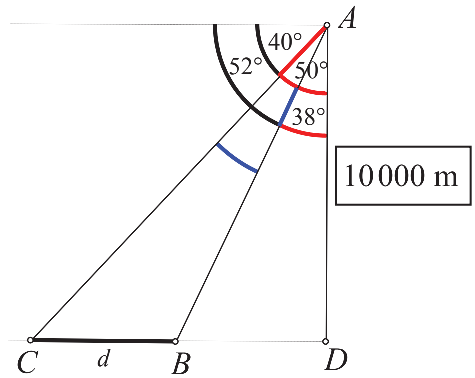
Larger triangle:

Using the definition of the tangent function:

$$\tan \theta = \frac{h}{5} \Rightarrow h = 5 \tan \theta = 5 \cdot 0.81 = 4.05$$

So, the height of the street light is 4.05 m.

29



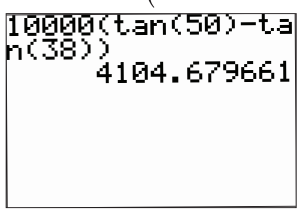
From the right triangle ACD, we will find side CD: $\frac{CD}{10\,000} = \tan 50^\circ \Rightarrow CD = 10\,000 \tan 50^\circ$

From the right triangle ABD, we will find side BD: $\frac{BD}{10\,000} = \tan 38^\circ \Rightarrow BD = 10\,000 \tan 38^\circ$

So, we have:

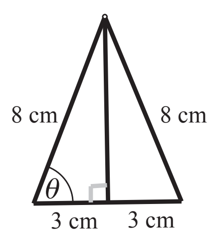
$$d = BD - CD = 10\,000 \tan 50^\circ - 10\,000 \tan 38^\circ$$

$$= 10\,000\left(\tan 50^\circ - \tan 38^\circ\right) \approx 4105$$



Therefore, the ships are approximately 4105 m apart.

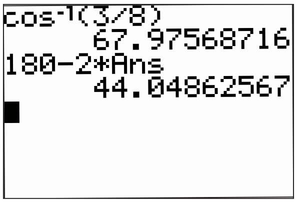
30 The altitude on the base of the triangle splits the triangle into two congruent right triangles.



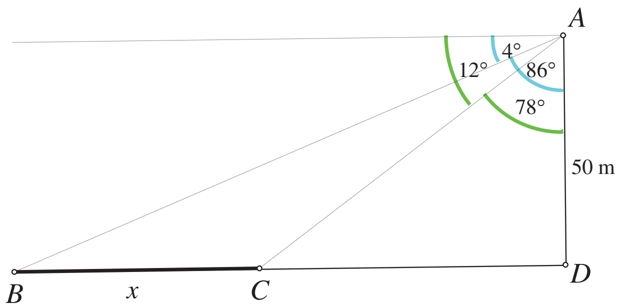
Using the definition of cosine, we can determine angle θ: $\cos \theta = \frac{3}{8} \Rightarrow \theta = \cos^{-1}\left(\frac{3}{8}\right) \approx 68.0^\circ$

The angle between two equal sides is $180^\circ - 2\theta \approx 44.0^\circ$.

Hence, the angles are: $68.0^\circ, 68.0^\circ, 44.0^\circ$.



31 Firstly, we will determine the distance between the positions of the boats; then, using distance and time, we will find the speed of the boat.



From right triangle *ACD*, we will find side *CD*:

$$\frac{CD}{50} = \tan 78^\circ \Rightarrow CD = 50 \tan 78^\circ$$

From right triangle *ABD*, we will find side *BD*:

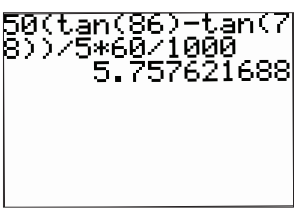
$$\frac{BD}{50} = \tan 86^\circ \Rightarrow BD = 50 \tan 86^\circ$$

So, we have:

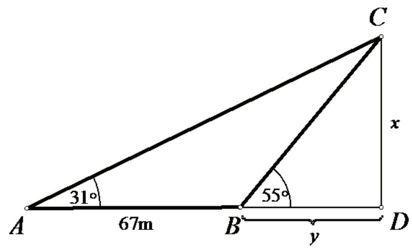
$$x = BD - CD = 50 \tan 86^\circ - 50 \tan 78^\circ$$
$$= 50\left(\tan 86^\circ - \tan 78^\circ\right) \approx 479.01 \text{ m}$$

Finally, we will find the speed:

$$\frac{x}{5 \text{ min}} \approx \frac{479.801 \text{ m}}{5} \cdot \frac{60}{1000} \text{ km/h} \approx 5.76 \text{ km/h}$$



32



We can express *x* using right triangles *ADC* and *BDC*.

In triangle *ADC*:

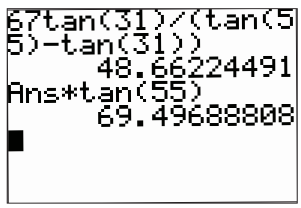
$$\tan 31^\circ = \frac{x}{67+y} \Rightarrow x = (67+y)\tan 31^\circ$$

In triangle *BDC*: $\tan 55^\circ = \dfrac{x}{y} \Rightarrow x = y \tan 55^\circ$

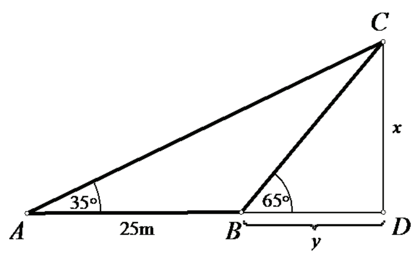
So, we have that:

$$(67+y)\tan 31^\circ = y \tan 55^\circ$$
$$\Rightarrow 67 \tan 31^\circ = y \tan 55^\circ - y \tan 31^\circ$$
$$\Rightarrow 67 \tan 31^\circ = y\left(\tan 55^\circ - \tan 31^\circ\right)$$
$$\Rightarrow y = \frac{67 \tan 31^\circ}{\tan 55^\circ - \tan 31^\circ} \approx 48.6622 \text{ m}$$

Now, we can find *x* using any of the above formulae: $x = y \tan 55^\circ \approx 69.5$ m.



33



We can express *x* using right triangles *ADC* and *BDC*.

In triangle *ADC*:

$$\tan 35^\circ = \frac{x}{25+y} \Rightarrow x = (25+y)\tan 35^\circ$$

In triangle *BDC*: $\tan 65^\circ = \dfrac{x}{y} \Rightarrow x = y \tan 65^\circ$

So, we have that:

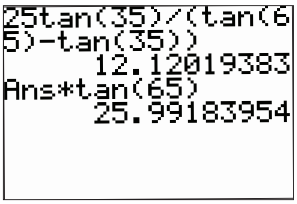
$$(25 + y)\tan 35^\circ = y \tan 65^\circ$$

$$\Rightarrow 25\tan 35^\circ = y\tan 65^\circ - y\tan 35^\circ$$

$$\Rightarrow 25\tan 35^\circ = y\left(\tan 65^\circ - \tan 35^\circ\right)$$

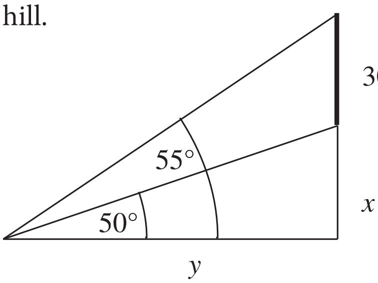
$$\Rightarrow y = \frac{25\tan 35^\circ}{\tan 65^\circ - \tan 35^\circ} \approx 12.12 \text{ m}$$

Now, we can find x using any of the above formulae: $x = y\tan 65^\circ \approx 26.0$ m.



So, the length of wire $= \sqrt{12.12^2 + 26^2} = 28.7$ m.

34 On the diagram, x represents the height of the hill.



From the right triangles, we have:

$$\tan 55^\circ = \frac{30 + x}{y} \Rightarrow y = \frac{30 + x}{\tan 55^\circ} \text{ and}$$

$$\tan 50^\circ = \frac{x}{y} \Rightarrow y = \frac{x}{\tan 50^\circ}$$

Hence,

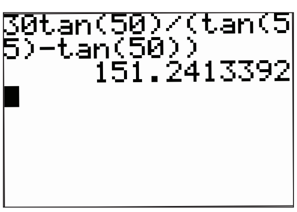
$$\frac{30 + x}{\tan 55^\circ} = \frac{x}{\tan 50^\circ} \Rightarrow$$

$$(30 + x)\tan 50^\circ = x\tan 55^\circ$$

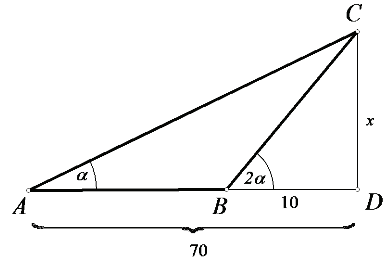
Finally:

$$30\tan 50^\circ = x\left(\tan 55^\circ - \tan 50^\circ\right) \Rightarrow$$

$$x = \frac{30\tan 50^\circ}{\tan 55^\circ - \tan 50^\circ} \approx 151 \text{ m}$$



35



From the right triangles, we have:

$\tan\alpha = \dfrac{x}{70}$ and $\tan 2\alpha = \dfrac{x}{10}$. Hence,

$x = 70\tan\alpha = 10\tan 2\alpha \Rightarrow 7\tan\alpha = \tan 2\alpha$.

Using the double angle formula for tangent, we have:

$$7\tan\alpha = \frac{2\tan\alpha}{1 - \tan^2\alpha} \Rightarrow$$

$$7 = \frac{2}{1 - \tan^2\alpha} \Rightarrow 7\left(1 - \tan^2\alpha\right) = 2.$$

Therefore, $1 - \tan^2\alpha = \dfrac{2}{7} \Rightarrow$

$$\tan^2\alpha = \frac{5}{7} \Rightarrow \tan\alpha = \sqrt{\frac{5}{7}}$$

(since α is an acute angle).

Now we can find the value of x:

$$x = 70\tan\alpha = 70\sqrt{\frac{5}{7}} \approx 59.2 \text{ m.}$$

36 From triangle ABC, we have: $\cos 2x = \dfrac{6}{10}$.

From triangle ABD, we have:

$\cos x = \dfrac{6}{BD} \Rightarrow BD = \dfrac{6}{\cos x}$. Hence,

we have to determine $\cos x$. Using the double angle formula for cosine, we have:

$$2\cos^2 x - 1 = \frac{3}{5} \Rightarrow \cos^2 x = \frac{4}{5} \Rightarrow \cos x = \sqrt{\frac{4}{5}}.$$

Therefore:

$$BD = \frac{6}{\cos x} = \frac{6}{\frac{2}{\sqrt{5}}} = 3\sqrt{5}.$$

37 Using the properties of cosine, we have:

$$\cos D\hat{E}A = \cos\left(90^\circ + 2x^\circ\right)$$

$$= \cos 90^\circ \cos 2x^\circ - \sin 90^\circ \sin 2x^\circ$$

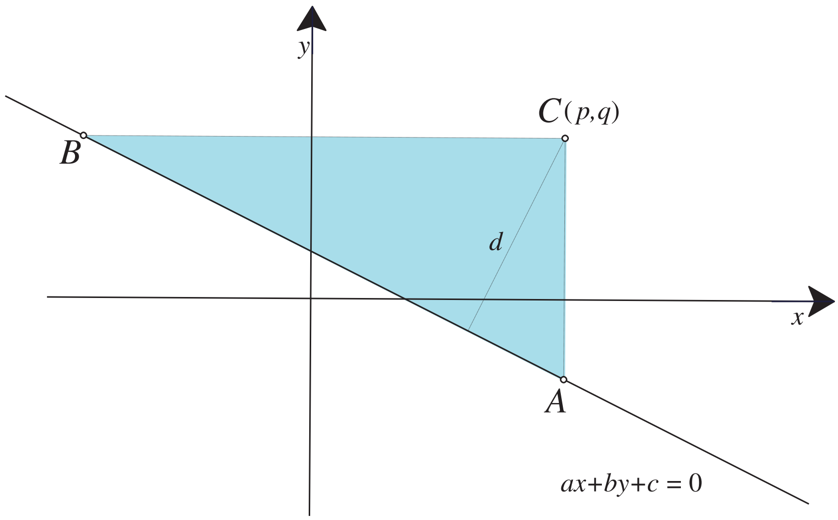
$$= -\sin 2x^\circ$$

To determine this value, it is enough to determine the values of $\sin x^\circ$ and $\cos x^\circ$, and use the double angle formula for sine. In triangle DEC, side $DC = \sqrt{1^2 + 3^2} = \sqrt{10}$; hence,

$\sin x^\circ = \frac{1}{\sqrt{10}}$ and $\cos x^\circ = \frac{3}{\sqrt{10}}$. Therefore:

$$\cos D\widehat{E}A = \cos\left(90^\circ + 2x^\circ\right) = -\sin 2x^\circ = -2\sin x^\circ \cos x^\circ = -2\frac{1}{\sqrt{10}}\frac{3}{\sqrt{10}} = -\frac{3}{5}.$$

38 We will introduce notation as given on the diagram below:



We will find the area of triangle ABC in two ways. However, for both methods, we have to find the magnitudes of the sides first.

Point A is on the line, its abscissa is p; hence, $A\left(p, -\frac{ap+c}{b}\right)$.

Point B is on the line, its ordinate is q; hence, $B\left(-\frac{bq+c}{a}, q\right)$.

$$\text{Side } AC = \left|q - \left(-\frac{ap+c}{b}\right)\right| = \left|\frac{qb+ap+c}{b}\right| = \frac{|ap+qb+c|}{b}.$$

$$\text{Side } BC = \left|p - \left(-\frac{bq+c}{a}\right)\right| = \left|\frac{pa+bq+c}{a}\right| = \frac{|ap+qb+c|}{a}.$$

$$\text{Side } AB = \sqrt{|AC|^2 + |BC|^2} = \sqrt{\left|\frac{ap+qb+c}{b}\right|^2 + \left|\frac{ap+bq+c}{a}\right|^2} \Rightarrow$$

$$AB = |ap+bq+c|\sqrt{\frac{1}{b^2} + \frac{1}{a^2}} = \frac{|ap+bq+c|}{ab}\sqrt{a^2+b^2}.$$

The area of the triangle equals the product of the legs, so: $A = \frac{1}{2}|AC||BC| = \frac{1}{2}\frac{|ap+qb+c|}{b}\frac{|ap+qb+c|}{a}$.

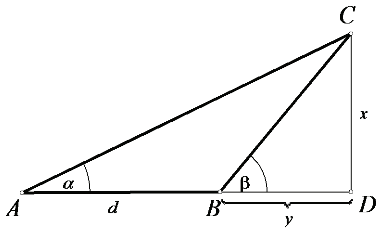
We can also find the area from the hypotenuse and its height:

$$A = \frac{1}{2}|AB|\,d = \frac{1}{2}\frac{|ap+qb+c|}{ab}\sqrt{a^2+b^2}\,d$$

So, $\frac{1}{2}\frac{|ap+qb+c|}{b}\frac{|ap+qb+c|}{a} = \frac{1}{2}\frac{|ap+qb+c|}{ab}\sqrt{a^2+b^2}\cdot d$.

Hence, $|ap+qb+c| = d\sqrt{a^2+b^2} \Rightarrow d = \frac{|ap+qb+c|}{\sqrt{a^2+b^2}}$.

39 From the diagram, we have:



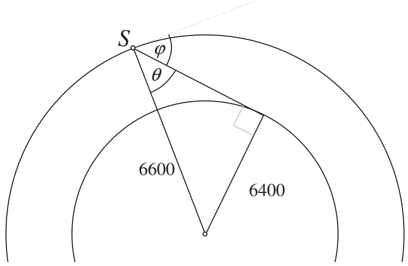
$$\tan\alpha = \frac{x}{d+y} \Rightarrow \cot\alpha = \frac{d+y}{x} \Rightarrow x\cot\alpha - d = y$$

$$\tan\beta = \frac{x}{y} \Rightarrow \cot\beta = \frac{y}{x} \Rightarrow x\cot\beta = y$$

Therefore:

$$x\cot\alpha - d = x\cot\beta \Rightarrow x(\cot\alpha - \cot\beta) = d \Rightarrow x = \frac{d}{\cot\alpha - \cot\beta}$$

40 The angle of depression is denoted by φ. From the right triangle on the diagram right, we will firstly find angle θ:



$$\sin\theta = \frac{6400}{6600} \Rightarrow \theta = \sin^{-1}\frac{64}{66} \approx 75.86^\circ$$

Hence, $\varphi = 90^\circ - \theta \approx 14^\circ$.

Exercise 8.2

1 $r = \sqrt{x^2+y^2} = \sqrt{12^2+9^2} = 15$

Then, $\sin\theta = \frac{y}{r} = \frac{9}{15} = \frac{3}{5}$

$\cos\theta = \frac{x}{r} = \frac{12}{15} = \frac{4}{5}$

$\tan\theta = \frac{y}{x} = \frac{9}{12} = \frac{3}{4}$

2 $r = \sqrt{x^2+y^2} = \sqrt{(-35)^2+12^2} = 37$

Then, $\sin\theta = \frac{y}{r} = \frac{12}{37}$

$\cos\theta = \frac{x}{r} = -\frac{35}{37}$

$\tan\theta = \frac{y}{x} = -\frac{12}{35}$

3 $r = \sqrt{x^2+y^2} = \sqrt{1^2+(-1)^2} = \sqrt{2}$

Then, $\sin\theta = \frac{y}{r} = \frac{-1}{\sqrt{2}} = -\frac{\sqrt{2}}{2}$

$\cos\theta = \frac{x}{r} = \frac{1}{\sqrt{2}} = \frac{\sqrt{2}}{2}$

$\tan\theta = \frac{-1}{1} = -1$

4 $r = \sqrt{x^2+y^2} = \sqrt{\left(-\sqrt{75}\right)^2+(-5)^2} = 10$

Then, $\sin\theta = \frac{y}{r} = \frac{-5}{10} = -\frac{1}{2}$

$\cos\theta = \frac{x}{r} = \frac{-\sqrt{75}}{10} = -\frac{\sqrt{3}}{2}$

$\tan\theta = \frac{y}{x} = \frac{-5}{-\sqrt{75}} = \frac{1}{\sqrt{3}} = \frac{\sqrt{3}}{3}$

5 **a)** The terminal side of the angle forms a $180^\circ - 120^\circ = 60^\circ$ angle with the x-axis. The sine values for 120° and 60° will be exactly the same, and the cosine and tangent values will be the same but of opposite sign. So:

$\sin 120^\circ = \sin 60^\circ = \frac{\sqrt{3}}{2}$

$\cos 120^\circ = -\cos 60^\circ = -\frac{1}{2}$

$\tan 120^\circ = -\tan 60^\circ = -\sqrt{3}$

The values of the other three functions are:

$\cot 120^\circ = \frac{1}{\tan 120^\circ} = -\frac{\sqrt{3}}{3}$

$\sec 120^\circ = \frac{1}{\cos 120^\circ} = -2$

$\csc 120^\circ = \frac{1}{\sin 120^\circ} = \frac{2\sqrt{3}}{3}$

b) The terminal side of the angle forms a $180^\circ - 135^\circ = 45^\circ$ angle with the x-axis. The sine values for 135° and 45° will be exactly the same, and the cosine and tangent values will be the same but of opposite sign. So:

$$\sin 135^\circ = \sin 45^\circ = \frac{\sqrt{2}}{2}$$

$$\cos 135^\circ = -\cos 45^\circ = -\frac{\sqrt{2}}{2}$$

$$\tan 135^\circ = -\tan 45^\circ = -1$$

The values of the other three functions are:

$$\cot 135^\circ = \frac{1}{\tan 135^\circ} = -1$$

$$\sec 135^\circ = \frac{1}{\cos 135^\circ} = -\sqrt{2}$$

$$\csc 135^\circ = \frac{1}{\sin 135^\circ} = \sqrt{2}$$

c) The terminal side of the angle lies in the fourth quadrant, and $330^\circ = 360^\circ - 30^\circ$. So, the sine and tangent values for 330° and 30° are the same but of opposite sign, and the cosine values will be exactly the same. So:

$$\sin 330^\circ = -\sin 30^\circ = -\frac{1}{2}$$

$$\cos 330^\circ = \cos 30^\circ = \frac{\sqrt{3}}{2}$$

$$\tan 330^\circ = -\tan 30^\circ = -\frac{\sqrt{3}}{3}$$

The values of the other three functions are:

$$\cot 330^\circ = \frac{1}{\tan 330^\circ} = -\sqrt{3}$$

$$\sec 330^\circ = \frac{1}{\cos 330^\circ} = \frac{2\sqrt{3}}{3}$$

$$\csc 330^\circ = \frac{1}{\sin 330^\circ} = -2$$

d) The terminal side of the angle lies on the negative y-axis. So, $\sin\left(270^\circ\right) = -1$, $\cos\left(270^\circ\right) = 0$, and the tangent value is not defined. $\cot\left(270^\circ\right) = 0$, $\csc\left(270^\circ\right) = -1$, and the secant is not defined.

e) The terminal side of the angle lies in the third quadrant, and $240^\circ = 180^\circ + 60^\circ$. So, the sine and cosine values for 240° and 60° are the same but of opposite sign, and the tangent values will be exactly the same. So:

$$\sin 240^\circ = -\sin 60^\circ = -\frac{\sqrt{3}}{2}$$

$$\cos 240^\circ = -\cos 60^\circ = -\frac{1}{2}$$

$$\tan 240^\circ = \tan 60^\circ = \sqrt{3}$$

The values of the other three functions are:

$$\cot 240^\circ = \frac{1}{\tan 240^\circ} = \frac{\sqrt{3}}{3}$$

$$\sec 240^\circ = \frac{1}{\cos 240^\circ} = -2$$

$$\csc 240^\circ = \frac{1}{\sin 240^\circ} = -\frac{2\sqrt{3}}{3}$$

f) The terminal side of the angle lies in the third quadrant, and $\frac{5\pi}{4} = \pi + \frac{\pi}{4}$. So, the sine and cosine values for $\frac{5\pi}{4}$ and $\frac{\pi}{4}$ are the same but of opposite sign, and the tangent values will be exactly the same. So:

$$\sin\frac{5\pi}{4} = -\sin\frac{\pi}{4} = -\frac{\sqrt{2}}{2}$$

$$\cos\frac{5\pi}{4} = -\cos\frac{\pi}{4} = -\frac{\sqrt{2}}{2}$$

$$\tan\frac{5\pi}{4} = \tan\frac{\pi}{4} = 1$$

The values of the other three functions are:

$$\cot\frac{5\pi}{4} = \frac{1}{\tan\frac{5\pi}{4}} = 1$$

$$\sec\frac{5\pi}{4} = \frac{1}{\cos\frac{5\pi}{4}} = -\sqrt{2}$$

$$\csc\frac{5\pi}{4} = \frac{1}{\sin\frac{5\pi}{4}} = -\sqrt{2}$$

g) The terminal side of the angle lies in the fourth quadrant. So, the sine and tangent values for $-\frac{\pi}{6}$ and $\frac{\pi}{6}$ are the same but of opposite sign, and the cosine values will be exactly the same. So:

$$\sin\left(-\frac{\pi}{6}\right) = -\sin\frac{\pi}{6} = -\frac{1}{2}$$

$$\cos\left(-\frac{\pi}{6}\right) = \cos\frac{\pi}{6} = \frac{\sqrt{3}}{2}$$

$$\tan\left(-\frac{\pi}{6}\right) = -\tan\frac{\pi}{6} = -\frac{\sqrt{3}}{3}$$

The values of the other three functions are:

$$\cot\left(-\frac{\pi}{6}\right) = \frac{1}{\tan\left(-\frac{\pi}{6}\right)} = -\sqrt{3}$$

$$\sec\left(-\frac{\pi}{6}\right) = \frac{1}{\cos\left(-\frac{\pi}{6}\right)} = \frac{2\sqrt{3}}{3}$$

$$\csc\left(-\frac{\pi}{6}\right) = \frac{1}{\sin\left(-\frac{\pi}{6}\right)} = -2$$

h) The terminal side of the angle lies in the third quadrant, and $\frac{7\pi}{6} = \pi + \frac{\pi}{6}$. So, the sine and cosine values for $\frac{7\pi}{6}$ and $\frac{\pi}{6}$ are the same but of opposite sign, and the tangent values will be exactly the same. So:

$$\sin\frac{7\pi}{6} = -\sin\frac{\pi}{6} = -\frac{1}{2}$$

$$\cos\frac{7\pi}{6} = -\cos\frac{\pi}{6} = -\frac{\sqrt{3}}{2}$$

$$\tan\frac{7\pi}{6} = \tan\frac{\pi}{6} = \frac{\sqrt{3}}{3}$$

The values of the other three functions are:

$$\cot\frac{7\pi}{6} = \frac{1}{\tan\frac{7\pi}{6}} = \sqrt{3}$$

$$\sec\frac{7\pi}{6} = \frac{1}{\cos\frac{7\pi}{6}} = -\frac{2\sqrt{3}}{3}$$

$$\csc\frac{7\pi}{6} = \frac{1}{\sin\frac{7\pi}{6}} = -2$$

i) The terminal side of the angle lies in the fourth quadrant. So, the sine and tangent values for -60° and 60° are the same but of opposite sign, and the cosine values will be exactly the same. So:

$$\sin\left(-60^\circ\right) = -\sin 60^\circ = -\frac{\sqrt{3}}{2}$$

$$\cos\left(-60^\circ\right) = \cos 60^\circ = \frac{1}{2}$$

$$\tan\left(-60^\circ\right) = -\tan 60^\circ = -\sqrt{3}$$

The values of the other three functions are:

$$\cot\left(-60^\circ\right) = \frac{1}{\tan\left(-60^\circ\right)} = -\frac{\sqrt{3}}{3}$$

$$\sec\left(-60^\circ\right) = \frac{1}{\cos\left(-60^\circ\right)} = 2$$

$$\csc\left(-60^\circ\right) = \frac{1}{\sin\left(-60^\circ\right)} = -\frac{2\sqrt{3}}{3}$$

j) The terminal side of the angle lies on the positive y-axis. So, $\sin\left(-\frac{3\pi}{2}\right) = 1$, $\cos\left(-\frac{3\pi}{2}\right) = 0$, and the tangent value is not defined. $\cot\left(-\frac{3\pi}{2}\right) = 0$, $\csc\left(-\frac{3\pi}{2}\right) = 1$, and the secant is not defined.

k) The terminal side of the angle lies in the fourth quadrant, and $\frac{5\pi}{3} = 2\pi - \frac{\pi}{3}$. So, the sine and tangent values for $\frac{5\pi}{3}$ and $\frac{\pi}{3}$ are the same but of opposite sign, and the cosine values will be exactly the same. So:

$$\sin\frac{5\pi}{3} = -\sin\frac{\pi}{3} = -\frac{\sqrt{3}}{2}$$

$$\cos\frac{5\pi}{3} = \cos\frac{\pi}{3} = \frac{1}{2}$$

$$\tan\frac{5\pi}{3} = -\tan\frac{\pi}{3} = -\sqrt{3}$$

The values of the other three functions are:

$$\cot\frac{5\pi}{3} = \frac{1}{\tan\frac{5\pi}{3}} = -\frac{\sqrt{3}}{3}$$

$$\sec\frac{5\pi}{3}=\frac{1}{\cos\frac{5\pi}{3}}=2$$

$$\csc\frac{5\pi}{3}=\frac{1}{\sin\frac{5\pi}{3}}=-\frac{2\sqrt{3}}{3}$$

l) The terminal side of the angle lies in the second quadrant, and $-210^\circ=-180^\circ-30^\circ$. So, the sine value for -210° and 30° will be exactly the same, and the cosine and tangent values are the same but of opposite sign. So:

$$\sin\left(-210^\circ\right)=\sin 30^\circ=\frac{1}{2}$$

$$\cos\left(-210^\circ\right)=-\cos 30^\circ=-\frac{\sqrt{3}}{2}$$

$$\tan\left(-210^\circ\right)=-\tan 30^\circ=-\frac{\sqrt{3}}{3}$$

The values of the other three functions are:

$$\cot\left(-210^\circ\right)=\frac{1}{\tan\left(-210^\circ\right)}=-\sqrt{3}$$

$$\sec\left(-210^\circ\right)=\frac{1}{\cos\left(-210^\circ\right)}=-\frac{2\sqrt{3}}{3}$$

$$\csc\left(-210^\circ\right)=\frac{1}{\sin\left(-210^\circ\right)}=2$$

m) The terminal side of the angle lies in the fourth quadrant. So, the sine and tangent values for $-\frac{\pi}{4}$ and $\frac{\pi}{4}$ are the same but of opposite sign, and the cosine values will be exactly the same. So:

$$\sin\left(-\frac{\pi}{4}\right)=-\sin\frac{\pi}{4}=-\frac{\sqrt{2}}{2}$$

$$\cos\left(-\frac{\pi}{4}\right)=\cos\frac{\pi}{4}=\frac{\sqrt{2}}{2}$$

$$\tan\left(-\frac{\pi}{4}\right)=-\tan\frac{\pi}{4}=-1$$

The values of the other three functions are:

$$\cot\left(-\frac{\pi}{4}\right)=\frac{1}{\tan\left(-\frac{\pi}{4}\right)}=-1$$

$$\sec\left(-\frac{\pi}{4}\right)=\frac{1}{\cos\left(-\frac{\pi}{4}\right)}=\sqrt{2}$$

$$\csc\left(-\frac{\pi}{4}\right)=\frac{1}{\sin\left(-\frac{\pi}{4}\right)}=-\sqrt{2}$$

n) The terminal side of the angle lies on the negative x-axis. So, $\sin(\pi)=0$, $\cos(\pi)=-1$, and $\tan(\pi)=0$. The cotangent and cosecant are not defined, and $\sec(\pi)=-1$.

o) Since $4.25\pi=4\pi+\frac{\pi}{4}$, the terminal sides of 4.25π and $\frac{\pi}{4}$ are the same. So:

$$\sin(4.25\pi)=\sin\frac{\pi}{4}=\frac{\sqrt{2}}{2}$$

$$\cos(4.25\pi)=\cos\frac{\pi}{4}=\frac{\sqrt{2}}{2}$$

$$\tan(4.25\pi)=\tan\frac{\pi}{4}=1$$

The values of the other three functions are:

$$\cot(4.25\pi)=\frac{1}{\tan(4.25\pi)}=1$$

$$\sec(4.25\pi)=\frac{1}{\cos(4.25\pi)}=\sqrt{2}$$

$$\csc(4.25\pi)=\frac{1}{\sin(4.25\pi)}=\sqrt{2}$$

6 From cosine, we can immediately determine the secant: $\sec\theta=\frac{17}{8}$. θ is an angle in the first quadrant. It follows from the definition $\cos\theta=\frac{x}{r}$ that, with θ in standard position, there must be a point on the terminal side of the angle that is 17 units from the origin (i.e. $r=17$) and which has an x-coordinate of 8. Using Pythagoras' theorem:

$|y|=\sqrt{17^2-8^2}=15$. Because θ is in the first quadrant, the y-coordinate of the point must be positive; thus, $y=15$. Therefore, $\sin\theta=\frac{15}{17}$, $\csc\theta=\frac{17}{15}$, $\tan\theta=\frac{15}{8}$, and $\cot\theta=\frac{8}{15}$.

7 When the sine and tangent are negative, then θ is an angle in the fourth quadrant. It follows from the definition $\tan\theta = \frac{y}{x}$ that, with θ in standard position, there must be a point on the terminal side of the angle which has an x-coordinate of 5 and a y-coordinate of -6. Using Pythagoras' theorem:

$r = \sqrt{5^2 + (-6)^2} = \sqrt{61}$.

Therefore, $\sin\theta = \frac{-6}{\sqrt{61}} = -\frac{6\sqrt{61}}{61}$ and

$\cos\theta = \frac{5}{\sqrt{61}} = \frac{5\sqrt{61}}{61}$.

8 If $\sin\theta = 0$, then the cotangent and cosecant are not defined. The terminal side of the angle lies on the negative x-axis, so $\cos\theta = -1$, and $\sec\theta = -1$. And $\tan\theta = 0$.

9 From secant, we can immediately determine the cosine: $\cos\theta = \frac{1}{2}$. θ is an angle in the fourth quadrant. It follows from the definition $\cos\theta = \frac{x}{r}$ that, with θ in standard position, there must be a point on the terminal side of the angle that is 2 units from the origin (i.e. $r = 2$) and which has an x-coordinate of 1. Using Pythagoras' theorem: $|y| = \sqrt{2^2 - 1^2} = \sqrt{3}$. Because θ is in the fourth quadrant, the y-coordinate of the point must be negative; thus, $y = -\sqrt{3}$. Therefore, $\sin\theta = -\frac{\sqrt{3}}{2}$,

$\csc\theta = -\frac{2\sqrt{3}}{3}$, $\tan\theta = -\sqrt{3}$, and

$\cot\theta = -\frac{\sqrt{3}}{3}$.

10 **a)** **i)** 150° is in the second quadrant. Using the identity $\sin(180^\circ - \theta) = \sin\theta$, we have: $180^\circ - 150^\circ = 30^\circ$; therefore, $\sin 150^\circ = \sin 30^\circ$.

ii) 95° is in the second quadrant. Using the identity $\sin(180^\circ - \theta) = \sin\theta$, we have: $180^\circ - 95^\circ = 85^\circ$; therefore, $\sin 95^\circ = \sin 85^\circ$.

b) **i)** 315° is in the fourth quadrant. Using the identity $\cos(360^\circ - \theta) = \cos\theta$, we have: $360^\circ - 315^\circ = 45^\circ$; therefore, $\cos 315^\circ = \cos 45^\circ$.

ii) 353° is in the fourth quadrant. Using the identity $\cos(360^\circ - \theta) = \cos\theta$, we have: $360^\circ - 353^\circ = 7^\circ$; therefore, $\cos 353^\circ = \cos 7^\circ$.

c) **i)** 240° is in the third quadrant. Using the identity $\tan(\theta - 180^\circ) = \tan\theta$, we have: $240^\circ - 180^\circ = 60^\circ$; therefore, $\tan 240^\circ = \tan 60^\circ$.

ii) 200° is in the third quadrant. Using the identity $\tan(\theta - 180^\circ) = \tan\theta$, we have: $200^\circ - 180^\circ = 20^\circ$; therefore, $\tan 200^\circ = \tan 20^\circ$.

11 For each triangle in parts **a**–**c**, we are given a pair of sides and the included angle (SAS).

a) $\text{Area} = \frac{1}{2}(4)(6)\sin(60^\circ)$

$= 12\left(\frac{\sqrt{3}}{2}\right) = 6\sqrt{3}\text{ units}^2$

b) $\text{Area} = \frac{1}{2}(8)(23)\sin(105^\circ)$

$= 92\sin(105^\circ) \approx 88.9\text{ units}^2$

c) $\text{Area} = \frac{1}{2}(30)(90)\sin(45^\circ)$

$= 1350\frac{\sqrt{2}}{2} = 675\sqrt{2}\text{ units}^2$

12 In the triangle given: $AB = c = 12$ cm, and $AC = b = 15$ cm. The formula for area is:

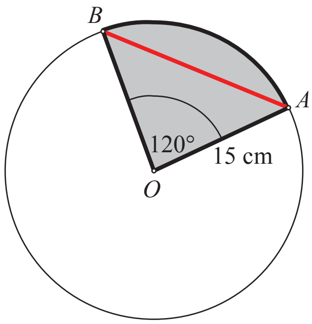
$A = \frac{1}{2}bc\sin\hat{A}$; hence:

$43 = \frac{1}{2}12\cdot 15\sin\hat{A} \Rightarrow \sin\hat{A} = \frac{43}{90}$

$\Rightarrow \sin^{-1}\left(\frac{43}{90}\right) \approx 28.5^\circ$.

There are two triangles possible with the given area and given sides: one has an acute angle $\hat{A} \approx 28.5^\circ$, and the other an obtuse angle $\hat{A} \approx 180^\circ - 28.540...^\circ \approx 151^\circ$.

13



a) The formula for the area of a sector is $A = \frac{1}{2}r^2\theta$, where θ is in radians.

120° in radian measure is:

$$120^\circ = 120^\circ\left(\frac{\pi}{180^\circ}\right) = \frac{120^\circ}{180^\circ}\pi = \frac{2\pi}{3}.$$

Therefore, $A = \frac{1}{2}15^2\frac{2\pi}{3} = 75\pi \approx 236\ \text{cm}^2$.

b) For triangle AOB we are given a pair of sides and the included angle, so:

$$A = \frac{1}{2}15\cdot 15\sin 120^\circ = \frac{225}{2}\frac{\sqrt{3}}{2}$$

$$= \frac{225\sqrt{3}}{4} \approx 97.4\ \text{cm}^2.$$

14 For parts **a** and **b**, the area of the shaded region (segment) can be found by subtracting the area of the triangle from the area of the sector.

a) Area of the sector:

$$A_{sector} = \frac{1}{2}r^2\theta = \frac{1}{2}10^2\cdot\frac{\pi}{3} = \frac{50\pi}{3}\ \text{cm}^2$$

Area of the triangle:

$$A_{triangle} = \frac{1}{2}10\cdot 10\sin\frac{\pi}{3} = 50\frac{\sqrt{3}}{2}$$

$$= 25\sqrt{3}\ \text{cm}^2$$

$$A_{segment} = A_{sector} - A_{triangle}$$

$$= \frac{50\pi}{3} - 25\sqrt{3} \approx 9.06\ \text{cm}^2.$$

b) Firstly, we need to convert the degrees to radian measure:

$$135^\circ = 135^\circ\left(\frac{\pi}{180^\circ}\right) = \frac{135^\circ}{180^\circ}\pi = \frac{3\pi}{4}$$

Area of the sector:

$$A_{sector} = \frac{1}{2}r^2\theta = \frac{1}{2}\cdot 12^2\cdot\frac{3\pi}{4} = 54\pi\ \text{cm}^2$$

Area of the triangle:

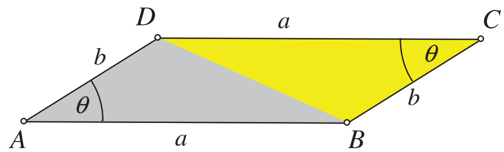
$$A_{triangle} = \frac{1}{2}12\cdot 12\sin 135^\circ$$

$$= 72\frac{\sqrt{2}}{2} = 36\sqrt{2}\ \text{cm}^2$$

$$A_{segment} = A_{sector} - A_{triangle}$$

$$= 54\pi - 36\sqrt{2} \approx 119\ \text{cm}^2$$

15 Triangles ABD and BCD are congruent.

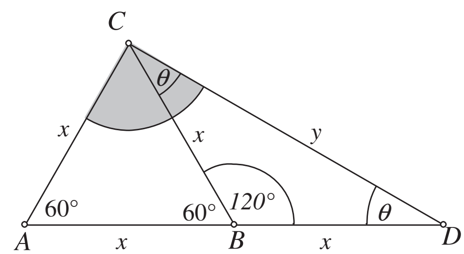


For area, it holds:

$$A = A_{ABC} + A_{BCD} = \frac{1}{2}ab\sin\theta + \frac{1}{2}ab\sin\theta$$

$$= ab\sin\theta$$

16 Method I:



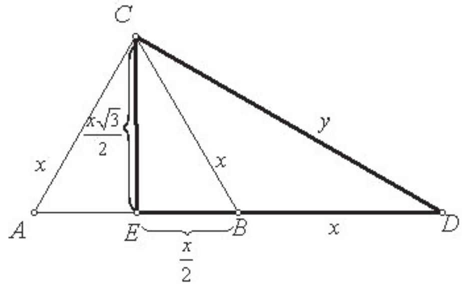
Triangle ABC is equilateral; hence, angle $D\hat{B}C = 180^\circ - 60^\circ = 120^\circ$. Triangle BDC is isosceles; hence, $\hat{D} = \frac{180^\circ - 120^\circ}{2} = 30^\circ$. Finally, we can deduce that the shaded angle in C is 90°, and ADC is a right triangle.

$$\sin 60^\circ = \frac{y}{2x} \Rightarrow \frac{\sqrt{3}}{2} = \frac{y}{2x} \Rightarrow y = x\sqrt{3}$$

(**Note:** We can find y using a Pythagorean identity: $y = \sqrt{(2x)^2 - x^2} = \sqrt{3x^2} = x\sqrt{3}$.)

Method II:

Using a Pythagorean identity, or an appropriate trigonometric value for an angle of 60°, we can determine the sides EB and EC. Now, we can determine y using a Pythagorean identity:



$$y = \sqrt{\left(\frac{x\sqrt{3}}{2}\right)^2 + \left(\frac{3x}{2}\right)^2} = \sqrt{\frac{3x^2}{4} + \frac{9x^2}{4}}$$
$$= \sqrt{3x^2} = x\sqrt{3}$$

17 We will find the area of all triangles:

$$A_{FJG} = \frac{1}{2}xh\sin\theta$$

$$A_{JHG} = \frac{1}{2}xf\sin\theta$$

$$A_{FHG} = \frac{1}{2}hf\sin 2\theta$$

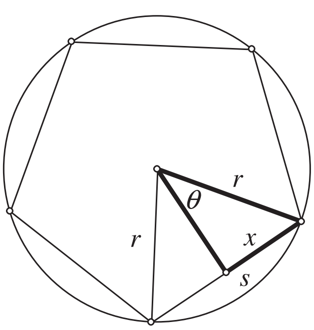
Since $A_{FJG} + A_{JHG} = A_{FHG}$, we have:

$$\frac{1}{2}xh\sin\theta + \frac{1}{2}xf\sin\theta = \frac{1}{2}hf\sin 2\theta \Rightarrow$$
$$x\sin\theta(h+f) = hf\sin 2\theta$$

Finally, using the double angle formula for sine, we have:

$$x\sin\theta\cdot(h+f) = 2hf\sin\theta\cos\theta \Rightarrow$$
$$x(h+f) = 2hf\cos\theta \Rightarrow x = \frac{2hf\cos\theta}{h+f}$$

18



The right triangle on the diagram has hypotenuse r and the leg opposite angle θ is $x = \frac{s}{2}$. The angle $\theta = \frac{360^\circ : n}{2} = \frac{180^\circ}{n}$. Hence,

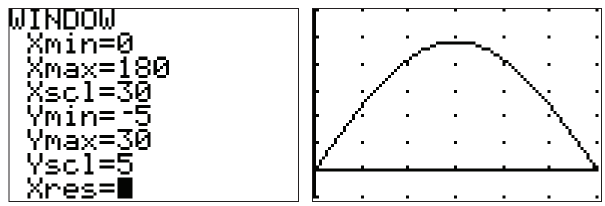
$$\frac{x}{r} = \sin\theta \Rightarrow x = r\sin\left(\frac{180^\circ}{n}\right) \Rightarrow$$
$$\frac{s}{2} = r\sin\left(\frac{180^\circ}{n}\right)$$

and, finally, $s = 2r\sin\left(\frac{180^\circ}{n}\right)$.

19 **a)** $A = \frac{1}{2}6\cdot 8\sin x = 24\sin x$

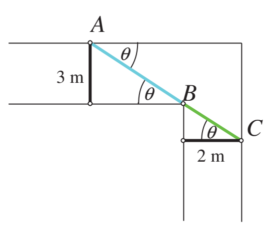
b) Domain: $0^\circ < x < 180^\circ$

For $0^\circ < x < 180^\circ$, $0 < \sin x \leqslant 1$, then $0 < 24\sin x \leqslant 24$, and the range of the function is: $0 < A \leqslant 24$.



c) From the range, we can see that $A_{max} = 24$. So, we have to solve the equation $24\sin x = 24 \Rightarrow \sin x = 1 \Rightarrow x = 90^\circ$. Hence, the maximum point is: $(90^\circ, 24)$, and the triangle is a right triangle. So, a right triangle gives a maximum area. A geometrical reason for that is the fact that, in a right triangle, the height of a leg equals the other leg, while in the other types of triangles, the height is shorter than the side.

20 **a)**



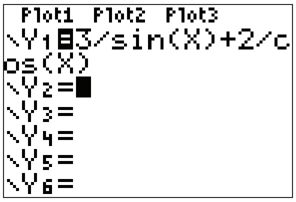
From the diagram, we can see that $L(\theta) = AB + BC$.

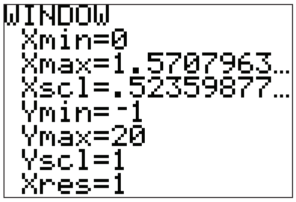
$$\frac{3}{AB} = \sin\theta \Rightarrow AB = 3\csc\theta$$

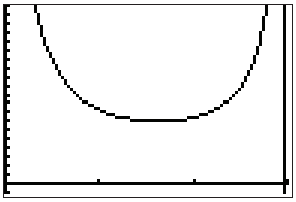
$$\frac{2}{CB} = \cos\theta \Rightarrow CB = 2\sec\theta$$

Hence: $L(\theta) = 3\csc\theta + 2\sec\theta$

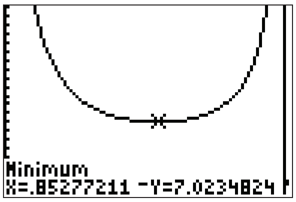
b)



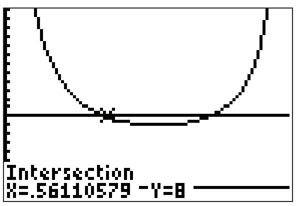


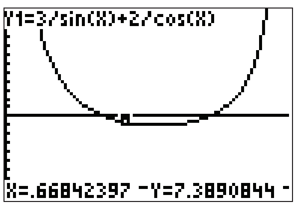


c)



The minimum value of function L is 7.02 m. This is the length of the longest rod that can be carried around the corner.





If we take a longer rod, for example, an 8 m rod, then for an angle of 0.561 we will be stuck in the corridor, and we can't enlarge the angle, because a larger angle will demand a rod shorter than 8 metres.

21 From the drawing, we have:

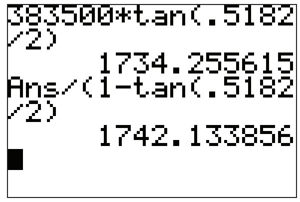
$$\frac{r}{383\,500+r} = \tan\left(\frac{0.5182^\circ}{2}\right) \Rightarrow$$

$$r = (383\,500+r)\tan\left(\frac{0.5182^\circ}{2}\right)$$

Hence,

$$r - r\tan\left(\frac{0.5182^\circ}{2}\right) = 383\,500\tan\left(\frac{0.5182^\circ}{2}\right)$$

$$\Rightarrow r = \frac{383\,500\tan\left(\frac{0.5182^\circ}{2}\right)}{1-\tan\left(\frac{0.5182^\circ}{2}\right)} \approx 1740 \text{ km}$$



22 a) For angles $0 \leqslant \theta < \frac{\pi}{2}$, the cosines are positive (notice that we exclude $\frac{\pi}{2}$ when cosine is 0 and hence the secant is not defined). So,

$\cos\theta = +\sqrt{1-\sin^2\theta} = \sqrt{1-x^2}$. Hence,

$\sec\theta = \frac{1}{\sqrt{1-x^2}}$, where $0 \leqslant x < 1$.

b) For angles $0 < \beta < \frac{\pi}{2}$, the sines are positive (notice that we exclude $\frac{\pi}{2}$ when the tangent is not defined). So,

$\frac{\sin\beta}{\cos\beta} = y \Rightarrow \frac{\sin\beta}{y} = \cos\beta$. Using the Pythagorean identity for sine and cosine, we have:

$$\sin^2\beta + \cos^2\beta = 1 \Rightarrow \sin^2\beta + \left(\frac{\sin\beta}{y}\right)^2 = 1$$

$$\Rightarrow y^2\sin^2\beta + \sin^2\beta = y^2$$

Finally:

$$\sin^2\beta = \frac{y^2}{y^2+1} \Rightarrow \sin\beta = \frac{y}{\sqrt{y^2+1}} = \frac{y\sqrt{y^2+1}}{y^2+1}$$

23 From right triangle OAP: $\cos\theta = \frac{OA}{1} = OA$

From right triangle OBP: $\tan\theta = \frac{PB}{1} = PB$

From right triangle OBP:

$\frac{1}{OB} = \cos\theta \Rightarrow \sec\theta = OB$

In right triangle OPC, the angle in O is $90^\circ - \theta$, so the angle in C is θ.

From right triangle OPC:

$\cot\theta = \cot\widehat{P} = \frac{CP}{1} = CP$

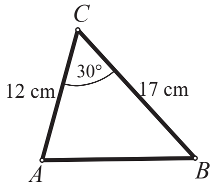
From right triangle OPC:

$\frac{1}{OC} = \sin\theta \Rightarrow \csc\theta = OC$

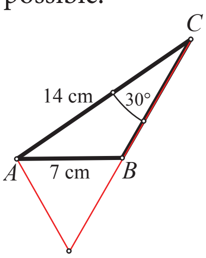
Exercise 8.3 and 8.4

1 Three angles (AAA) are given and no side. The sum of the angles equals 180°; hence, there are infinitely many triangles.

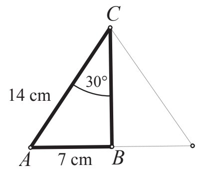
2 Two sides and their included angle (SAS) are given; therefore, there is one unique triangle.



3 Two sides and a non-included angle (SSA) are given; so, no triangle, one triangle, or two triangles are possible. In this case, one side is half the other and the angle opposite the shorter side is 30°; therefore, there is only one triangle possible.

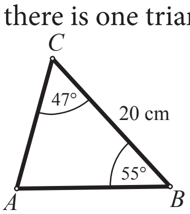


Possible position of the given elements.

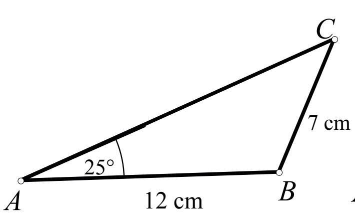


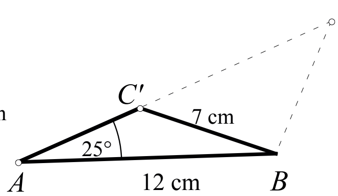
Actual position of the given elements (half of an equilateral triangle).

4 Two angles and a side (ASA) are given; therefore, there is one triangle.

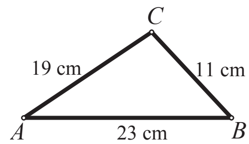


5 Two sides and a non-included angle (SSA) are given; so, no triangle, one triangle, or two triangles are possible. In this case, triangles ABC and ABC' have the given elements. Therefore, there are two distinct triangles.



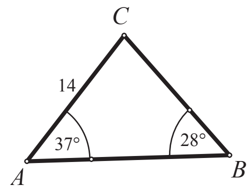


6 Three sides (SSS) are given. The sum of the two sides is greater than the third, so there is one triangle.



In questions 7–20, the diagrams are not drawn to scale.

7 AAS is given.
The third angle is $A\hat{C}B = 180^\circ - 37^\circ - 28^\circ = 115^\circ$.



Using the law of sines:

$$\frac{14}{\sin 28^\circ} = \frac{BC}{\sin 37^\circ} \Rightarrow BC = \frac{14 \sin 37^\circ}{\sin 28^\circ} \approx 17.9$$

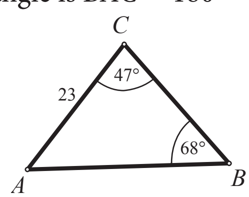
$$\frac{14}{\sin 28^\circ} = \frac{AB}{\sin 115^\circ} \Rightarrow AB = \frac{14 \sin 115^\circ}{\sin 28^\circ} \approx 27.0$$

Therefore, $A\hat{C}B = 115^\circ$, $BC \approx 17.9$ and $AB \approx 27.0$.

```
180-37-28
                115
14*sin(37)/sin(2
8)
        17.94658291
```

```
                115
14*sin(37)/sin(2
8)
        17.94658291
14*sin(115)/sin(
28)
        27.02678932
```

8 AAS is given.
The third angle is $B\hat{A}C = 180^\circ - 68^\circ - 47^\circ = 65^\circ$.



Using the law of sines:

$$\frac{23}{\sin 68^\circ} = \frac{AB}{\sin 47^\circ} \Rightarrow AB = \frac{23 \sin 47^\circ}{\sin 68^\circ} \approx 18.1$$

$$\frac{23}{\sin 68^\circ} = \frac{BC}{\sin 65^\circ} \Rightarrow BC = \frac{23 \sin 65^\circ}{\sin 68^\circ} \approx 22.5$$

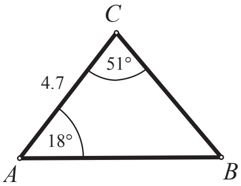
Therefore, $B\hat{A}C = 65°$, $AB \approx 18.1$ and $BC \approx 22.5$.

```
180-68-47
                65
23*sin(47)/sin(6
8)
       18.14217866
```

```
                65
23*sin(47)/sin(6
8)
       18.14217866
23*sin(65)/sin(6
8)
       22.48214203
```

9 AAS is given.

The third angle is $A\hat{B}C = 180^{\circ} - 18^{\circ} - 51^{\circ} = 111^{\circ}$.



Using the law of sines:

$$\frac{4.7}{\sin 111^{\circ}} = \frac{AB}{\sin 51^{\circ}} \Rightarrow AB = \frac{4.7 \sin 51^{\circ}}{\sin 111^{\circ}} \approx 3.91$$

$$\frac{4.7}{\sin 111^{\circ}} = \frac{BC}{\sin 18^{\circ}} \Rightarrow BC = \frac{4.7 \sin 18^{\circ}}{\sin 111^{\circ}} \approx 1.56$$

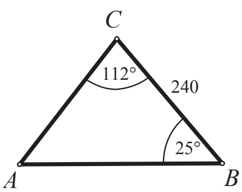
Therefore, $A\hat{B}C = 111°$, $AB \approx 3.91$ and $BC \approx 1.56$.

```
180-18-51
               111
4.7*sin(51)/sin(
111)
       3.912449228
```

```
               111
4.7*sin(51)/sin(
111)
       3.912449228
4.7*sin(18)/sin(
111)
        1.55570943
```

10 ASA is given.

The third angle is $B\hat{A}C = 180^{\circ} - 112^{\circ} - 25^{\circ} = 43^{\circ}$.



Using the law of sines:

$$\frac{240}{\sin 43^{\circ}} = \frac{AB}{\sin 112^{\circ}} \Rightarrow AB = \frac{240 \sin 112^{\circ}}{\sin 43^{\circ}} \approx 326$$

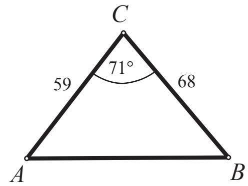
$$\frac{240}{\sin 43^{\circ}} = \frac{AC}{\sin 25^{\circ}} \Rightarrow AC = \frac{240 \sin 25^{\circ}}{\sin 43^{\circ}} \approx 149$$

Therefore, $B\hat{A}C = 43°$, $AB \approx 326$ and $AC \approx 149$.

```
180-112-25
                43
240sin(112)/sin(
43)
       326.2824929
```

```
                43
240sin(112)/sin(
43)
       326.2824929
240sin(25)/sin(4
3)
       148.7223266
```

11 SAS is given.



Using the law of cosines, we can find the third side:

$$AB = \sqrt{59^2 + 68^2 - 2 \cdot 58 \cdot 68 \cos 71^{\circ}} \approx 74.1$$

```
59²+68²-2*59*68*
cos(71)
       5492.641129
√(Ans
       74.11235476
Ans→C
       74.11235476
```

We use the law of sines to solve for one of the other angles, say $A\hat{B}C$:

$$\frac{\sin A\hat{B}C}{59} = \frac{\sin 71}{AB} \Rightarrow$$

$$\sin A\hat{B}C = \frac{59 \sin 71^{\circ}}{AB} \Rightarrow$$

$$A\hat{B}C = \sin^{-1}\left(\frac{59 \sin 71^{\circ}}{AB}\right) \approx 48.8^{\circ}$$

Then, $B\hat{A}C \approx 180^{\circ} - 71^{\circ} - 48.8^{\circ} = 60.2^{\circ}$.

Therefore, $AB \approx 74.1$, $A\hat{B}C \approx 48.8°$ and $B\hat{A}C = 60.2°$.

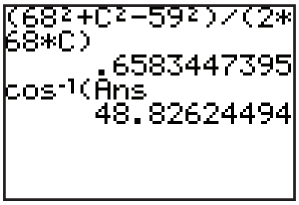
```
       5492.641129
√(Ans
       74.11235476
59*sin(71)/Ans
       .7527165496
sin⁻¹(Ans
       48.82624494
```

```
       74.11235476
59*sin(71)/Ans
       .7527165496
sin⁻¹(Ans
       48.82624494
180-71-Ans
       60.17375506
```

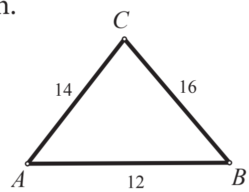
Note: We can solve for any angle using the law of cosines. For example, $A\hat{B}C$:

$$\cos A\hat{B}C = \frac{68^2 + AB^2 - 59^2}{2 \cdot 68 \cdot AB} \Rightarrow$$

$$A\hat{B}C = \cos^{-1}\left(\frac{68^2 + AB^2 - 59^2}{2 \cdot 68 \cdot AB}\right) \approx 48.8^\circ$$



12 SSS is given.



We use the law of cosines to solve for any two angles.

Solve for $B\hat{A}C$:

$$\cos B\hat{A}C = \frac{12^2 + 14^2 - 16^2}{2 \cdot 12 \cdot 14} = \frac{1}{4} \Rightarrow$$

$$B\hat{A}C = \cos^{-1}\left(\frac{1}{4}\right) \approx 75.5^\circ$$

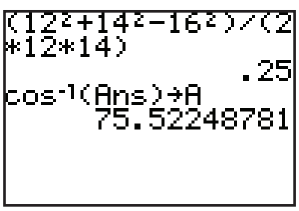
Solve for $A\hat{B}C$:

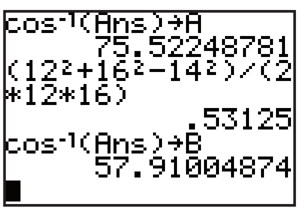
$$\cos A\hat{B}C = \frac{12^2 + 16^2 - 14^2}{2 \cdot 12 \cdot 16} = \frac{17}{32} \Rightarrow$$

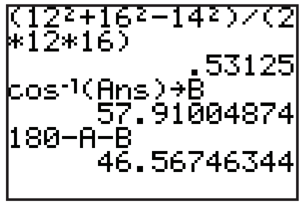
$$A\hat{B}C = \cos^{-1}\left(\frac{17}{32}\right) \approx 57.9^\circ$$

Solve for $A\hat{C}B$:

$$A\hat{C}B \approx 180^\circ - 75.5^\circ - 57.9^\circ = 46.6^\circ$$







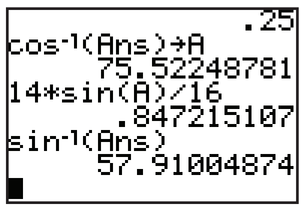
Therefore, $B\hat{A}C = 75.5°$, $A\hat{B}C \approx 57.9°$ and $A\hat{C}B \approx 46.6°$.

Note: We can solve for angles first using the law of cosines, then the law of sines. However, we have to be careful not to find the wrong angle using the law of sines. It is best to use the law of cosines to find the largest angle in a triangle (opposite the largest side, in this case angle $B\hat{A}C$), and then the law of sines to solve for any other angle (those angles should be acute!).

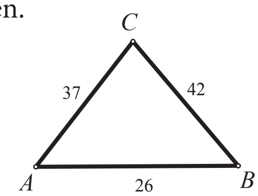
So, after finding $B\hat{A}C \approx 75.5^\circ$, we can use the law of sines:

$$\frac{\sin A\hat{B}C}{14} = \frac{\sin B\hat{A}C}{16} \Rightarrow$$

$$\sin A\hat{B}C = \frac{14 \sin B\hat{A}C}{16} \Rightarrow A\hat{B}C \approx 57.9^\circ$$



13 SSS is given.



We use the law of cosines to solve for any two angles.

Solve for the largest angle $B\hat{A}C$:

$$\cos B\hat{A}C = \frac{26^2 + 37^2 - 42^2}{2 \cdot 26 \cdot 37} = 0.146\,04... \Rightarrow$$

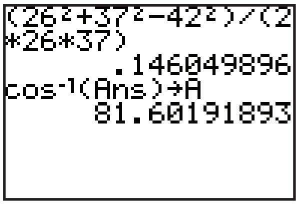
$$B\hat{A}C = \cos^{-1}(0.146\,04...) \approx 81.6^\circ$$

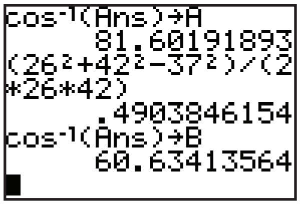
Solve for $A\hat{B}C$:

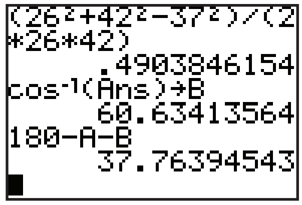
$$\cos A\hat{B}C = \frac{26^2 + 42^2 - 37^2}{2 \cdot 26 \cdot 42} = 0.490\,38... \Rightarrow A\hat{B}C = \cos^{-1}(0.490\,38...) \approx 60.6^\circ$$

Solve for $A\hat{C}B$:

$A\hat{C}B \approx 180^\circ - 81.6^\circ - 60.6^\circ = 37.8^\circ$





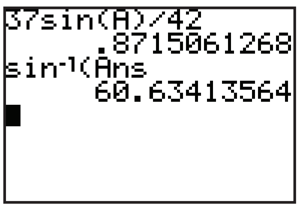


Therefore, $B\hat{A}C \approx 81.6°$, $A\hat{B}C \approx 60.6°$ and $A\hat{C}B = 37.8°$.

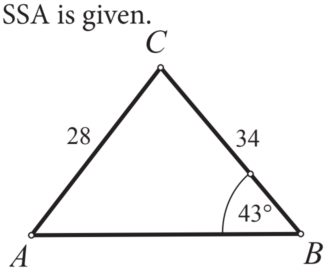
Note: As with question 12, we could have used the law of cosines first, and then the law of sines.

So, after finding $B\hat{A}C \approx 81.6^\circ$, we can use the law of sines:

$$\frac{\sin A\hat{B}C}{37} = \frac{\sin B\hat{A}C}{42} \Rightarrow \sin A\hat{B}C = \frac{37 \sin B\hat{A}C}{42} \Rightarrow A\hat{B}C \approx 60.6^\circ$$



14 SSA is given.

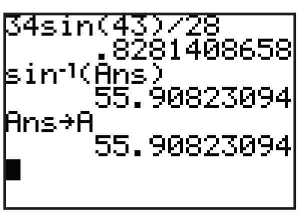


The given angle is opposite the shorter side, so we have the ambiguous case.

(**Hint:** The ambiguous case means there may be none, one or two triangles. The number of triangles depends on the length of the side opposite the given angle.)

Solve for $C\hat{A}B$ using the law of sines:

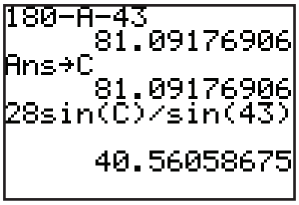
$$\frac{\sin C\hat{A}B}{34} = \frac{\sin 43^\circ}{28} \Rightarrow \sin C\hat{A}B = \frac{34 \sin 43^\circ}{28} \Rightarrow \begin{cases} C\hat{A}B = \sin^{-1}(0.828\,14...) \approx 55.9^\circ \\ C\hat{A}B = 180^\circ - \sin^{-1}(0.828\,14...) \approx 124.1^\circ \end{cases}$$



So, there are two possible triangles.

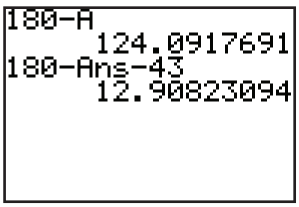
Triangle 1:
A triangle with angles $C\hat{A}B \approx 55.9^\circ$ and $B\hat{C}A \approx 180^\circ - 55.9^\circ - 43^\circ = 81.1^\circ$; and the third side:

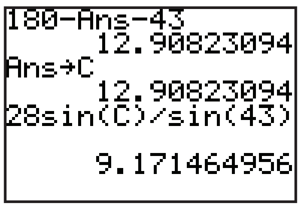
$$\frac{AB}{\sin B\hat{C}A} = \frac{28}{\sin 43^\circ} \Rightarrow AB = \frac{28 \sin B\hat{C}A}{\sin 43^\circ} \approx 40.6.$$



Triangle 2:
A triangle with angles $C\hat{A}B \approx 124.1^\circ$ and $B\hat{C}A \approx 180^\circ - 124.1^\circ - 43^\circ = 12.9^\circ$; and the third side:

$$\frac{AB}{\sin B\hat{C}A} = \frac{28}{\sin 43^\circ} \Rightarrow AB = \frac{28 \sin B\hat{C}A}{\sin 43^\circ} \approx 9.17.$$



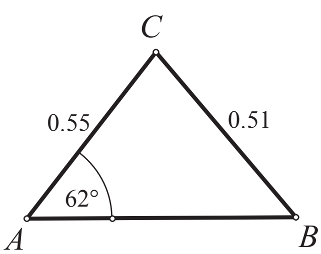


Therefore, the solutions are:

Triangle 1 with: $C\hat{A}B \approx 55.9^\circ$, $B\hat{C}A \approx 81.1^\circ$, $AB \approx 40.6$.

Triangle 2 with: $C\hat{A}B \approx 124.1^\circ$, $B\hat{C}A \approx 12.9^\circ$, $AB \approx 9.17$.

15

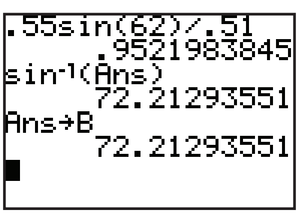


SSA is given.

The given angle is opposite the shorter side, so we have the ambiguous case.

Solve for $A\hat{B}C$ using the law of sines:

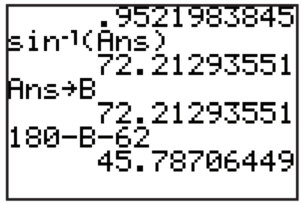
$$\frac{\sin A\hat{B}C}{0.55} = \frac{\sin 62^\circ}{0.51} \Rightarrow \sin A\hat{B}C = \frac{0.55 \sin 62^\circ}{0.51} \Rightarrow \begin{cases} A\hat{B}C = \sin^{-1}(0.952\,19\ldots) \approx 72.2^\circ \\ A\hat{B}C = 180^\circ - \sin^{-1}(0.952\,19\ldots) \approx 107.8^\circ \end{cases}$$



So, there are two possible triangles.

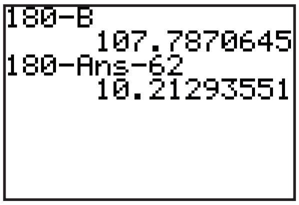
Triangle 1:
A triangle with angles $A\hat{B}C \approx 72.2^\circ$ and $B\hat{C}A \approx 180^\circ - 72.2^\circ - 62^\circ = 45.8^\circ$; and the third side:

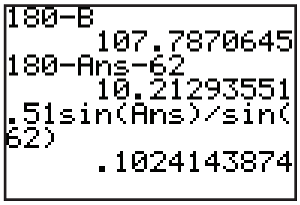
$$\frac{AB}{\sin B\hat{C}A} = \frac{0.51}{\sin 62^\circ} \Rightarrow AB = \frac{0.51 \sin B\hat{C}A}{\sin 62^\circ} \approx 0.414.$$



Triangle 2:
A triangle with angles $A\hat{B}C \approx 107.8^\circ$ and $B\hat{C}A \approx 180^\circ - 107.8^\circ - 62^\circ = 10.2^\circ$; and the third side:

$$\frac{AB}{\sin B\hat{C}A} = \frac{0.51}{\sin 62^\circ} \Rightarrow AB = \frac{0.51 \sin B\hat{C}A}{\sin 62^\circ} \approx 0.102.$$



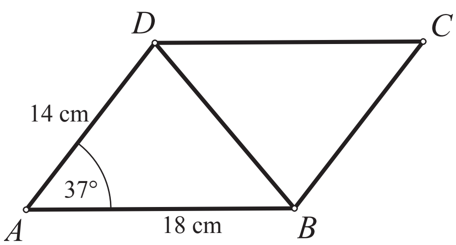


Therefore, the solutions are:

Triangle 1 with: $A\hat{B}C \approx 72.2^\circ$, $B\hat{C}A \approx 45.8^\circ$, $AB \approx 0.414$.

Triangle 2 with: $A\hat{B}C \approx 107.8^\circ$, $B\hat{C}A \approx 10.2^\circ$, $AB \approx 0.102$.

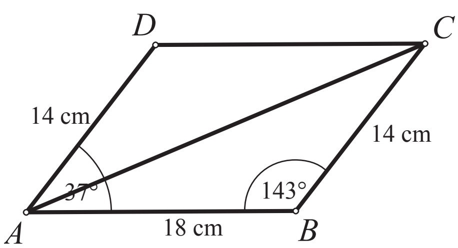
16



We can find the shorter diagonal BD from triangle ABD, where SAS is given.

Use the law of cosines:

$$BD = \sqrt{18^2 + 14^2 - 2 \cdot 18 \cdot 14 \cos 37^\circ} \approx 10.8 \text{ cm}$$



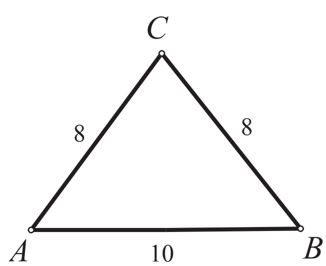
We can find the longer diagonal AC from triangle ABC, where SAS is given.

First determine angle $A\hat{B}C$: $A\hat{B}C \approx 180^\circ - 37^\circ = 143^\circ$

Use the law of cosines:

$$AC = \sqrt{18^2 + 14^2 - 2 \cdot 18 \cdot 14 \cos 143^\circ} \approx 30.4 \text{ cm}$$

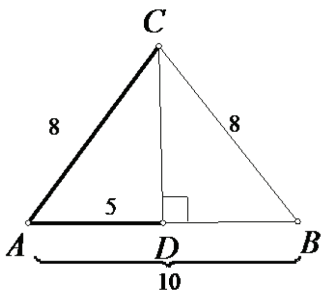
17 SSS is given.
Method I:



Use the law of cosines to solve for any angle:

$$\cos B\hat{A}C = \frac{8^2 + 10^2 - 8^2}{2 \cdot 8 \cdot 10} = \frac{5}{8} \Rightarrow B\hat{A}C = \cos^{-1}\left(\frac{5}{8}\right) \approx 51.3^\circ$$

Angle $A\hat{B}C = B\hat{A}C \approx 51.3^\circ$, and $A\hat{C}B \approx 180^\circ - 2 \cdot 51.3^\circ = 77.4^\circ$.



Method II:

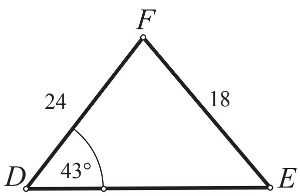
The altitude CD divides an isosceles triangle into two right-angled triangles, as shown.

The angle $B\hat{A}C$ is adjacent to leg AD, so using right triangle ADC we have:

$$\cos B\hat{A}C = \frac{5}{8} \Rightarrow B\hat{A}C = \cos^{-1}\left(\frac{5}{8}\right) = 51.3^\circ,$$

$$A\hat{B}C = B\hat{A}C \approx 51.3^\circ, \text{ and } A\hat{C}B \approx 180^\circ - 2\cdot 51.3^\circ = 77.4^\circ.$$

18

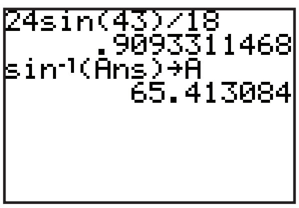


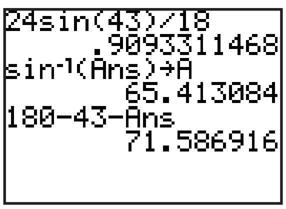
SSA is given.

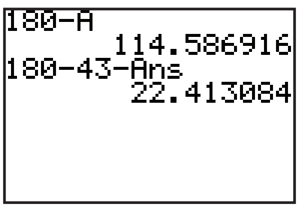
Firstly, we will determine two possible measures of $D\hat{E}F$ using the law of sines:

$$\frac{\sin D\hat{E}F}{24} = \frac{\sin 43^\circ}{18} \Rightarrow \sin D\hat{E}F = \frac{24\cdot\sin 43^\circ}{18} \Rightarrow \begin{cases} D\hat{E}F = \sin^{-1}\left(\dfrac{24\cdot\sin 43^\circ}{18}\right) \approx 65.4^\circ \\ D\hat{E}F = 180^\circ - \sin^{-1}\left(\dfrac{24\cdot\sin 43^\circ}{18}\right) \approx 114.6^\circ \end{cases}$$

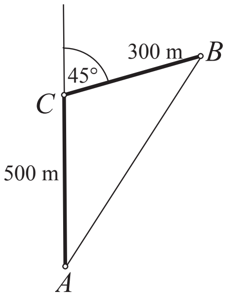
So, angle $D\hat{F}E = 180^\circ - 43^\circ - D\hat{E}F \approx 71.6^\circ$, or $D\hat{F}E = 180^\circ - 43^\circ - D\hat{E}F \approx 22.4^\circ$.





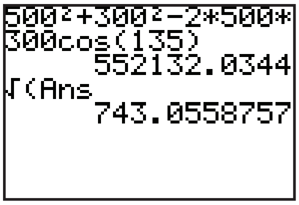


19



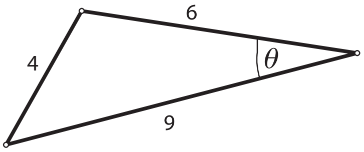
In triangle ACB, SAS is given (the angle $A\hat{C}B = 180^\circ - 45^\circ = 145^\circ$). We can find side AB using the law of cosines:

$$AB = \sqrt{500^2 + 300^2 - 2\cdot 500\cdot 300\cos 135^\circ} \approx 743$$



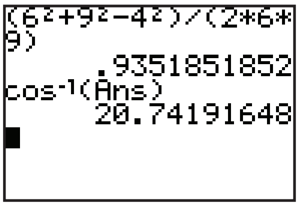
Therefore, the distance between points A and B is approximately 743 m.

20



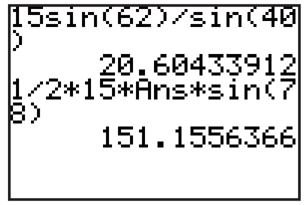
In the triangle, SSS is given, so we can determine all the angles. The smallest angle in the triangle is the angle opposite the shortest side, i.e. the angle opposite the side of 4. Using the law of cosines, we have:

$$\cos\theta = \frac{6^2 + 9^2 - 4^2}{2\cdot 6\cdot 9} \Rightarrow \theta = \cos^{-1}\left(\frac{101}{108}\right) \approx 20.7^\circ.$$



21 SAA is given, so we can determine all the other elements of the triangle.

To use the area formula, $A = \frac{1}{2}ab \sin C$, we need to know two sides and the included angle. Therefore, we will determine side RP using the law of sines:

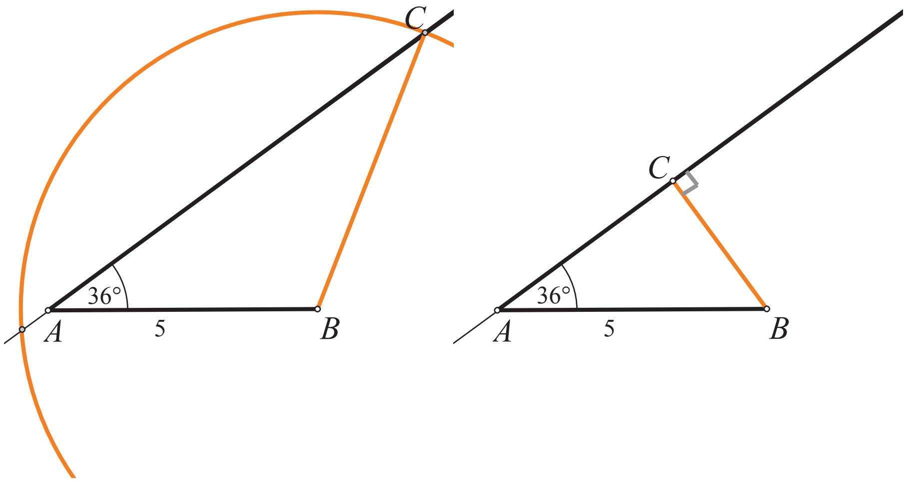


$$\frac{RP}{\sin 62^\circ} = \frac{15}{\sin 40^\circ} \Rightarrow RP = \frac{15 \sin 62^\circ}{\sin 40^\circ} \approx 20.6 \text{ cm}$$

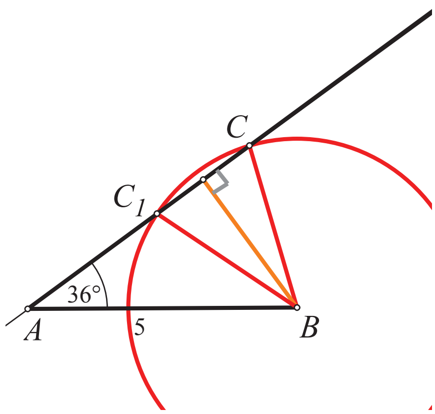
Hence, $A = \frac{1}{2} 15 \cdot 20.6 \sin 78^\circ \approx 151 \text{ cm}^2$.

22 When BC is given, triangle ABC is determined by SSA.

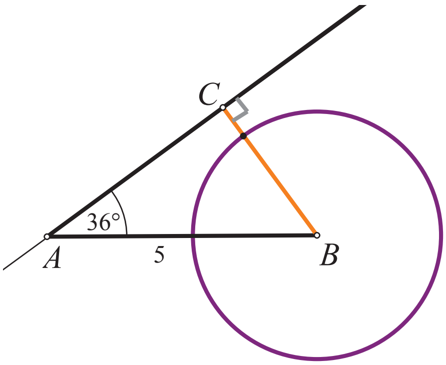
a) One triangle is possible when $BC \geqslant 5$ and $\frac{BC}{5} = \sin 36^\circ \Rightarrow BC = 5 \sin 36^\circ$.



b) Two triangles are possible when $5 \sin 36^\circ < BC < 5$.

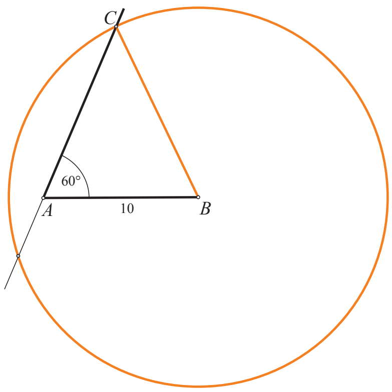


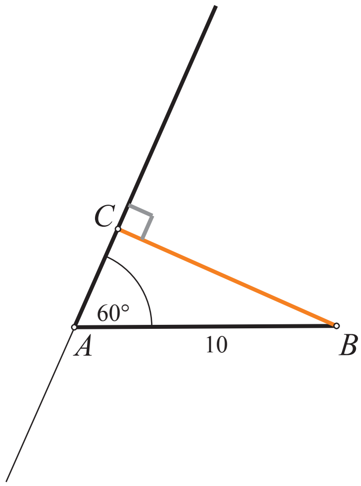
c) No triangle is possible when $BC < 5\sin 36^\circ$.



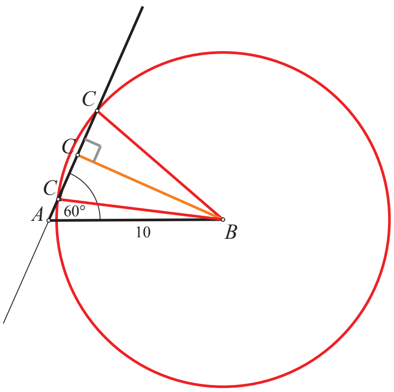
23 When BC is given, triangle ABC is determined by SSA.

a) One triangle is possible when $BC \geqslant 10$ and $\dfrac{BC}{10} = \sin 60^\circ \Rightarrow BC = 10\sin 60^\circ = 5\sqrt{3}$.

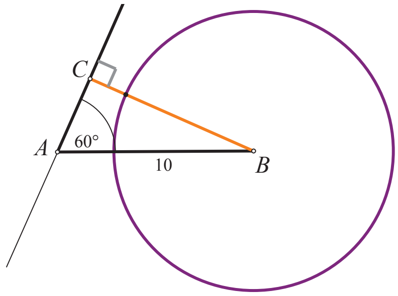




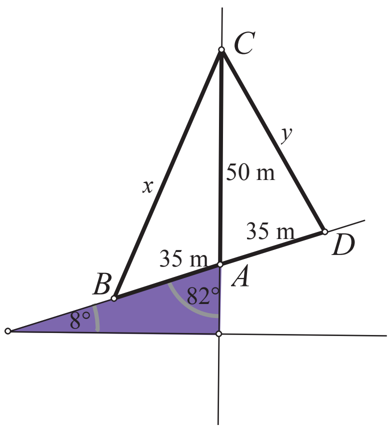
b) Two triangles are possible when BC is the smaller side and $5\sqrt{3} < BC < 10$.



c) No triangle is possible when $BC < 5\sqrt{3}$.



24



In triangles *ABC* and *ACD*, we are given SAS, so we can solve for the third side.

First determine angle *A* in the shaded right triangle:
$\hat{A} = 90^\circ - 8^\circ = 82^\circ$.

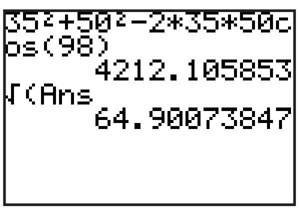
So, angle $A\hat{B}C$ equals: $A\hat{B}C = 180^\circ - 82^\circ = 98^\circ$.

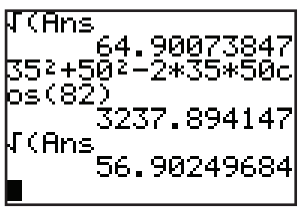
Using the law of cosines, we have:

$$x = \sqrt{35^2 + 50^2 - 2 \cdot 35 \cdot 50 \cos 98^\circ} \approx 64.9 \text{ m}$$

In triangle *ACD*, angle $D\hat{A}C$ equals angle *A* (vertically opposite angles) so: $D\hat{A}C = 82^\circ$.

Using the law of cosines, we have: $y = \sqrt{35^2 + 50^2 - 2 \cdot 35 \cdot 50 \cos 82^\circ} \approx 56.9 \text{ m}$





Therefore, the lengths of the supporting wires are approximately 64.9 m (x) and 56.9 m (y).

25 a) The largest angle is opposite the largest side, so side $x + 2$. We will use the law of cosines to find x:

$$(x+2)^2 = x^2 + (x-2)^2 - 2x(x-2)\cos 120^\circ$$

$$x^2 + 4x + 4 = x^2 + x^2 - 4x + 4 - 2x(x-2)\frac{-1}{2}$$

$$x^2 + 4x + 4 = x^2 + x^2 - 4x + 4 - 2x(x-2)\frac{-1}{2} \Rightarrow 2x^2 - 10x = 0 \Rightarrow x_1 = 0, x_2 = 5$$

Since the side has to be positive, the only possibility is $x = 5$.

b) The sides adjacent to the angle of 120° are $x - 2 = 3$ and $x = 5$. Hence, the area is:

$$A = \frac{1}{2} 3 \cdot 5 \cdot \sin 120^\circ = \frac{15}{2} \cdot \frac{\sqrt{3}}{2} = \frac{15\sqrt{3}}{4}$$

c) Denote the angles and sides as: $\hat{A} = 120, a = x + 2 = 7, \hat{B}, b = x = 5, \hat{C}, c = x - 2 = 3$. Using the sine rule, we will express sin B and sin C:

$$\frac{\sin B}{b} = \frac{\sin A}{a} \Rightarrow \frac{\sin B}{5} = \frac{\sin 120^\circ}{7} \Rightarrow \sin B = \frac{5\frac{\sqrt{3}}{2}}{7} = \frac{5\sqrt{3}}{14}$$

$$\frac{\sin C}{c} = \frac{\sin A}{a} \Rightarrow \frac{\sin c}{3} = \frac{\sin 120^\circ}{7} \Rightarrow \sin B = \frac{3\frac{\sqrt{3}}{2}}{7} = \frac{3\sqrt{3}}{14}$$

Hence: $\sin A + \sin B + \sin C = \frac{\sqrt{3}}{2} + \frac{5\sqrt{3}}{14} + \frac{3\sqrt{3}}{14} = \frac{15\sqrt{3}}{14}$

26 Using the law of cosines, we can find any angle: $\cos^2 A = \frac{6^2 + 7^2 - 8^2}{2 \cdot 6 \cdot 7} = \frac{1}{4}$

Using the Pythagorean identity for sine and cosine, we can find the sine of the angle.

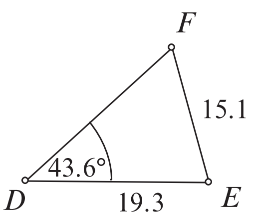
$\sin A = \sqrt{1 - \left(\frac{1}{4}\right)^2} = \sqrt{\frac{15}{16}} = \frac{\sqrt{15}}{4}$ (Notice that the sine of an angle in a triangle is always positive.)

Hence, the area is: $A = \frac{1}{2} 6 \cdot 7 \cdot \frac{\sqrt{15}}{4} = \frac{21\sqrt{15}}{4}$.

27 a) If $c^2 > a^2 + b^2$, then $0 > a^2 + b^2 - c^2$. Since b and c are positive, then $0 > \frac{a^2 + b^2 - c^2}{2ab}$. So, $0 > \cos C$ and C is an obtuse angle; hence, the triangle is obtuse.

b) If $c^2 < a^2 + b^2$, then $0 < a^2 + b^2 - c^2$. Since b and c are positive, then $0 < \frac{a^2 + b^2 - c^2}{2ab}$. So, $0 < \cos C$ and C is an acute angle. Angle C is the largest angle in the triangle; hence, all angles are acute, and the triangle is acute.

28



In the triangle, SSA is given. We will use the law of sines to firstly find the angles and then the side.

Using the law of sines, we have:

$$\frac{\sin \hat{F}}{19.3} = \frac{\sin 43.6^\circ}{15.1} \Rightarrow \sin \hat{F} = \frac{19.3 \sin 43.6^\circ}{15.1}$$

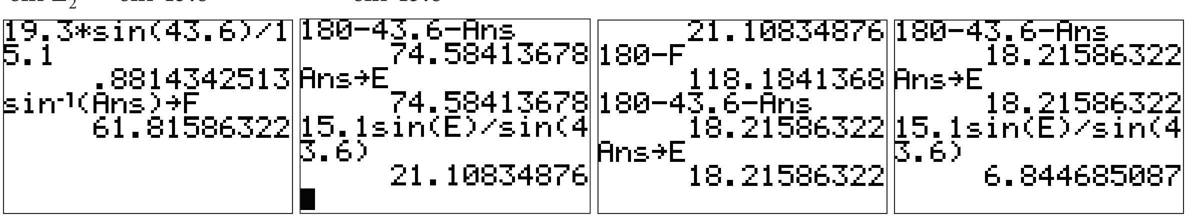
There are two possible solutions for the angle: $\hat{F}_1 \approx 61.8^\circ$ or $\hat{F}_2 = 180 - \hat{F}_1 \approx 118^\circ$.

In the first case, $\hat{F}_1 \approx 61.8^\circ$, then $\hat{E}_1 = 180^\circ - 43.6^\circ - \hat{F}_1 \approx 74.6^\circ$, and DF:

$$\frac{DF}{\sin \hat{E}_1} = \frac{15.1}{\sin 43.6^\circ} \Rightarrow DF = \frac{15.1 \sin \hat{E}_1}{\sin 43.6^\circ} \approx 21.1.$$

In the second case, $\hat{F}_2 \approx 118^\circ$, then $\hat{E}_2 = 180^\circ - 43.6^\circ - \hat{F}_2 \approx 18.2^\circ$, and DF:

$$\frac{DF}{\sin \hat{E}_2} = \frac{15.1}{\sin 43.6^\circ} \Rightarrow DF = \frac{15.1 \sin \hat{E}_2}{\sin 43.6^\circ} \approx 6.84.$$



29 **a)** $A = \dfrac{1}{2} ZY \cdot WZ \cdot \sin\theta = 112 \Rightarrow \dfrac{1}{2} 20 \cdot WZ \dfrac{4}{5} = 112 \Rightarrow WZ = 14$

b) $\cos\theta = \sqrt{1 - \left(\dfrac{4}{5}\right)^2} = \dfrac{3}{5}$

$WY = \sqrt{20^2 + 14^2 - 2 \cdot 20 \cdot 14 \dfrac{3}{5}} = \sqrt{260} = 2\sqrt{65}$

c) From triangle XYW: $WY = \sqrt{x^2 + 9x^2 - 2 \cdot 3x^2 \cos 120^\circ} = \sqrt{13x^2}$

Hence: $\sqrt{13x^2} = \sqrt{260} \Rightarrow x^2 = 20 \Rightarrow x = 2\sqrt{5}$

d) We will determine angles $Z\hat{Y}W$ and $X\hat{Y}W$.

Using the cosine rule in triangle ZYW, we have:

$$\cos Z\hat{Y}W = \frac{20^2 + 4 \cdot 65 - 14^2}{2 \cdot 20 \cdot 2\sqrt{65}} = 0.7194\ldots \Rightarrow Z\hat{Y}W \approx 44.0^\circ$$

```
20→A
                20
2√(65)→B
        16.1245155
14→C
                14
```

```
(A²+B²-C²)/(2AB)
       .7194014606
cos⁻¹(Ans)
       43.99491399
Ans→X
       43.99491399
```

Using the cosine rule in triangle XYW, we have:

$$\cos X\hat{Y}W = \frac{4 \cdot 65 + 180 - 20}{2 \cdot 2\sqrt{65} \cdot 6\sqrt{5}} = 0.9707\ldots \Rightarrow X\hat{Y}W \approx 13.9^\circ$$

```
2√(65)→A
        16.1245155
3*2*√(5)→B
       13.41640786
2*√(5)→C
       4.472135955
```

```
(A²+B²-C²)/(2AB)
       .9707253434
cos⁻¹(Ans)
       13.89788625
Ans→Y
       13.89788625
▮
```

So: $X\hat{Y}Z = Z\hat{Y}W + X\hat{Y}W \approx 57.9^\circ$

```
       .9707253434
cos⁻¹(Ans)
       13.89788625
Ans→Y
       13.89788625
X+Y
       57.89280024
▮
```

30 Using the law of sines and the double angle formula for sine, we have:

$$\frac{\sin 2\theta}{15} = \frac{\sin\theta}{12} \Rightarrow \frac{2\sin\theta\cos\theta}{15} = \frac{\sin\theta}{12} \Rightarrow \cos\theta = \frac{5}{8} \Rightarrow \theta \approx 51.3^\circ$$

31 **a)** $RS = RP - PS = q - PS$

From right triangle PSQ, we have: $\cos P = \dfrac{PS}{PQ} = \dfrac{PS}{r} \Rightarrow PS = r\cos P$.

Hence, $RS = q - r\cos P$.

b) In triangle SRQ, we have: $p^2 = SR^2 + SQ^2$. Since, from triangle PSQ, $SQ = r\sin P$, we have:

$$p^2 = (q - r\cos P)^2 + (r\sin P)^2 = q^2 - 2qr\cos P + r^2\cos^2 P + r^2\sin P$$

$$p^2 = q^2 - 2qr\cos P + r^2\left(\cos^2 P + \sin P\right)$$

Using the Pythagorean identity for sine and cosine, we have the cosine rule:
$p^2 = q^2 - 2qr\cos P + r^2 \cdot 1 = q^2 + r^2 - 2qr\cos P$.

c) Using the cosine rule for angle $P\hat{Q}R$, we have:

$$\cos P\hat{Q}R = \frac{r^2 + p^2 - q^2}{2rp} \Rightarrow \frac{1}{2} = \frac{r^2 + p^2 - q^2}{2rp} \Rightarrow rp = r^2 + p^2 - q^2$$

Hence, we have to solve the equation for p. This is a quadratic equation in p:

$$p^2 - rp + r^2 - q^2 = 0 \Rightarrow p_{1,2} = \frac{r \pm \sqrt{r^2 - 4\left(r^2 - q^2\right)}}{2} = \frac{r \pm \sqrt{r^2 - 4r^2 + 4q^2}}{2}$$

Hence, $p = \frac{1}{2}\left(r \pm \sqrt{4q^2 - 3r^2}\right)$.

32 a) Starting from the formula for area, we have:

$$4A = 2ab\sin C \Rightarrow 16A^2 = 4a^2b^2\sin^2 C \Rightarrow 16A^2 = 4a^2b^2\left(1 - \cos^2 C\right)$$
$$\Rightarrow 16A^2 = 4a^2b^2 - 4a^2b^2\cos^2 C$$

Using the cosine formula, we have: $\cos C = \frac{a^2 + b^2 - c^2}{2ab} \Rightarrow \cos^2 C = \frac{\left(a^2 + b^2 - c^2\right)^2}{4a^2b^2}$

Finally, we have:

$$16A^2 = 4a^2b^2 - 4a^2b^2\frac{\left(a^2 + b^2 - c^2\right)^2}{4a^2b^2} = 4a^2b^2 - \left(a^2 + b^2 - c^2\right)^2$$

b) Using the difference of squares, we will transform the expression for $16A^2$:

$$\begin{aligned} 4a^2b^2 - \left(a^2 + b^2 - c^2\right)^2 &= \left(2ab - \left(a^2 + b^2 - c^2\right)\right)\left(2ab + \left(a^2 + b^2 - c^2\right)\right) \\ &= \left(-\left(a^2 - 2ab + b^2\right) + c^2\right)\left(\left(a^2 + 2ab + b^2\right) - c^2\right) \\ &= \left(c^2 - (a-b)^2\right)\left((a+b)^2 - c^2\right) \\ &= (c - (a-b))(c + (a-b))((a+b) - c)((a+b) + c) \end{aligned}$$

Use the difference of squares formula.

So: $16A^2 = (c - a + b)(c + a - b)(a + b - c)(a + b + c)$

Now, we will introduce s into the formula. It holds that:

$2s = a + b + c$

$2s - 2a = -a + b + c$

$2s - 2b = a - b + c$

$2s - 2c = a + b - c$

Therefore: $16A^2 = (2s - 2a)(2s - 2b)(2s - 2c)(2s)$

c) Factorizing the right side of the equation:

$$16A^2 = 2^4(s-a)(s-b)(s-c)s \Rightarrow A = \sqrt{s(s-a)(s-b)(s-c)}$$

Exercise 8.5

1 **a)** $m = \tan 70^\circ \approx 2.75$

b) The line goes through the origin, so $c = 0$.

$y = mx + c = \tan 70^\circ x = x \tan 70°$

2 **a)** $m = \tan(-20^\circ) \approx -0.364$

b) The line goes through the origin, so $c = 0$.

$y = mx = \tan\left(-20^\circ\right) x = x \tan(-20°)$

3 **a)** The angle with the positive x-axis is $\theta = -45^\circ$, so: $m = \tan(-45^\circ) = -1$.

b) The line goes through the point $(0, 2)$, so $c = 2$.

$y = mx + c = -x + 2$

4 **a)** The angle with the positive x-axis is $\theta = 90^\circ - 68^\circ = 22^\circ$, so: $m = \tan(22^\circ) \approx 0.404$.

b) The line goes through the point $\left(0, -\frac{3}{2}\right)$, so $c = -\frac{3}{2}$.

$y = mx + c = \tan\left(22^\circ\right) x - \frac{3}{2} = x \tan 22°x - \frac{3}{2}$

For questions 5–7, we are asked to find the acute angle that a line through two given points makes with the x-axis. Firstly, we will need to find the slope of each line. The slope of a line through two points (x_1, y_1) and (x_2, y_2) is given by the formula: $m = \frac{y_2 - y_1}{x_2 - x_1}$.

5 $m = \frac{2-4}{-1-1} = 1 \Rightarrow \tan\theta = 1 \Rightarrow \theta = 45^\circ$

6 $m = \frac{-5-1}{6-(-3)} = -\frac{2}{3} \Rightarrow \tan\theta = -\frac{2}{3} \Rightarrow \theta \approx -33.7^\circ$. Therefore, the acute angle is approximately 33.7°.

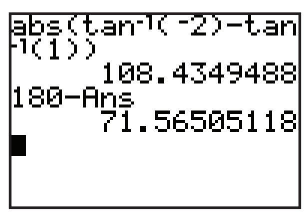
7 $m = \frac{-10 - \frac{1}{2}}{-4-2} = \frac{3}{4} \Rightarrow \tan\theta = \frac{3}{4} \Rightarrow \theta \approx 60.3^\circ$

In questions 8–9, we are asked to find the acute angle between two given lines. The angle between two lines is given by the formula: $\left|\tan^{-1}(m_1) - \tan^{-1}(m_2)\right|$.

8 The slopes are $m_1 = -2$ and $m_2 = 1$, hence:

$\left|\tan^{-1}(m_1) - \tan^{-1}(m_2)\right| = \left|\tan^{-1}(-2) - \tan^{-1}(1)\right| \approx \left|-63.4^\circ - 45^\circ\right| = 108.4^\circ$.

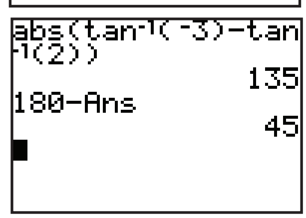
Therefore, the acute angle between the lines is: $\theta \approx 180^\circ - 108.4^\circ = 71.6^\circ$.



9 The slopes are $m_1 = -3$ and $m_2 = 2$, hence:

$\left|\tan^{-1}(m_1) - \tan^{-1}(m_2)\right| = \left|\tan^{-1}(-3) - \tan^{-1}(2)\right| = 135^\circ$.

Therefore, the acute angle between the lines is: $\theta \approx 180^\circ - 135^\circ = 45^\circ$.



10 **a)** The slope of the line is $m = \tan 30^\circ = \frac{\sqrt{3}}{3}$. The line goes through the origin, so $c = 0$. Therefore, the equation of the line is: $y = \frac{\sqrt{3}}{3}x$.

b) The equation of line L_2 can be written in the form $2y = -x + 6 \Rightarrow y = -\frac{1}{2}x + 3$, so its slope is $m = -\frac{1}{2}$. Hence, the angle between L_1 and L_2 is:

$$\left|\tan^{-1}\left(\frac{\sqrt{3}}{3}\right) - \tan^{-1}\left(-\frac{1}{2}\right)\right| = \left|30^\circ - \tan^{-1}\left(-\frac{1}{2}\right)\right| \approx 56.6^\circ.$$

11 Method I:
Angle $D\hat{C}A = 180^\circ - 40^\circ = 140^\circ$ and angle $C\hat{A}D = 180^\circ - 160^\circ = 20^\circ$, so, in triangle DCA, ASA is given.

We can determine side CA using the law of sines, but we notice that triangle DCA is isosceles; hence, $CA = CD = 30$ cm.

From right triangle ABC, we can determine side AB: $\sin 40^\circ = \frac{AB}{30} \Rightarrow AB = 30 \sin 40^\circ \approx 19.3$ cm

Method II:
Angle $D\hat{C}A = 180^\circ - 40^\circ = 140^\circ$ and angle $C\hat{A}D = 180^\circ - 160^\circ = 20^\circ$, so, in triangle DCA, ASA is given.

We can determine side AD using the law of sines:

$$\frac{AD}{\sin 140^\circ} = \frac{30}{\sin 20^\circ} \Rightarrow AD = \frac{30 \sin 140^\circ}{\sin 20^\circ} \approx 56.38$$

From right triangle ABD, we can determine side AB: $\sin 20^\circ = \frac{AB}{AD} \Rightarrow AB = AD \sin 20^\circ \approx 19.3$ cm

12 In triangle OPR, SSS is given, so we can find angle $P\hat{R}O$ using the law of cosines:

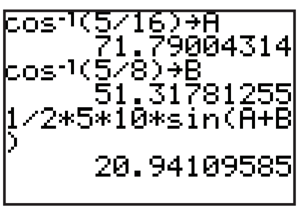
$$\cos P\hat{R}O = \frac{5^2 + 8^2 - 8^2}{2 \cdot 5 \cdot 8} = \frac{5}{16} \Rightarrow P\hat{R}O = \cos^{-1}\left(\frac{5}{16}\right) \approx 71.8^\circ$$

In triangle ORS, SSS is given, so we can find angle $S\hat{R}O$ using the law of cosines:

$$\cos S\hat{R}O = \frac{10^2 + 8^2 - 8^2}{2 \cdot 10 \cdot 8} = \frac{5}{8} \Rightarrow S\hat{R}O = \cos^{-1}\left(\frac{5}{8}\right) \approx 51.3^\circ$$

In triangle PRS, SAS is given. Angle $P\hat{R}S = P\hat{R}O + O\hat{R}S \approx 123^\circ$; therefore:

$$A = \frac{1}{2} \cdot 5 \cdot 10 \cdot \sin(P\hat{R}S) = 20.9 \text{ cm}^2.$$



Note: Triangles OPR and PRS are isosceles, so we can determine angles $P\hat{R}O$ and $S\hat{R}O$ as the adjacent angle to the leg (one-half of the chord) and the hypotenuse (radius) in a right triangle formed inside those isosceles triangles.

13 We will determine FG as a side of triangle FGB, where $F\hat{B}G = 78^\circ + 44^\circ = 122^\circ$.

Side BG is in triangle AGB, where ASA is given. So, using the law of sines, we can find BG:

$$\frac{BG}{\sin 81^\circ} = \frac{250}{\sin(180^\circ - 81^\circ - 44^\circ)} \Rightarrow BG = \frac{250 \sin 81^\circ}{\sin 55^\circ} \approx 301.436 \text{ m}$$

Side FB is in triangle ABF, where ASA is given. So, using the law of sines, we can find FB:

$$\frac{FB}{\sin 35^\circ} = \frac{250}{\sin(180^\circ - 35^\circ - 78^\circ)} \Rightarrow FB = \frac{250 \sin 35^\circ}{\sin 67^\circ} \approx 155.778 \text{ m}$$

Now, in triangle FGB, SAS is given, so we can determine FG using the law of cosines:

$$FG = \sqrt{FB^2 + BG^2 - 2 \cdot FB \cdot BG \cos 122^\circ} \approx 406 \text{ m}$$

```
180-81-44
                55
250sin(81)/sin(5
5)→A
        301.436207
180-35-78
                67
```

```
5)→A
        301.436207
180-35-78
                67
250sin(35)/sin(6
7)→B
       155.7776785
```

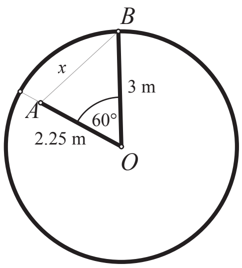
```
A²+B²-2*A*Bcos(1
22)
       164897.3442
√(Ans
         406.07554
```

Note: We can also find FG as a side of triangle FGA. Using the same method, we can find $F\hat{A}G = 35^\circ + 81^\circ = 116^\circ$.

$$AG = \frac{250 \sin 44^\circ}{\sin 55^\circ} \approx 212.005 \text{ m}, \; AF = \frac{250 \sin 78^\circ}{\sin 67^\circ} \approx 265.655 \text{ m, and finally:}$$

$$FG = \sqrt{FA^2 + AG^2 - 2 \cdot FA \cdot AG \cos 116^\circ} \approx 406 \text{ m.}$$

14



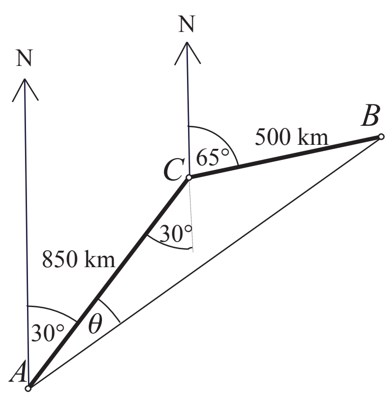
In triangle ABO, angle $A\hat{O}B = 60^\circ$, so SAS is given.

We can determine side x using the law of cosines:

$$x = \sqrt{2.25^2 + 3^2 - 2.25 \cdot 3 \cos 60^\circ} \approx 2.70$$

Therefore, the distance between the tips of the hands of the clock is approximately 2.70 m.

15



$$A\hat{C}B = 30^\circ + (180^\circ - 65^\circ) = 145^\circ$$

a) In triangle ABC, SAS is given, so we can find AB using the law of cosines: $AB = \sqrt{850^2 + 500^2 - 2 \cdot 850 \cdot 500 \cdot \cos 145^\circ} \approx 1291.8$ km

b) We can use the cosine law to find angle θ:

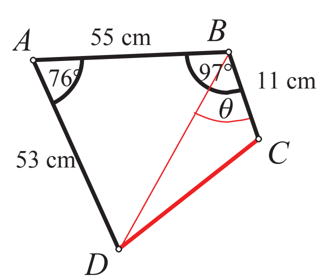
$$\cos \theta = \frac{850^2 + AB^2 - 500^2}{2 \cdot 850 \cdot AB} \approx 0.9750... \Rightarrow \theta \approx 12.8^\circ$$

Therefore, the bearing from A to B is $12.8^\circ + 30^\circ = 42.8^\circ$.

```
850²+500²-2*850*
500*cos(145)
       1668779.238
√(Ans)
       1291.812385
Ans→A
       1291.812385
```

```
Ans→A
       1291.812385
(850²+A²-500²)/(
2*850*A)
       .9750456312
cos⁻¹(Ans
       12.82679678
```

16 To find CD we will determine side DB and angle $D\hat{B}C = \theta$.



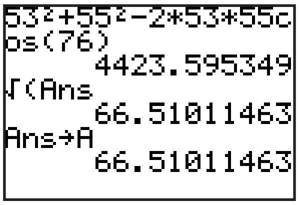
In triangle ABD, SAS is given, so we can find DB using the law of cosines:

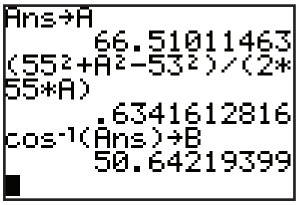
$$DB = \sqrt{53^2 + 55^2 - 2 \cdot 53 \cdot 55 \cdot \cos 76^\circ} \approx 66.51 \text{ cm}$$

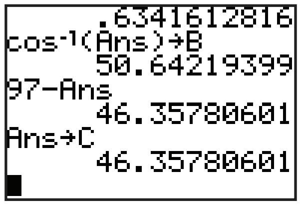
$$\cos A\hat{B}D = \frac{55^2 + DB^2 - 53^2}{2 \cdot 55 \cdot DB} \approx 0.6421... \Rightarrow \theta \approx 50.64^\circ$$

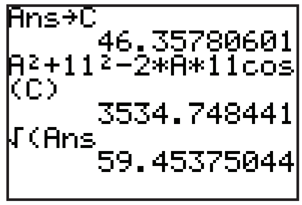
$$\theta = 97^\circ - A\hat{B}D \approx 46.36^\circ$$

$$DC = \sqrt{DB^2 + 11^2 - 2 \cdot DB \cdot 11 \cdot \cos\theta} \approx 59.5$$









Therefore, the length of tube CD must be approximately 59.5 cm.

Note: We can also solve for CD if we determine side AC and angle $D\hat{A}C$ (to find this angle, we must first find angle $C\hat{A}B$). (Results: $AC \approx 57.39$ cm, $D\hat{A}C \approx 65.0^\circ$)

17 Triangle ABC is a right triangle with legs of 12 cm, so the area is:

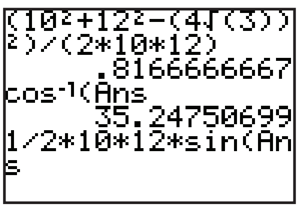
$$A = \frac{1}{2} 12 \cdot 12 = 72 \text{ cm}^2.$$

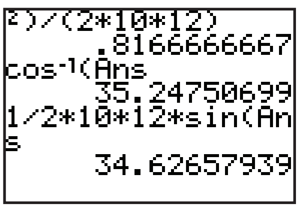
In right triangle ABD, for side BD, it holds that: $\tan 60^\circ = \frac{12}{BD} \Rightarrow BD = \frac{12}{\sqrt{3}} = 4\sqrt{3}$ cm; therefore, ABD is a right triangle with legs of 12 cm and $4\sqrt{3}$ cm, and so the area is: $A = \frac{1}{2} 12 \cdot 4\sqrt{3} = 24\sqrt{3}$ cm^2.

In triangle BCD all three sides are given, so we can find one of the angles by use of the law of cosines:

$$\cos D\hat{C}B = \frac{10^2 + 12^2 - \left(4\sqrt{3}\right)^2}{2 \cdot 10 \cdot 12} \Rightarrow D\hat{C}B \approx 35.2475^\circ$$

Therefore, the area is: $A = \frac{1}{2} 10 \cdot 12 \sin D\hat{C}B \approx 34.6 \text{ cm}^2$.



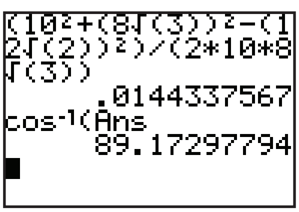


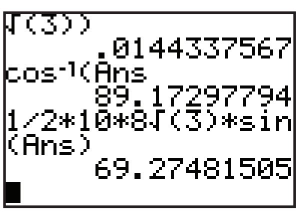
In triangle ADC, side AD equals $\sqrt{\left(4\sqrt{3}\right)^2 + 12^2} = 8\sqrt{3}$ cm and side AC equals $\sqrt{12^2 + 12^2} = 12\sqrt{2}$ cm.

We can find one of the angles by use of the law of cosines:

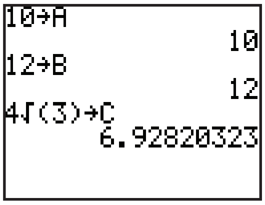
$$\cos C\hat{D}A = \frac{10^2 + \left(8\sqrt{3}\right)^2 - \left(12\sqrt{2}\right)^2}{2 \cdot 10 \cdot 8\sqrt{3}} \Rightarrow C\hat{D}A \approx 89.1729...^\circ$$

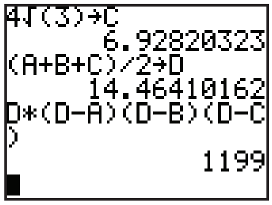
Therefore, the area is: $A = \frac{1}{2} 10 \cdot 8\sqrt{3} \sin C\hat{D}A \approx 69.3 \text{ cm}^2$.

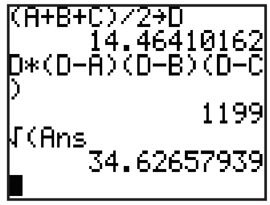




Note: We can find the area of a triangle given three sides using Hero's formula (or Heron's formula): $A = \sqrt{s(s-a)(s-b)(s-c)}$, where $s = \frac{a+b+c}{2}$. For triangle BCD, we have:





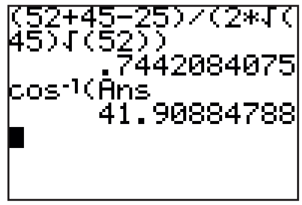


18 The sides of triangle DEF are:

$DF = \sqrt{3^2+4^2} = 5$ cm

$FE = \sqrt{3^2+6^2} = \sqrt{45}$ cm

$DE = \sqrt{4^2+6^2} = \sqrt{52}$ cm

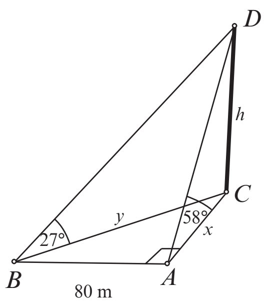


Using the law of cosines, we can find angle $D\hat{E}F$:

$$\cos D\hat{E}F = \frac{52+45-25}{2\cdot\sqrt{42}\sqrt{45}} \Rightarrow D\hat{E}F \approx 41.9^\circ$$

19 From triangle ACD, we can find side $AC = x$: $\tan 58^\circ = \frac{h}{x} \Rightarrow x = \frac{h}{\tan 58^\circ}$

From triangle BCD, we can find side $BC = y$: $\tan 27^\circ = \frac{h}{y} \Rightarrow y = \frac{h}{\tan 27^\circ}$



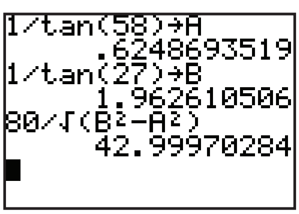
Triangle ABC is a right triangle with a right angle at A. So, from Pythagoras' theorem, we have:

$$80^2 = y^2 - x^2 = \left(\frac{h}{\tan 27^\circ}\right)^2 - \left(\frac{h}{\tan 58^\circ}\right)^2 = h^2\left(\frac{1}{\tan^2 27^\circ} - \frac{1}{\tan^2 58^\circ}\right) \Rightarrow$$

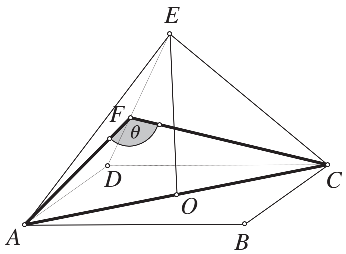
$$h^2 = \frac{80^2}{\left(\frac{1}{\tan^2 27^\circ} - \frac{1}{\tan^2 58^\circ}\right)} \Rightarrow h = \frac{80}{\sqrt{\frac{1}{\tan^2 27^\circ} - \frac{1}{\tan^2 58^\circ}}} \approx 43.0$$

Therefore, the height of the building is approximately 43.0 m.

Note: To simplify the calculation, we can find $\frac{1}{\tan 27^\circ}$ and $\frac{1}{\tan 58^\circ}$ and store the values in our GDC. Then we can find h:



20 We have to determine angle θ.



Using the cosine rule, we can determine the angle if we know the sides *AC, AF* and *CF*. (Note that sides *AF* and *CF* are equal.)

Side *AC* is a diagonal of the base; hence: $AC = 8\sqrt{2}$.

Side *AF* is the height of triangle *ADE*. We can determine *AF* by finding the angles in the triangle, or the area. Here, we will use the area.

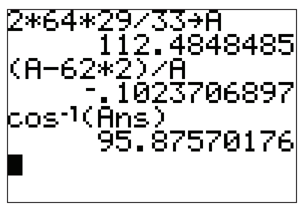
Triangle *ADE* is isosceles. Side $AD = 8$, and $AE = \sqrt{\left(4\sqrt{2}\right)^2 + 10^2} = \sqrt{132}$.

Hence, the height on side *AD* is: $\sqrt{\sqrt{132}^2 - 4^2} = \sqrt{116}$ and the area is

$A = \dfrac{8\sqrt{116}}{2} = 8\sqrt{29}$. We can also find the area using side *DE* and its height *AF*; so:

$A = 8\sqrt{29} = \dfrac{\sqrt{132}AF}{2} = \sqrt{33}AF \Rightarrow 8\sqrt{29} = \sqrt{33}AF = AF = \dfrac{8\sqrt{29}}{\sqrt{33}}$.

Now we have all the necessary elements to find θ:

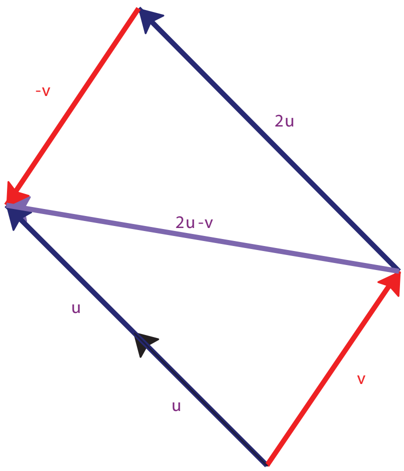


$$\cos\theta = \frac{2AF^2 - AC^2}{2AF^2} = \frac{2\,\dfrac{64\cdot 29}{33} - 62\cdot 2}{2\,\dfrac{64\cdot 29}{33}} = -0.102\,37\ldots \Rightarrow \theta \approx 95.9^\circ$$

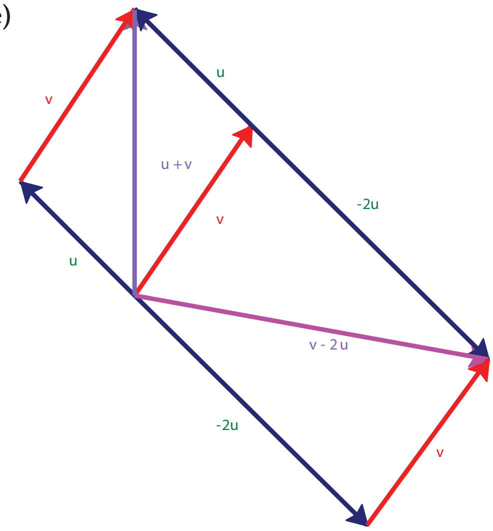
Chapter 9

Exercise 9.1 and 9.2

1 **a)**, **b)**, **d)**



c), **e)**



2 For $A(3, 4)$ and $B(7, -1)$:

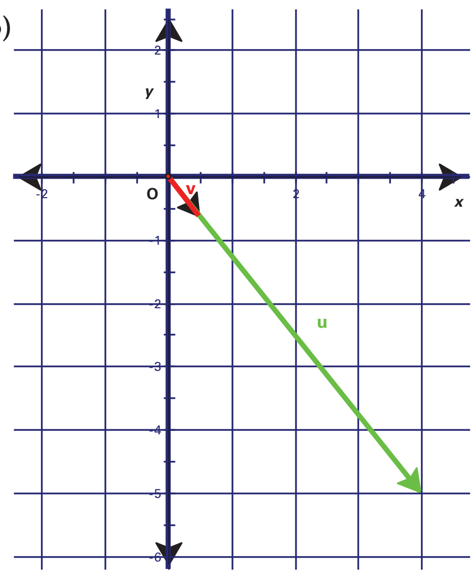
a) $\left|\overrightarrow{AB}\right| = \sqrt{(7-3)^2 + (-1-4)^2} = \sqrt{4^2 + (-5)^2} = \sqrt{41}$

b) $\mathbf{u} = (4, -5)$

c) $\mathbf{v} = \dfrac{1}{\sqrt{41}}(4, -5) = \left(\dfrac{4\sqrt{41}}{41}, \dfrac{-5\sqrt{41}}{41}\right) = (0.62, -0.78)$

d) $|\mathbf{v}| = \sqrt{\left(\dfrac{4\sqrt{41}}{41}\right)^2 + \left(\dfrac{-5\sqrt{41}}{41}\right)^2} = 1$

e) and **b)**



Vectors **u** and **v** have the same direction, but different magnitude.

3 For $A(-2, 3)$ and $B(5, 1)$:

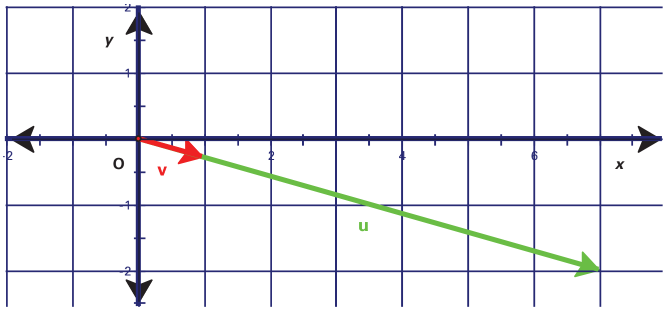
a) $|\overrightarrow{AB}| = \sqrt{(5+2)^2 + (1-3)^2} = \sqrt{7^2 + (-2)^2} = \sqrt{53}$

b) $\mathbf{u} = (7, -2)$

c) $\mathbf{v} = \frac{1}{\sqrt{53}}(7, -2) = \left(\frac{7\sqrt{53}}{53}, \frac{-2\sqrt{53}}{53}\right) = (0.96, -0.27)$

d) $|\mathbf{v}| = \sqrt{\left(\frac{7\sqrt{53}}{53}\right)^2 + \left(\frac{-2\sqrt{53}}{53}\right)^2} = 1$

e) and **b)**



Vectors **u** and **v** have the same direction, but different magnitude.

4 For $A(3, 5)$ and $B(0, 5)$:

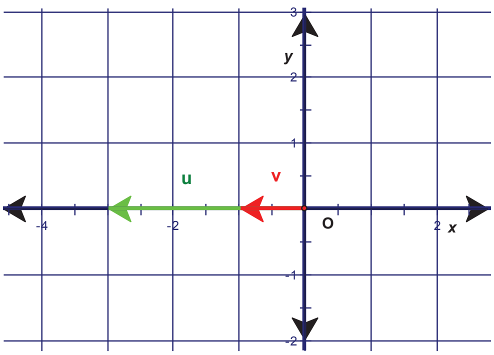
a) $|\overrightarrow{AB}| = \sqrt{(3-0)^2 + (5-5)^2} = 3$

b) $\mathbf{u} = (-3, 0)$

c) $\mathbf{v} = \frac{1}{3}(-3, 0) = (-1, 0)$

d) $|\mathbf{v}| = \sqrt{(-1)^2 + 0^2} = 1$

e) and **b)**



5 For $A(2, -4)$ and $B(2, 1)$:

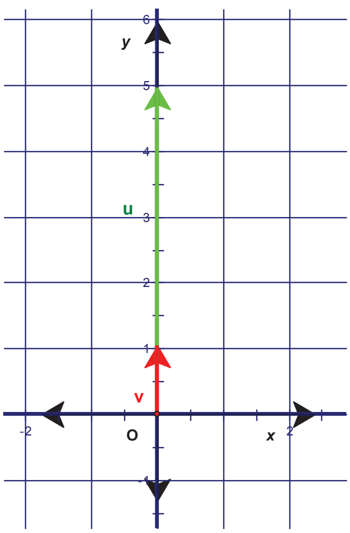
a) $|\overrightarrow{AB}| = \sqrt{(2-2)^2 + (-4-1)^2} = \sqrt{0^2 + (-5)^2} = 5$

b) $\mathbf{u} = (0, 5)$

c) $\mathbf{v} = \frac{1}{5}(0, 5) = (0, 1)$

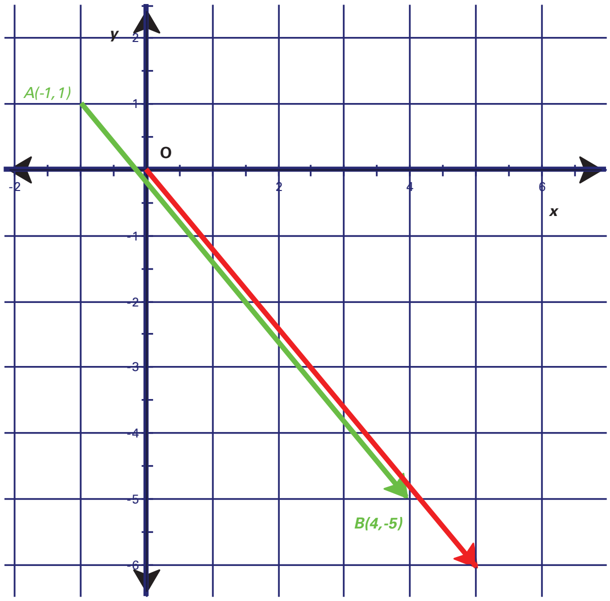
d) $|\mathbf{v}| = \sqrt{0^2 + 1^2} = 1$

e) and **b)**



6 **a)** $\overrightarrow{PQ} = (2+3, -5-1) = (5, -6)$ **b)** $\left|\overrightarrow{PQ}\right| = \sqrt{5^2 + (-6)^2} = \sqrt{61}$

c)

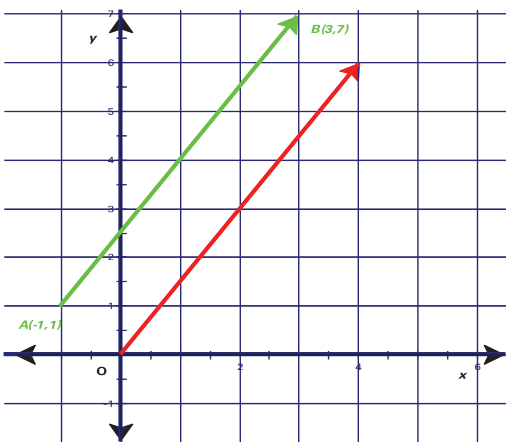


d) For $A(-1, 1)$ and $B(x, y)$:

$\overrightarrow{AB} = (5, -6) \Rightarrow (x+1, y-1) = (5, -6) \Rightarrow x = 4, y = -5 \Rightarrow B(4, -5)$

7 **a)** $\overrightarrow{PQ} = (7-3, 8-2) = (4, 6)$ **b)** $\left|\overrightarrow{PQ}\right| = \sqrt{4^2 + 6^2} = \sqrt{52} = 2\sqrt{13}$

c)

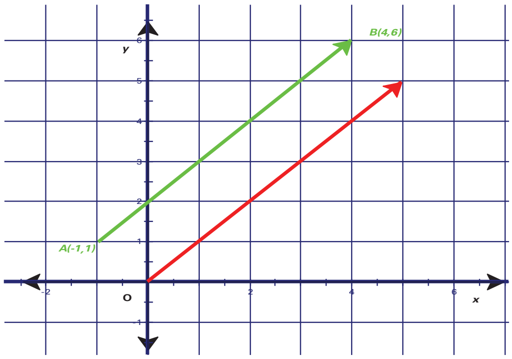


d) For $A(-1, 1)$ and $B(x, y)$:

$\overrightarrow{AB} = (4, 6) \Rightarrow (x+1, y-1) = (4, 6) \Rightarrow x = 3, y = 7 \Rightarrow B(3, 7)$

8 **a)** $\overrightarrow{PQ} = (7-2, 7-2) = (5, 5)$ **b)** $|\overrightarrow{PQ}| = \sqrt{5^2 + 5^2} = 5\sqrt{2}$

c)

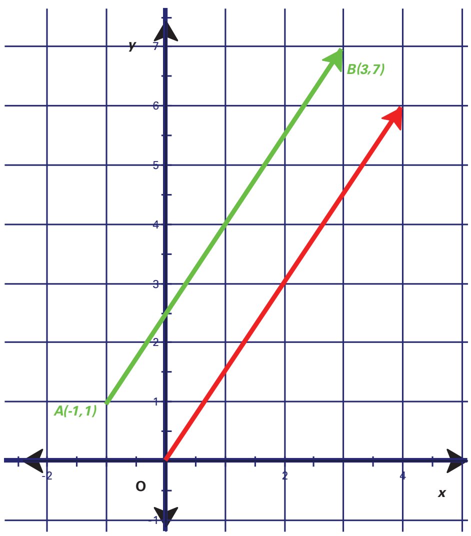


d) For $A(-1, 1)$ and $B(x, y)$:

$\overrightarrow{AB} = (5, 5) \Rightarrow (x+1, y-1) = (5, 5) \Rightarrow x = 4, y = 6 \Rightarrow B(4, 6)$

9 **a)** $\overrightarrow{PQ} = (-2+6, -2+8) = (4, 6)$ **b)** $|\overrightarrow{PQ}| = \sqrt{4^2 + 6^2} = \sqrt{52} = 2\sqrt{13}$

c)



d) For $A(-1, 1)$ and $B(x, y)$:

$\overrightarrow{AB} = (4, 6) \Rightarrow (x+1, y-1) = (4, 6) \Rightarrow x = 3, y = 7 \Rightarrow B(3, 7)$

10 $\mathbf{a} = \mathbf{u} - \mathbf{v}, \mathbf{c} = \mathbf{v} + \mathbf{u}$

11 For $\mathbf{v} = 3\mathbf{i} - 2\mathbf{j}$ and $A(-2, 1)$, $B(x, y)$:

$\overrightarrow{AB} = \mathbf{v} \Rightarrow (x+2, y-1) = (3, -2) \Rightarrow x + 2 = 3, y - 1 = -2 \Rightarrow x = 1, y = -1$

The terminal point is $B(1, -1)$.

12 For $\mathbf{v} = (-3, 1)$ and $A(x, y)$, $B(5, 0)$:

$\overrightarrow{AB} = \mathbf{v} \Rightarrow (5 - x, 0 - y) = (-3, 1) \Rightarrow 5 - x = -3, -y = 1 \Rightarrow x = 8, y = -1$

The initial point is $A(8, -1)$.

13 For $\mathbf{v} = (6, 7)$ and $A(-2, 1)$, $B(x, y)$:

$\overrightarrow{AB} = \mathbf{v} \Rightarrow (x+2, y-1) = (6, 7) \Rightarrow x + 2 = 6, y - 1 = 7 \Rightarrow x = 4, y = 8$

The terminal point is $B(4, 8)$.

14 For $\mathbf{v} = 2\mathbf{i} + 7\mathbf{j}$ and $A(x, y)$, $B(-3, 2)$:

$\overrightarrow{AB} = \mathbf{v} \Rightarrow (-3 - x, 2 - y) = (2, 7) \Rightarrow -3 - x = 2, 2 - y = 7 \Rightarrow x = -5, y = -5$

The initial point is $A(-5, -5)$.

15 For $\mathbf{u} = 3\mathbf{i} - \mathbf{j}$ and $\mathbf{v} = -\mathbf{i} + 3\mathbf{j}$:

a) $\mathbf{u} + \mathbf{v} = 2\mathbf{i} + 2\mathbf{j}$; $\mathbf{u} - \mathbf{v} = 4\mathbf{i} - 4\mathbf{j}$; $2\mathbf{u} + 3\mathbf{v} = 3\mathbf{i} + 7\mathbf{j}$; $2\mathbf{u} - 3\mathbf{v} = 9\mathbf{i} - 11\mathbf{j}$

b) $|\mathbf{u} + \mathbf{v}| = \sqrt{2^2 + 2^2} = \sqrt{8} = 2\sqrt{2}, |\mathbf{u} - \mathbf{v}| = \sqrt{4^2 + 4^2} = \sqrt{32} = 4\sqrt{2}$

$|\mathbf{u}| + |\mathbf{v}| = \sqrt{3^3 + (-1)^2} + \sqrt{(-1)^2 + 3^2} = \sqrt{10} + \sqrt{10} = 2\sqrt{10}$

$|\mathbf{u}| - |\mathbf{v}| = \sqrt{3^3 + (-1)^2} - \sqrt{(-1)^2 + 3^2} = \sqrt{10} - \sqrt{10} = 0$

c) $|2\mathbf{u} + 3\mathbf{v}| = |3\mathbf{i} + 7\mathbf{j}| = \sqrt{3^2 + 7^2} = \sqrt{58}$; $|2\mathbf{u} - 3\mathbf{v}| = |9\mathbf{i} - 11\mathbf{j}| = \sqrt{9^2 + 11^2} = \sqrt{202}$

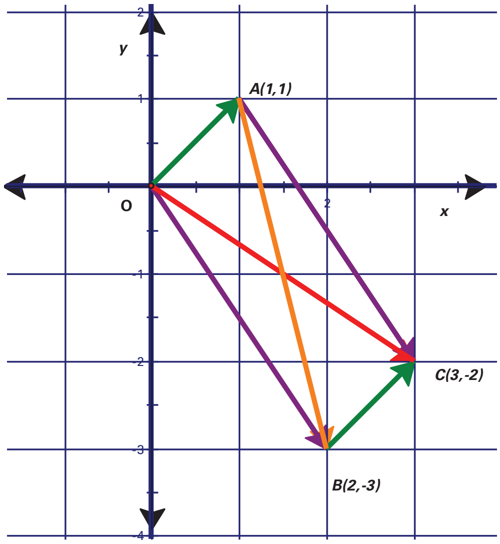
$2|\mathbf{u}| + 3|\mathbf{v}| = 2\sqrt{10} + 3\sqrt{10} = 5\sqrt{10}$; $2|\mathbf{u}| - 3|\mathbf{v}| = 2\sqrt{10} - 3\sqrt{10} = -\sqrt{10}$

16 For $\mathbf{u} = (1, 5)$ and $\mathbf{v} = (3, -4)$:

$2\mathbf{u} - 3\mathbf{x} + \mathbf{v} = 5\mathbf{x} - 2\mathbf{v} \Rightarrow 8\mathbf{x} = 2\mathbf{u} + 3\mathbf{v} \Rightarrow \mathbf{x} = \frac{1}{8}(2\mathbf{u} + 3\mathbf{v}) \Rightarrow \mathbf{x} = \frac{1}{8}(11, -2) = \left(\frac{11}{8}, -\frac{1}{4}\right)$

17 $\left.\begin{array}{r} \mathbf{u} - 2\mathbf{v} = 2\mathbf{i} - 3\mathbf{j} \\ \mathbf{u} + 3\mathbf{v} = \mathbf{i} + \mathbf{j} \end{array}\right\} 5\mathbf{v} = -\mathbf{i} + 4\mathbf{j} \Rightarrow \mathbf{v} = -\frac{1}{5}\mathbf{i} + \frac{4}{5}\mathbf{j}, \mathbf{u} = \frac{8}{5}\mathbf{i} - \frac{7}{5}\mathbf{j}$

18



Diagonals are OC and BA:

$\overline{OC} = \overline{OA} + \overline{AC} = (2\mathbf{i} - 3\mathbf{j}) + (\mathbf{i} + \mathbf{j}) = 3\mathbf{i} - 2\mathbf{j}$

$\overline{BA} = \overline{BO} + \overline{OA} = -(\mathbf{i} + \mathbf{j}) + (2\mathbf{i} - 3\mathbf{j}) = \mathbf{i} - 4\mathbf{j}$

Length of the diagonals:

$|OC| = \sqrt{3^2 + (-2)^2} = \sqrt{13}$

$|BA| = \sqrt{1^2 + (-4)^2} = \sqrt{17}$

19 $\overrightarrow{PR} = \overrightarrow{PQ} + \overrightarrow{QR} = \mathbf{u} + \mathbf{v}$

$\overrightarrow{PM} = \overrightarrow{PS} + \frac{1}{2}\overrightarrow{SR} = \mathbf{v} + \frac{1}{2}\mathbf{u}$

$\overrightarrow{QS} = \overrightarrow{QR} + \overrightarrow{RS} = \mathbf{v} - \mathbf{u}$

$\overrightarrow{QN} = \frac{1}{2}\overrightarrow{QS} = \frac{1}{2}\mathbf{v} - \frac{1}{2}\mathbf{u}$

20 For the points $A(2, 0)$, $B(8, 4)$, $C(12, 12)$, $D(x, y)$ to form a parallelogram:

$\overrightarrow{AD} = \overrightarrow{BC} \Rightarrow (x - 2, y - 0) = (12 - 8, 12 - 4) \Rightarrow$

$x - 2 = 4 \Rightarrow x = 6$

$y - 0 = 8 \Rightarrow y = 8$

The point is $(6, 8)$.

21 For the points $A(0, 3)$, $B(x, 2)$, $C(5, 4)$, $D(2, y)$ to form a parallelogram:

$\overrightarrow{AD} = \overrightarrow{BC} \Rightarrow (2-0, y-3) = (5-x, 4-2) \Rightarrow$

$2 = 5 - x \Rightarrow x = 3$

$y - 3 = 2 \Rightarrow y = 5$

The point is (3, 5).

22 From $\begin{pmatrix} 8 \\ 46 \end{pmatrix} = r\begin{pmatrix} 1 \\ 9 \end{pmatrix} + s\begin{pmatrix} 1 \\ -4 \end{pmatrix}$ we have the system: $\begin{cases} r + s = 8 \\ 9r - 4s = 46 \end{cases} \Rightarrow 13r = 78 \Rightarrow r = 6, s = 2$

So: $\begin{pmatrix} 8 \\ 46 \end{pmatrix} = 6\begin{pmatrix} 1 \\ 9 \end{pmatrix} + 2\begin{pmatrix} 1 \\ -4 \end{pmatrix}$.

23 From $\begin{pmatrix} 4 \\ 7 \end{pmatrix} = r\begin{pmatrix} 2 \\ 3 \end{pmatrix} + s\begin{pmatrix} 2 \\ 1 \end{pmatrix}$ we have the system: $\begin{cases} 2r + 2s = 4 \\ 3r + s = 7 \end{cases} \Rightarrow 4r = 10 \Rightarrow r = \frac{5}{2}, s = -\frac{1}{2}$

So: $(4, 7) = \frac{5}{2}(2, 3) - \frac{1}{2}(2, 1)$.

24 From $\begin{pmatrix} 5 \\ -5 \end{pmatrix} = r\begin{pmatrix} 1 \\ -1 \end{pmatrix} + s\begin{pmatrix} -1 \\ 1 \end{pmatrix}$ we have the system: $\begin{cases} r - s = 5 \\ -r + s = -5 \end{cases} \Rightarrow s = r - 5$

So: $(5, -5) = r(1, -1) + (r - 5)(-1, 1)$.

25 From $\begin{pmatrix} -11 \\ 0 \end{pmatrix} = r\begin{pmatrix} 2 \\ 5 \end{pmatrix} + s\begin{pmatrix} 3 \\ 2 \end{pmatrix}$ we have the system: $\begin{cases} 2r + 3s = -11 \\ 5r + 2s = 0 \end{cases} \Rightarrow 11r = 22 \Rightarrow r = 2, s = -5$

So: $(-11, 0) = 2(2, 5) - 5(3, 2)$.

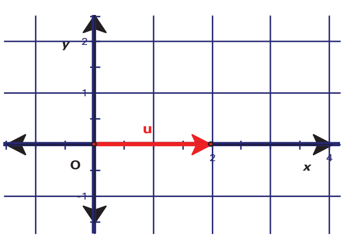
26 Let $\mathbf{w} = (x, y)$. Then, from $\begin{pmatrix} x \\ y \end{pmatrix} = r\begin{pmatrix} 1 \\ 1 \end{pmatrix} + s\begin{pmatrix} -1 \\ 1 \end{pmatrix}$, we have the system:

$\begin{cases} r - s = x \\ r + s = y \end{cases} \Rightarrow 2r = x + y \Rightarrow r = \frac{x+y}{2}, 2s = y - x \Rightarrow s = \frac{y-x}{2}$

So: $\mathbf{w} = \begin{pmatrix} x \\ y \end{pmatrix} = \frac{x+y}{2}\begin{pmatrix} 1 \\ 1 \end{pmatrix} + \frac{y-x}{2}\begin{pmatrix} -1 \\ 1 \end{pmatrix}$.

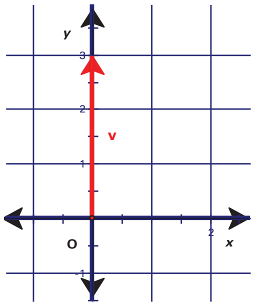
Exercise 9.3

1 **a)**



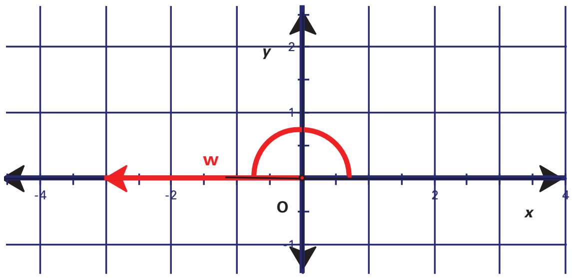
$\tan\theta = \frac{0}{2} = 0 \Rightarrow \theta = \arctan 0 = 0^\circ$

b)



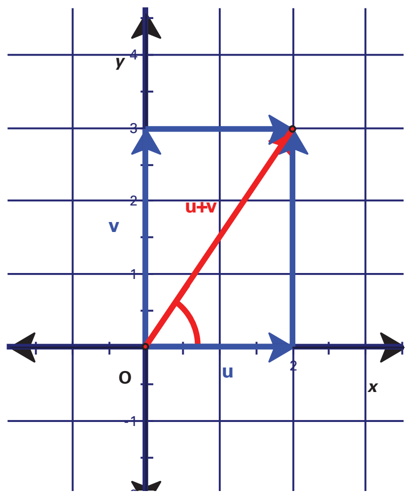
$$\tan \theta = \frac{3}{0} = \infty \Rightarrow \theta = \arctan \infty = 90^\circ$$

c)



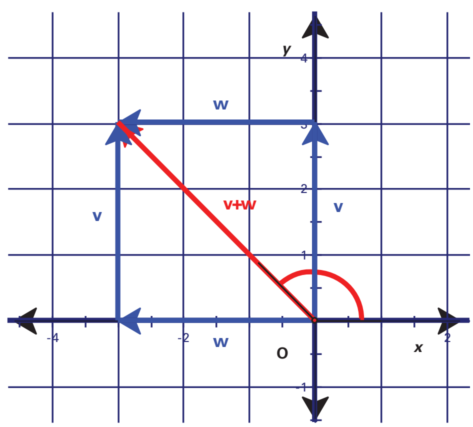
$$\tan \theta = \frac{0}{-3} = 0 \Rightarrow \theta = \arctan 0 = 180^\circ$$

d) $\mathbf{u} + \mathbf{v} = (2, 0) + (0, 3) = (2, 3)$



$$\tan \theta = \frac{3}{2} \Rightarrow \theta = \arctan \frac{3}{2} = 56.3^\circ$$

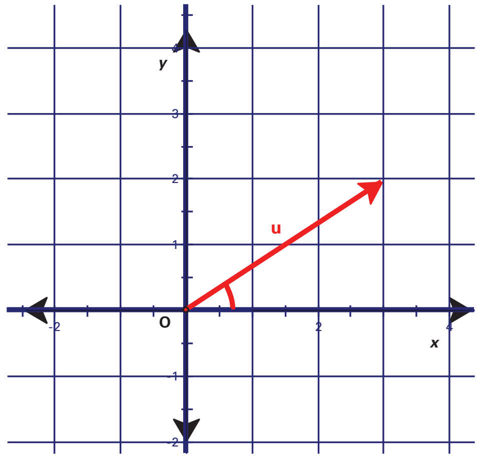
e) $\mathbf{v} + \mathbf{w} = (0, 3) + (-3, 0) = (-3, 3)$



$$\tan \theta = \frac{3}{-3} \Rightarrow \theta = \arctan(-1) = 135^\circ$$

2 **a)** $|\mathbf{u}| = \sqrt{3^2 + 2^2} = \sqrt{13}$

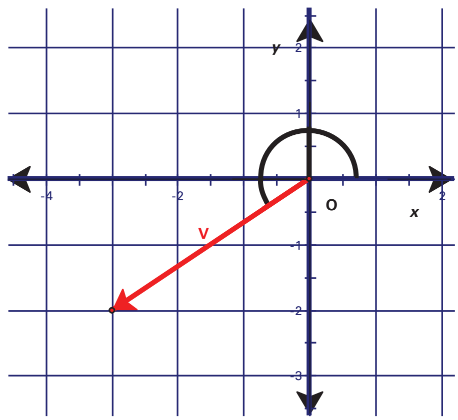
$\tan\theta = \frac{2}{3} \Rightarrow \theta = \arctan\left(\frac{2}{3}\right) = 33.69^\circ$



b) $|\mathbf{v}| = \sqrt{(-3)^2 + (-2)^2} = \sqrt{13}$

$\tan\theta = \frac{-2}{-3} = \frac{2}{3} \Rightarrow \theta = \arctan\left(\frac{2}{3}\right) = 180^\circ + 33.69^\circ$

$= 213.69^\circ$



c) $|2\mathbf{u}| = 2\,|\mathbf{u}| = 2\sqrt{13}$

The direction angle for 2**u** is the same as that for **u**, i.e. 33.69°.

d) $|3\mathbf{v}| = 3\,|\mathbf{v}| = 3\sqrt{13}$

The direction angle for 3**v** is the same as that for **v**, i.e. 213.69°.

e) $2\mathbf{u} + 3\mathbf{v} = 2\begin{pmatrix} 3 \\ 2 \end{pmatrix} + 3\begin{pmatrix} -3 \\ -2 \end{pmatrix} = \begin{pmatrix} -3 \\ -2 \end{pmatrix} = \mathbf{v}$

So, the magnitude is $|\mathbf{v}| = \sqrt{13}$ and the direction angle is 213.69°.

f) $2\mathbf{u} - 3\mathbf{v} = 2\begin{pmatrix} 3 \\ 2 \end{pmatrix} - 3\begin{pmatrix} -3 \\ -2 \end{pmatrix} = \begin{pmatrix} 15 \\ 10 \end{pmatrix} = 5\mathbf{u}$

So, the magnitude is $|2\mathbf{u} - 3\mathbf{v}| = |5\mathbf{u}| = 5\,|\mathbf{u}| = 5\sqrt{13}$ and the direction angle is the same as that for **u**, i.e. 33.69°.

3 For $\mathbf{u} = (-4, 7)$, $\mathbf{v} = (2, 5)$:

a) $|\mathbf{u}| = \sqrt{(-4)^2 + 7^2} = \sqrt{65}$

$\tan\theta = \dfrac{7}{-4} \Rightarrow \theta = \arctan\left(-\dfrac{7}{4}\right) + \pi = 119.7^\circ$

b) $|\mathbf{v}| = \sqrt{2^2 + 5^2} = \sqrt{29}$

$\tan\theta = \dfrac{5}{2} \Rightarrow \theta = \arctan\left(\dfrac{5}{2}\right) = 68.2^\circ$

c) $|3\mathbf{u}| = 3|\mathbf{u}| = 3\sqrt{65}$

Direction vector for $3\mathbf{u}$ is the same as for $\mathbf{u}$, i.e. 119.7°.

d) $|-2\mathbf{v}| = 2|\mathbf{v}| = 2\sqrt{29}$

Direction vector for $-2\mathbf{v}$ is 180° more than for $\mathbf{v}$, because they are opposite, i.e. 248°.

e) $3\mathbf{u} + 2\mathbf{v} = 3\begin{pmatrix}-4\\7\end{pmatrix} + 2\begin{pmatrix}2\\5\end{pmatrix} = (-8, 31)$

$|3\mathbf{u} + 2\mathbf{v}| = \sqrt{(-8)^2 + 31^2} = \sqrt{1025} = 5\sqrt{41}$

$\theta = \arctan\left(\dfrac{31}{-8}\right) + \pi = 104.5^\circ$

f) $\mathbf{u} - \mathbf{v} = \begin{pmatrix}-4\\7\end{pmatrix} - \begin{pmatrix}2\\5\end{pmatrix} = (-6, 2)$

$|\mathbf{u} - \mathbf{v}| = \sqrt{(-6)^2 + 2^2} = \sqrt{40} = 2\sqrt{10}$

$\theta = \arctan\left(\dfrac{2}{-6}\right) + \pi = 161.6^\circ$

4 **a)** For $|\mathbf{u}| = 310$ and $\theta = 62^\circ$, we have:

$\mathbf{u} = |\mathbf{u}|(\cos\theta\mathbf{i} + \sin\theta\mathbf{j}) = 310\left(\cos 62^\circ\mathbf{i} + \sin 62^\circ\mathbf{j}\right) = (145.54, 273.71)$

b) For $|\mathbf{u}| = 43.2$ and $\theta = 19.6^\circ$, we have:

$\mathbf{u} = |\mathbf{u}|(\cos\theta\mathbf{i} + \sin\theta\mathbf{j}) = 43.2\left(\cos 19.6^\circ\mathbf{i} + \sin 19.6^\circ\mathbf{j}\right) = (40.70, 14.49)$

c) For $|\mathbf{u}| = 12$, $\theta = 135^\circ$, we have:

$\mathbf{u} = |\mathbf{u}|(\cos\theta\mathbf{i} + \sin\theta\mathbf{j}) = 12\left(\cos 135^\circ\mathbf{i} + \sin 135^\circ\mathbf{j}\right) = 12\left(-\dfrac{\sqrt{2}}{2}\mathbf{i} + \dfrac{\sqrt{2}}{2}\mathbf{j}\right) = (-6\sqrt{2}, 6\sqrt{2})$

d) For $|\mathbf{u}| = 240$, $\theta = 300^\circ$, we have:

$\mathbf{u} = |\mathbf{u}|(\cos\theta\mathbf{i} + \sin\theta\mathbf{j}) = 240\left(\cos 300^\circ\mathbf{i} + \sin 300^\circ\mathbf{j}\right) = 240\left(\dfrac{1}{2}\mathbf{i} - \dfrac{\sqrt{3}}{2}\mathbf{j}\right) = (120, -120\sqrt{3})$

5 For $A(2,1)$, $B(4,7)$, $C(-1,1)$, $D(x, y)$:

$\overrightarrow{AB} = 2\overrightarrow{CD} \Rightarrow \begin{pmatrix}4-2\\7-1\end{pmatrix} = 2\begin{pmatrix}x+1\\y-1\end{pmatrix} \Rightarrow \begin{pmatrix}1\\3\end{pmatrix} = \begin{pmatrix}x+1\\y-1\end{pmatrix} \Rightarrow$

$x + 1 = 1 \Rightarrow x = 0$

$y - 1 = 3 \Rightarrow y = 4$

Point $D(0, 4)$.

6 **a)** For $\mathbf{u} = (3, 4)$, the unit vector in the same direction is:

$$\frac{\mathbf{u}}{|\mathbf{u}|} = \frac{1}{\sqrt{3^2+4^2}}(3\mathbf{i}+4\mathbf{j}) = \frac{1}{5}(3\mathbf{i}+4\mathbf{j}) = \left(\frac{3}{5}, \frac{4}{5}\right)$$

b) For $\mathbf{u} = 2\mathbf{i} - 5\mathbf{j}$, the unit vector in the same direction is:

$$\frac{\mathbf{u}}{|\mathbf{u}|} = \frac{1}{\sqrt{2^2+5^2}}(2\mathbf{i}-5\mathbf{j}) = \frac{1}{\sqrt{29}}(2\mathbf{i}-5\mathbf{j}) = \frac{2}{\sqrt{29}}\mathbf{i} - \frac{5}{\sqrt{29}}\mathbf{j}$$

7 **a)** $\cos 150^\circ \mathbf{i} + \sin 150^\circ \mathbf{j} = \left(-\frac{\sqrt{3}}{2}, \frac{1}{2}\right)$

b) $\cos 315^\circ \mathbf{i} + \sin 315^\circ \mathbf{j} = \left(\frac{\sqrt{2}}{2}, -\frac{\sqrt{2}}{2}\right)$

8 The vector of magnitude 7 that is parallel to $\mathbf{u} = 3\mathbf{i} - 4\mathbf{j}$ is:

$$7\frac{\mathbf{u}}{|\mathbf{u}|} = \frac{7}{\sqrt{3^2+(-4)^2}}(3\mathbf{i}-4\mathbf{j}) = \frac{7}{5}(3\mathbf{i}-4\mathbf{j}) = \frac{21}{5}\mathbf{i} - \frac{28}{5}\mathbf{j} = \left(\frac{21}{5}, -\frac{28}{5}\right)$$

9 The vector of magnitude 3 that is parallel to $\mathbf{u} = 2\mathbf{i} + 3\mathbf{j}$ is:

$$3\frac{\mathbf{u}}{|\mathbf{u}|} = \frac{3}{\sqrt{2^2+3^2}}(2\mathbf{i}+3\mathbf{j}) = \frac{3}{\sqrt{13}}(2\mathbf{i}+3\mathbf{j}) = \frac{6\sqrt{13}}{13}\mathbf{i} + \frac{9\sqrt{13}}{13}\mathbf{j} = \left(\frac{6\sqrt{13}}{13}, \frac{9\sqrt{13}}{13}\right)$$

10 One vector perpendicular to $\mathbf{u} = 3\mathbf{i} - 4\mathbf{j}$ is $\mathbf{v} = 4\mathbf{i} + 3\mathbf{j}$ because $\mathbf{u}\cdot\mathbf{v} = 3\cdot 4 + (-4)\cdot 3 = 0$; so, the vector of magnitude 7 is:

$$7\frac{\mathbf{v}}{|\mathbf{v}|} = \frac{7}{\sqrt{4^2+3^2}}(4\mathbf{i}+3\mathbf{j}) = \frac{7}{5}(4\mathbf{i}+3\mathbf{j}) = \frac{28}{5}\mathbf{i} + \frac{21}{5}\mathbf{j} = \left(\frac{28}{5}, \frac{21}{5}\right)$$

11 One vector perpendicular to $\mathbf{u} = 2\mathbf{i} + 3\mathbf{j}$ is $\mathbf{v} = 3\mathbf{i} - 2\mathbf{j}$, so the vector of magnitude 3 is:

$$3\frac{\mathbf{v}}{|\mathbf{v}|} = \frac{3}{\sqrt{3^2+(-2)^2}}(3\mathbf{i}-2\mathbf{j}) = \frac{3}{\sqrt{13}}(3\mathbf{i}-2\mathbf{j}) = \frac{9\sqrt{13}}{13}\mathbf{i} - \frac{6\sqrt{13}}{13}\mathbf{j} = \left(\frac{9\sqrt{13}}{13}, -\frac{6\sqrt{13}}{13}\right)$$

12 **a)** The direction angle of a vector (in standard position) is the angle it makes with the positive x-axis.

The direction angle for the plane's still-air velocity is: $90^\circ - 170^\circ = -80^\circ$.
The velocity vector for the plane is: $\vec{P} = 840\left(\cos(-80^\circ)\mathbf{i} + \sin(-80^\circ)\mathbf{j}\right) = (840\cos 80^\circ, -840\sin 80^\circ)$.

The direction angle for the wind's velocity is: $90^\circ - 120^\circ = -30^\circ$.
The velocity vector for the wind is: $\vec{W} = 60\left(\cos(-30^\circ)\mathbf{i} + \sin(-30^\circ)\mathbf{j}\right) = (60\cos 30^\circ, -60\sin 30^\circ)$.

b) The true velocity of the plane is:

$\vec{V} = \vec{P} + \vec{W} = (840\cos 80^\circ + 60\cos 30^\circ, -840\sin 80^\circ - 60\sin 30^\circ) = (197.83, -857.24)$.

c) The speed is the magnitude of the velocity vector:

$s = \sqrt{197.83^2 + (-857.24)^2} = 879.77$ km/h.

The direction angle for the true velocity of the plane is:

$\theta = \arctan\left(\frac{-857.24}{197.83}\right) = -77^\circ$.

Therefore, the bearing of the plane is: $90^\circ - \theta = 90^\circ + 77^\circ = 167^\circ$.

13 **a)** The direction angle for the plane's still-air velocity is: $90^\circ - 340^\circ = -250^\circ = 110^\circ$.
The velocity vector for the plane is: $\vec{P} = 520\left(\cos 110^\circ \mathbf{i} + \sin 110^\circ \mathbf{j}\right) = (520\cos 110^\circ, 520\sin 110^\circ)$.

The direction angle for the wind's velocity is: $90^\circ - 320^\circ = -230^\circ = 130^\circ$.
The velocity vector for the wind is: $\vec{W} = 64\left(\cos 130^\circ \mathbf{i} + \sin 130^\circ \mathbf{j}\right) = (64\cos 130^\circ, 64\sin 130^\circ)$.

b) The actual ground velocity of the plane is:

$\vec{V} = \vec{P} + \vec{W} = (520\cos 110^\circ + 64\cos 130^\circ, 520\sin 110^\circ + 64\sin 130^\circ) = (-218.99, 537.67)$.

Therefore, the speed is: $s = \sqrt{(-218.99)^2 + 537.67^2} = 580.56$ km/h.

The direction angle for the true velocity is: $\theta = \arctan\left(\dfrac{537.67}{-218.99}\right) = 112.16^\circ$, which gives us a bearing of 337.84°.

14 The force vector is $\vec{F} = 25(\cos 15^\circ \mathbf{i} + \sin 15^\circ \mathbf{j}) = (24.15, 6.47)$.

15 The velocity vector of the boat for still water is $\vec{B} = 30\mathbf{j}$. The velocity vector of the current is $\vec{C} = 6\mathbf{i}$. Therefore, the true velocity vector of the boat is $\vec{V} = \vec{B} + \vec{C} = 6\mathbf{i} + 30\mathbf{j}$. It is collinear with the vector $\dfrac{1}{30}\vec{V} = \dfrac{6}{30}\mathbf{i} + \dfrac{30}{30}\mathbf{j} = \dfrac{1}{5}\mathbf{i} + \mathbf{j}$. So, while moving 1 km (the width of the river) in the northerly direction, the boat will also move $\dfrac{1}{5}$ km $= 200$ m in an easterly direction.

16 The vector for the applied force is: $\vec{A} = 2500(\cos 38^\circ \mathbf{i} + \sin 38^\circ \mathbf{j}) = (1970.03, 1539.15)$.

The force exerted by the ship is: $\vec{S} = 10\,000\mathbf{i} = (10\,000, 0)$.

Let $\vec{F}$ be the vector of the force needed. The force needed, together with the applied force, must counter the force exerted by the ship, so:

$\vec{F} + \vec{A} = \vec{S} \Rightarrow \vec{F} = \vec{S} - \vec{A} = (8029.97, -1539.15)$.

Therefore, the magnitude of the force needed to pull the ship in the direction given is:

$\left|\vec{F}\right| = \sqrt{8029.97^2 + (-1539.15)^2} = 8176.15$ N, and the direction angle is:

$\theta = \arctan\left(\dfrac{-1539.15}{8029.97}\right) = -10.85^\circ$.

17 **a)** For a bearing of 072°, the direction angle of the boat is $90^\circ - 72^\circ = 18^\circ$. The velocity of the boat with respect to the water is: $\vec{B} = 40(\cos 18^\circ \mathbf{i} + \sin 18^\circ \mathbf{j}) = (38.04, 12.36)$.

b) The speed of the water stream is 12.36 km/h and the true speed of the boat is 38.04 km/h.

18 The tensions in the strings, **T** and **S**, are the magnitudes of the vectors $\vec{S}$ and $\vec{T}$. We have:

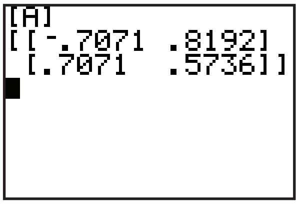
$\vec{S} = \mathbf{S}(\cos 135^\circ \mathbf{i} + \sin 135^\circ \mathbf{j}), \vec{T} = \mathbf{T}(\cos 35^\circ \mathbf{i} + \sin 35^\circ \mathbf{j})$

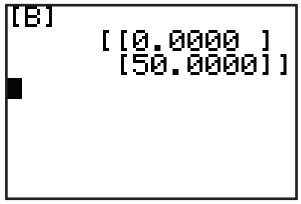
$\vec{S} + \vec{T} = 50\mathbf{j} \Rightarrow (\mathbf{S}\cos 135^\circ \mathbf{i} + \mathbf{S}\sin 135^\circ \mathbf{j}) + (\mathbf{T}\cos 35^\circ \mathbf{i} + \mathbf{T}\sin 35^\circ \mathbf{j}) = 50\mathbf{j}$

$$\Rightarrow \begin{cases} \mathbf{S}\cos 135^\circ + \mathbf{T}\cos 35^\circ = 0 \\ \mathbf{S}\sin 135^\circ + \mathbf{T}\sin 35^\circ = 50 \end{cases}$$

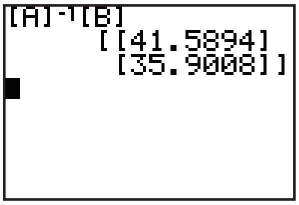
This system of equations can be solved easily with a GDC.

For:





we have:



So, the tensions in the strings are **S** = 41.6 N and **T** = 35.9 N.

19 The velocity vector of the runner relative to the water is: $\mathbf{v} = -8\mathbf{i} + 35\mathbf{j}$.

The speed is: $\sqrt{(-8)^2 + 35^2} = 35.9\,\text{km/h}$.

The direction angle is: $\theta = \arctan\left(\frac{35}{-8}\right) = 102.88^\circ$, which is N 12.88° W.

20 To reach a point exactly north of the starting point:

$\theta = \arctan\left(\frac{0.2}{1}\right) = 11.31^\circ$ in a north westerly direction

21 For forces $\mathbf{F} = (-10, 3)$, $\mathbf{G} = (-4, 1)$, and $\mathbf{H} = (4, -10)$, the resultant force **R** is:

$$\mathbf{R} = \begin{pmatrix} -10 \\ 3 \end{pmatrix} + \begin{pmatrix} -4 \\ 1 \end{pmatrix} + \begin{pmatrix} 4 \\ -10 \end{pmatrix} = \begin{pmatrix} -10 \\ -6 \end{pmatrix}$$

The force required to keep the system in equilibrium is:

$$\mathbf{P} = -\mathbf{R} = -\begin{pmatrix} -10 \\ -6 \end{pmatrix} = (10, 6).$$

22 For the wind velocity vector $\mathbf{w} = -60\mathbf{i}$, the plane's air velocity $\mathbf{p} = 60\mathbf{i} + b\mathbf{j}$, and the speed of 300 km/h, we have:

$$|\mathbf{p}| = \sqrt{60^2 + b^2} = 300 \Rightarrow b = \sqrt{300^2 - 60^2} = 293.94 \Rightarrow$$

$$\mathbf{p} = 60\mathbf{i} + 293.94\mathbf{j}$$

$$\theta = \arctan\left(\frac{293.94}{60}\right) = 78.46^\circ$$

$$90^\circ - 78.46^\circ = 11.54^\circ$$

The pilot should steer the plane 11.54° in a north easterly direction. The plane is moving at a speed of 293.94 km/h.

23 For vertices of the parallelogram $P(2, 2)$, $Q(10, 2)$, $R(12, 6)$, $S(x, y)$:

a) If P and R are on the same diagonal:

$$\overrightarrow{QP} = \overrightarrow{RS} \Rightarrow \begin{pmatrix} 2-10 \\ 2-2 \end{pmatrix} = \begin{pmatrix} x-12 \\ y-6 \end{pmatrix}$$

$$x - 12 = -8 \Rightarrow x = 4$$

$$y - 6 = 0 \Rightarrow y = 6$$

Point $S(4, 6)$.

b) If P and R are on the common side:

$$\overrightarrow{RP} = \overrightarrow{QS} \Rightarrow \begin{pmatrix} 2-12 \\ 2-6 \end{pmatrix} = \begin{pmatrix} x-10 \\ y-2 \end{pmatrix}$$

$x - 10 = -10 \Rightarrow x = 0$

$y - 2 = -4 \Rightarrow y = -2$

Or:

$$\overrightarrow{RP} = \overrightarrow{SQ} \Rightarrow \begin{pmatrix} 2-12 \\ 2-6 \end{pmatrix} = \begin{pmatrix} 10-x \\ 2-y \end{pmatrix}$$

$10 - x = -10 \Rightarrow x = 20$

$2 - y = -4 \Rightarrow y = 6$

Point $S(0, -2)$ or $S(20, 6)$.

24 Let points $A(x_a, y_a)$, $B(x_b, y_b)$, $C(x_c, y_c)$, $D(x_d, y_d)$ be the vertices of the parallelogram. For the diagonals $\overrightarrow{AC}$ and $\overrightarrow{BD}$ we have:

$$\overrightarrow{AC} + \overrightarrow{BD} = \begin{pmatrix} x_c - x_a \\ y_c - y_a \end{pmatrix} + \begin{pmatrix} x_d - x_b \\ y_d - y_b \end{pmatrix} = \begin{pmatrix} x_c - x_b + x_d - x_a \\ y_c - y_b + y_d - y_a \end{pmatrix} = \begin{pmatrix} x_c - x_b \\ y_c - y_b \end{pmatrix} + \begin{pmatrix} x_d - x_a \\ y_d - y_a \end{pmatrix}$$

$$= \overrightarrow{BC} + \overrightarrow{AD} = 2\overrightarrow{AD} \Rightarrow \frac{1}{2}\overrightarrow{AC} + \frac{1}{2}\overrightarrow{BD} = \overrightarrow{AD}$$

This shows that the diagonals $\overrightarrow{AC}$ and $\overrightarrow{BD}$ intersect.

25 Let the points $A(x_a, y_a)$, $B(x_b, y_b)$, $C(x_c, y_c)$ be the vertices of a triangle ABC.

The midpoint of side AC is $M_1\left(\frac{x_a + x_c}{2}, \frac{y_a + y_c}{2}\right)$.

The midpoint of side BC is $M_2\left(\frac{x_b + x_c}{2}, \frac{y_b + y_c}{2}\right)$.

The vector joining the midpoints:

$$\overrightarrow{M_1M_2} = \begin{pmatrix} \frac{x_b + x_c}{2} - \frac{x_a + x_c}{2} \\ \frac{y_b + y_c}{2} - \frac{y_a + y_c}{2} \end{pmatrix} = \begin{pmatrix} \frac{x_b - x_a}{2} \\ \frac{y_b - y_a}{2} \end{pmatrix} = \frac{1}{2}\begin{pmatrix} x_b - x_a \\ y_b - y_a \end{pmatrix} = \frac{1}{2}\overrightarrow{AB} \Rightarrow$$

$M_1M_2 \parallel AB$ and $|M_1M_2| = \frac{1}{2}|AB|$.

26 Let the points $A(x_a, y_a)$, $B(x_b, y_b)$, $C(x_c, y_c)$, $D(x_d, y_d)$ be the vertices of a quadrilateral.

The midpoint of side AB is $M_1\left(\frac{x_a + x_b}{2}, \frac{y_a + y_b}{2}\right)$.

The midpoint of side BC is $M_2\left(\frac{x_b + x_c}{2}, \frac{y_b + y_c}{2}\right)$.

The midpoint of side CD is $M_3\left(\frac{x_c + x_d}{2}, \frac{y_c + y_d}{2}\right)$.

The midpoint of side DA is $M_4\left(\frac{x_d + x_a}{2}, \frac{y_d + y_a}{2}\right)$.

The vectors joining the midpoints:

$$\overrightarrow{M_1M_2} = \begin{pmatrix} \frac{x_b + x_c}{2} - \frac{x_a + x_b}{2} \\ \frac{y_b + y_c}{2} - \frac{y_a + y_b}{2} \end{pmatrix} = \begin{pmatrix} \frac{x_c - x_a}{2} \\ \frac{y_c - y_a}{2} \end{pmatrix}$$

$$\overrightarrow{M_3M_4} = \begin{pmatrix} \frac{x_d + x_a}{2} - \frac{x_c + x_d}{2} \\ \frac{y_d + y_a}{2} - \frac{y_c + y_d}{2} \end{pmatrix} = \begin{pmatrix} \frac{x_a - x_c}{2} \\ \frac{y_a - y_c}{2} \end{pmatrix}$$

$$\overrightarrow{M_1M_2} = -\overrightarrow{M_3M_4} \quad \Rightarrow$$

$M_1M_2 \parallel M_3M_4$ and $|M_1M_2| = |M_3M_4|$.

This proves that $M_1M_2M_3M_4$ is a parallelogram.

27 a) The velocity vector of the boat for still water is $\mathbf{b} = 30\mathbf{j}$. The velocity vector of the current is $\mathbf{c} = 10\mathbf{i}$. The true velocity vector of the boat is: $\mathbf{v} = \mathbf{b} + \mathbf{c} = 10\mathbf{i} + 30\mathbf{j}$. It is collinear with the vector $5\mathbf{v} = 5(10\mathbf{i} + 30\mathbf{j}) = 50\mathbf{i} + 150\mathbf{j}$. So, while moving 150 m (which is the width of the river) in a northerly direction, the boat will also move 50 m eastward.

b) The trip lasts 5 minutes.

c) The angle at which the boat must be steered is: $\theta = \arctan\left(\frac{150}{50}\right) = 71.57^\circ \Rightarrow \text{N } 18.43^\circ \text{ W}$

The length of the route across the river: $\sqrt{150^2 + 50^2} = 158.11$ m

Time taken for the trip is: $\frac{158.11}{30} = 5.27$ minutes

28 a) The true velocity of the jet is:

$\mathbf{p} = 400(\cos 60^\circ\,\mathbf{i} + \sin 60^\circ\,\mathbf{j}) + 20\mathbf{i} = 200\mathbf{i} + 200\sqrt{3}\mathbf{j} + 20\mathbf{i} = 220\mathbf{i} + 200\sqrt{3}\mathbf{j}$

b) The true speed of the jet is:

$|\mathbf{p}| = \sqrt{220^2 + \left(200\sqrt{3}\right)^2} = 410.37$ km/h

The direction in which the jet moves is: $\theta = \arctan\left(\frac{200\sqrt{3}}{220}\right) = 57.58^\circ \Rightarrow \text{N } 32.42^\circ \text{ E}$

29 $\mathbf{G} = 200(\cos 30^\circ\,\mathbf{i} + \sin 30^\circ\,\mathbf{j}) = 100\sqrt{3}\mathbf{i} + 100\mathbf{j}$

$\mathbf{F} = 200(\cos 135^\circ\,\mathbf{i} + \sin 135^\circ\,\mathbf{j}) = -100\sqrt{2}\mathbf{i} + 100\sqrt{2}\mathbf{j}$

Box: $\mathbf{B} = -300\mathbf{j}$

Resultant force: $\mathbf{R} = \mathbf{G} + \mathbf{F} + \mathbf{B} = 100\sqrt{3}\mathbf{i} + 100\mathbf{j} - 100\sqrt{2}\mathbf{i} + 100\sqrt{2}\mathbf{j} - 300\mathbf{j}$

$= (100\sqrt{3} - 100\sqrt{2})\mathbf{i} + (100\sqrt{2} - 200)\mathbf{j}$

$|\mathbf{R}| = \sqrt{(100\sqrt{3} - 100\sqrt{2})^2 + (100\sqrt{2} - 200)^2} = 66.6$ N

$\theta = \arctan\left(\frac{100\sqrt{2} - 200}{100\sqrt{3} - 100\sqrt{2}}\right) = -61.5^\circ \Rightarrow \text{N } 151.5^\circ \text{E}$

Exercise 9.4

1 **a)** $\mathbf{u} = \mathbf{i} + \sqrt{3}\mathbf{j},\ \mathbf{v} = \sqrt{3}\mathbf{i} - \mathbf{j}$

$\mathbf{u} \cdot \mathbf{v} = 1 \times \sqrt{3} + \sqrt{3} \times (-1) = 0$

$$\cos\theta = \frac{\mathbf{u} \cdot \mathbf{v}}{|\mathbf{u}| \cdot |\mathbf{v}|} = \frac{0}{\sqrt{1^2 + \left(\sqrt{3}\right)^2} \cdot \sqrt{\left(\sqrt{3}\right)^2 + 1^2}} = 0 \Rightarrow \theta = 90^\circ$$

b) $\mathbf{u} = (2, 5),\ \mathbf{v} = (4, 1)$

$\mathbf{u} \cdot \mathbf{v} = 2 \times 4 + 5 \times 1 = 13$

$$\cos\theta = \frac{\mathbf{u} \cdot \mathbf{v}}{|\mathbf{u}| \cdot |\mathbf{v}|} = \frac{13}{\sqrt{2^2 + 5^2} \cdot \sqrt{4^2 + 1^2}} = \frac{13}{\sqrt{29} \cdot \sqrt{17}} \Rightarrow \theta \approx 54^\circ$$

c) $\mathbf{u} = 2\mathbf{i} - 3\mathbf{j},\ \mathbf{v} = 4\mathbf{i} - \mathbf{j}$

$\mathbf{u} \cdot \mathbf{v} = 2 \times 4 + (-3) \times (-1) = 11$

$$\cos\theta = \frac{\mathbf{u} \cdot \mathbf{v}}{|\mathbf{u}| \cdot |\mathbf{v}|} = \frac{11}{\sqrt{2^2 + (-3)^2} \cdot \sqrt{4^2 + (-1)^2}} = \frac{11}{\sqrt{13} \cdot \sqrt{17}} \Rightarrow \theta = 42^\circ$$

d) $\mathbf{u} = 2\mathbf{j},\ \mathbf{v} = -\mathbf{i} + \sqrt{3}\mathbf{j}$

$\mathbf{u} \cdot \mathbf{v} = 0 \times (-1) + 2 \times \sqrt{3} = 2\sqrt{3}$

$$\cos\theta = \frac{\mathbf{u} \cdot \mathbf{v}}{|\mathbf{u}| \cdot |\mathbf{v}|} = \frac{2\sqrt{3}}{\sqrt{2^2} \cdot \sqrt{(-1)^2 + \left(\sqrt{3}\right)^2}} = \frac{2\sqrt{3}}{4} = \frac{\sqrt{3}}{2} \Rightarrow \theta = 30^\circ$$

e) $\mathbf{u} = (-3, 0),\ \mathbf{v} = (0, 7)$

$\mathbf{u} \cdot \mathbf{v} = (-3) \times 0 + 0 \times 7 = 0$

$$\cos\theta = \frac{\mathbf{u} \cdot \mathbf{v}}{|\mathbf{u}| \cdot |\mathbf{v}|} = 0 \Rightarrow \theta = 90^\circ$$

f) $\mathbf{u} = (3, 0),\ \mathbf{v} = (\sqrt{3}, 1)$

$\mathbf{u} \cdot \mathbf{v} = 3 \times \sqrt{3} + 0 \times 1 = 3\sqrt{3}$

$$\cos\theta = \frac{\mathbf{u} \cdot \mathbf{v}}{|\mathbf{u}| \cdot |\mathbf{v}|} = \frac{3\sqrt{3}}{\sqrt{3^2} \cdot \sqrt{\left(\sqrt{3}\right)^2 + 1^2}} = \frac{3\sqrt{3}}{6} = \frac{\sqrt{3}}{2} \Rightarrow \theta = 30^\circ$$

g) $\mathbf{u} = -6\mathbf{j},\ \mathbf{v} = -2\mathbf{i} + 2\sqrt{3}\mathbf{j}$

$\mathbf{u} \cdot \mathbf{v} = 0 \times (-2) - 6 \times 2\sqrt{3} = -12\sqrt{3}$

$$\cos\theta = \frac{\mathbf{u} \cdot \mathbf{v}}{|\mathbf{u}| \cdot |\mathbf{v}|} = \frac{-12\sqrt{3}}{\sqrt{(-6)^2} \cdot \sqrt{(-2)^2 + \left(2\sqrt{3}\right)^2}} = \frac{-12\sqrt{3}}{24} = -\frac{\sqrt{3}}{2} \Rightarrow \theta = 150^\circ$$

h) $\mathbf{u} = 2\mathbf{i} + 2\mathbf{j},\ \mathbf{v} = -4\mathbf{i} - 4\mathbf{j}$

$\mathbf{u} \cdot \mathbf{v} = 2 \times (-4) + 2 \times (-4) = -16$

$$\cos\theta = \frac{\mathbf{u} \cdot \mathbf{v}}{|\mathbf{u}| \cdot |\mathbf{v}|} = \frac{-16}{\sqrt{2^2 + 2^2} \cdot \sqrt{(-4)^2 + (-4)^2}} = \frac{-16}{2\sqrt{2} \cdot 4\sqrt{2}} = -1 \Rightarrow \theta = 180^\circ$$

2 For $\mathbf{u} = 3\mathbf{i} - 2\mathbf{j}$, $\mathbf{v} = \mathbf{i} + 3\mathbf{j}$ and $\mathbf{w} = 4\mathbf{i} + 5\mathbf{j}$, we have:

a) $\mathbf{u}\cdot(\mathbf{v}+\mathbf{w}) = (3\mathbf{i}-2\mathbf{j})\left[(\mathbf{i}+3\mathbf{j})+(4\mathbf{i}+5\mathbf{j})\right] = (3\mathbf{i}-2\mathbf{j})\cdot(5\mathbf{i}+8\mathbf{j}) = 3\times5+(-2)\times8 = -1$

b) $\mathbf{u}\cdot\mathbf{v}+\mathbf{u}\cdot\mathbf{w} = (3\mathbf{i}-2\mathbf{j})\cdot(\mathbf{i}+3\mathbf{j})+(3\mathbf{i}-2\mathbf{j})\cdot(4\mathbf{i}+5\mathbf{j}) = 3\times1+(-2)\times3+3\times4+(-2)\times5 = -1$

c) $\mathbf{u}\cdot(\mathbf{v}\cdot\mathbf{w}) = (3\mathbf{i}-2\mathbf{j})\left[(\mathbf{i}+3\mathbf{j})\cdot(4\mathbf{i}+5\mathbf{j})\right] = (3\mathbf{i}-2\mathbf{j})\cdot(1\times4+3\times5) = (3\mathbf{i}-2\mathbf{j})\cdot19 = 57\mathbf{i}-38\mathbf{j}$

d) $(\mathbf{u}\cdot\mathbf{v})\cdot\mathbf{w} = \left[(3\mathbf{i}-2\mathbf{j})\cdot(\mathbf{i}+3\mathbf{j})\right]\cdot(4\mathbf{i}+5\mathbf{j}) = \left[3\times1+(-2)\times3\right]\cdot(4\mathbf{i}+5\mathbf{j}) = (-3)(4\mathbf{i}+5\mathbf{j}) = -12\mathbf{i}-15\mathbf{j}$

e) $(\mathbf{u}\cdot\mathbf{v})(\mathbf{u}\cdot\mathbf{w}) = \left[(3\mathbf{i}-2\mathbf{j})\cdot(\mathbf{i}+3\mathbf{j})\right]\cdot\left[(3\mathbf{i}-2\mathbf{j})\cdot(4\mathbf{i}+5\mathbf{j})\right]$

$= \left[3\times1+(-2)\times3\right]\cdot\left[3\times4+(-2)\times5\right] = (-3)(2) = -6$

f) $(\mathbf{u}+\mathbf{v})\cdot(\mathbf{u}-\mathbf{v}) = \left[(3\mathbf{i}-2\mathbf{j})+(\mathbf{i}+3\mathbf{j})\right]\cdot\left[(3\mathbf{i}-2\mathbf{j})-(\mathbf{i}+3\mathbf{j})\right]$

$= \left[4\mathbf{i}+\mathbf{j}\right]\cdot\left[2\mathbf{i}-5\mathbf{j}\right] = 4\times2+1\times(-5) = 3$

g) Scalar multiplication is distributive over addition of vectors. Scalar multiplication is not associative.

3 For $\mathbf{u} = \begin{pmatrix} -\frac{1}{2} \\ 2 \end{pmatrix}$, $\mathbf{v} = \begin{pmatrix} -2 \\ \frac{1}{2} \end{pmatrix}$: $\mathbf{u}\cdot\mathbf{v} = -\frac{1}{2}\times(-2)+2\times\left(\frac{1}{2}\right) = 2$

Vectors are neither orthogonal nor parallel.

For $\mathbf{u} = \begin{pmatrix} 8 \\ 4 \end{pmatrix}$, $\mathbf{v} = \begin{pmatrix} 6 \\ -12 \end{pmatrix}$: $\mathbf{u}\cdot\mathbf{v} = 8\times6+4\times(-12) = 0$

Vectors are orthogonal.

For $\mathbf{u} = \begin{pmatrix} 2\sqrt{3} \\ 2 \end{pmatrix}$, $\mathbf{v} = \begin{pmatrix} 1 \\ -\sqrt{3} \end{pmatrix}$: $\mathbf{u}\cdot\mathbf{v} = 2\sqrt{3}\times1+2\times\left(-\sqrt{3}\right) = 0$

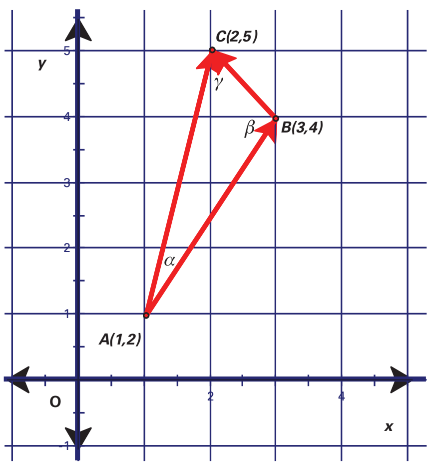
Vectors are orthogonal.

4 **a)** Work $= \mathbf{F}\cdot\overrightarrow{MN} = (400\mathbf{i}-50\mathbf{j})\cdot\left[(12-2)\mathbf{i}+(43-3)\,\mathbf{j}\right] = 400\times10+(-50)\times40 = 2000$

b) Work $= \mathbf{F}\cdot\overrightarrow{MN} = (30\mathbf{i}+150\mathbf{j})\cdot\left[(15-0)\mathbf{i}+(70-30)\,\mathbf{j}\right] = 30\times15+150\times40 = 6450$

c) Work $= \mathbf{F}\cdot\overrightarrow{MN} = (5\mathbf{i}+25\mathbf{j})\cdot\left[(1-0)\mathbf{i}+(6-0)\,\mathbf{j}\right] = 5\times1+25\times6 = 155$

5 **a)**



$$\cos\alpha = \frac{\overrightarrow{AC}\cdot\overrightarrow{AB}}{\left|\overrightarrow{AC}\right|\cdot\left|\overrightarrow{AB}\right|} = \frac{(\mathbf{i}+3\mathbf{j})\cdot(2\mathbf{i}+2\mathbf{j})}{|\mathbf{i}+3\mathbf{j}|\cdot|2\mathbf{i}+2\mathbf{j}|} = \frac{1\times2+3\times2}{\sqrt{1^2+3^2}\cdot\sqrt{2^2+2^2}} = \frac{8}{\sqrt{10}\sqrt{8}} = \frac{2}{\sqrt{5}} \Rightarrow \alpha \approx 26.6^\circ$$

$$\cos\beta = \frac{\overrightarrow{BC}\cdot(\overrightarrow{-AB})}{|\overrightarrow{BC}|\cdot|\overrightarrow{-AB}|} = \frac{(-\mathbf{i}+\mathbf{j})\cdot(-2\mathbf{i}-2\mathbf{j})}{|-\mathbf{i}+\mathbf{j}|\cdot|-2\mathbf{i}-2\mathbf{j}|} = \frac{(-1)\times(-2)+1\times(-2)}{\sqrt{(-1)^2+(-1)^2}\cdot\sqrt{(-2)^2+(-2)^2}} = 0 \Rightarrow \beta = 90^\circ$$

So, $\gamma \approx 180^\circ - 26.6^\circ - 90^\circ = 63.4^\circ$.

b) $$\cos\alpha = \frac{\overrightarrow{AC}\cdot\overrightarrow{AB}}{|\overrightarrow{AC}|\cdot|\overrightarrow{AB}|} = \frac{(-11\mathbf{i}-6\mathbf{j})\cdot(-4\mathbf{i}-11\mathbf{j})}{|-11\mathbf{i}-6\mathbf{j}|\cdot|-4\mathbf{i}-4\mathbf{j}|} = \frac{(-11)\times(-4)+(-6)\times(-11)}{\sqrt{(-11)^2+(-6)^2}\cdot\sqrt{(-4)^2+(-11)^2}}$$

$$= \frac{110}{\sqrt{157}\sqrt{137}} \Rightarrow \alpha \approx 41.4^\circ$$

$$\cos\beta = \frac{\overrightarrow{BC}\cdot(\overrightarrow{-AB})}{|\overrightarrow{BC}|\cdot|\overrightarrow{-AB}|} = \frac{(-7\mathbf{i}+5\mathbf{j})\cdot(4\mathbf{i}+11\mathbf{j})}{|-7\mathbf{i}+5\mathbf{j}|\cdot|4\mathbf{i}+11\mathbf{j}|} = \frac{(-7)\times4+5\times11}{\sqrt{(-7)^2+5^2}\cdot\sqrt{4^2+11^2}} = \frac{27}{\sqrt{74}\cdot\sqrt{137}} \Rightarrow \beta \approx 74.4^\circ$$

So, $\gamma \approx 180^\circ - 41.4^\circ - 74.4^\circ = 64.2^\circ$.

c) $$\cos\alpha = \frac{\overrightarrow{AC}\cdot\overrightarrow{AB}}{|\overrightarrow{AC}|\cdot|\overrightarrow{AB}|} = \frac{(-10\mathbf{i}-4\mathbf{j})\cdot(-2\mathbf{i}-4\mathbf{j})}{|-10\mathbf{i}-4\mathbf{j}|\cdot|-2\mathbf{i}-4\mathbf{j}|} = \frac{(-10)\times(-2)+(-4)\times(-4)}{\sqrt{(-10)^2+(-4)^2}\cdot\sqrt{(-2)^2+(-4)^2}}$$

$$= \frac{36}{\sqrt{116}\sqrt{20}} = \frac{9}{\sqrt{29}\cdot\sqrt{5}} \Rightarrow \alpha \approx 41.6^\circ$$

$$\cos\beta = \frac{\overrightarrow{BC}\cdot(\overrightarrow{-AB})}{|\overrightarrow{BC}|\cdot|\overrightarrow{-AB}|} = \frac{(-8\mathbf{i})\cdot(2\mathbf{i}+4\mathbf{j})}{|-8\mathbf{i}|\cdot|2\mathbf{i}+4\mathbf{j}|} = \frac{(-8)\times2+0\times11}{\sqrt{(-8)^2}\cdot\sqrt{2^2+4^2}} = \frac{-16}{8\cdot\sqrt{20}} = -\frac{1}{\sqrt{5}} \Rightarrow \beta \approx 116.6^\circ$$

So, $\gamma \approx 180^\circ - 41.6^\circ - 116.6^\circ = 21.8^\circ$.

6 Perpendicular to $\mathbf{u}$ is any vector $\mathbf{v} = a\mathbf{i} + b\mathbf{j}$, such that $\mathbf{u}\cdot\mathbf{v} = 0$.

a) $\mathbf{u}\cdot\mathbf{v} = (3\mathbf{i}+5\mathbf{j})\cdot(a\mathbf{i}+b\mathbf{j}) = 3a+5b = 0$

For $a = 5t \Rightarrow b = -3t \Rightarrow \mathbf{v} = (5t, -3t), t \in \mathbb{R}, t \neq 0$.

b) $\mathbf{u}\cdot\mathbf{v} = \left(\frac{1}{2}\mathbf{i} - \frac{3}{4}\mathbf{j}\right)\cdot(a\mathbf{i}+b\mathbf{j}) = \frac{a}{2} - \frac{3b}{4} = 0$

For $a = 3t \Rightarrow b = 2t \Rightarrow \mathbf{v} = (3t, 2t), t \in \mathbb{R}, t \neq 0$.

7 For any point $C(x, y)$ on the circle, vectors $\overrightarrow{AC}$ and $\overrightarrow{BC}$ must be perpendicular. Hence, $\overrightarrow{AC}\cdot\overrightarrow{BC} = 0$.

a) $\overrightarrow{AC} = (x-1, y-2), \overrightarrow{BC} = (x-3, y-4)$

$\overrightarrow{AC}\cdot\overrightarrow{BC} = 0 \Rightarrow (x-1)(x-3)+(y-2)(y-4) = 0 \Rightarrow x^2 - 4x + y^2 - 6y + 11 = 0$

b) $\overrightarrow{AC} = (x-3, y-4), \overrightarrow{BC} = (x+1, y+7)$

$\overrightarrow{AC}\cdot\overrightarrow{BC} = 0 \Rightarrow (x-3)(x+1)+(y-4)(y+7) = 0 \Rightarrow x^2 - 2x + y^2 + 3y - 31 = 0$

8 The triangle ABC is right angled if one of the products $\overrightarrow{AB}\cdot\overrightarrow{AC}$, $\overrightarrow{BA}\cdot\overrightarrow{BC}$, or $\overrightarrow{CA}\cdot\overrightarrow{CB}$ is equal to zero.

$\overrightarrow{AB}\cdot\overrightarrow{AC} = (\mathbf{i}+3\mathbf{j})\cdot(5\mathbf{i}+\mathbf{j}) = 8$

$\overrightarrow{BA}\cdot\overrightarrow{BC} = (-\mathbf{i}-3\mathbf{j})\cdot(4\mathbf{i}-2\mathbf{j}) = 2$

$\overrightarrow{CA}\cdot\overrightarrow{CB} = (-5\mathbf{i}-\mathbf{j})\cdot(-4\mathbf{i}+2\mathbf{j}) = 18$

There are no perpendicular vectors, so the triangle is not right angled.

9 $\mathbf{a} \perp \mathbf{b} \Rightarrow \mathbf{a} \cdot \mathbf{b} = 0 \Rightarrow (t\mathbf{i} - 3\mathbf{j}) \cdot (5\mathbf{i} + 7\mathbf{j}) = 0 \Rightarrow 5t - 21 = 0 \Rightarrow t = \frac{21}{5}$

10 $(-6, b) \perp (b, b^2) \Rightarrow (-6\mathbf{i} + b\mathbf{j}) \cdot (b\mathbf{i} + b^2\mathbf{j}) = 0 \Rightarrow -6b + b^3 = 0 \Rightarrow b(b^2 - 6) = 0 \Rightarrow b = 0, \pm\sqrt{6}$
For $b = 0$, the second vector would be the zero vector, so the solution is $\sqrt{6}$ and $-\sqrt{6}$.

11 Let $\mathbf{v} = (a, b)$ be the required unit vector. Then:

$\mathbf{u} = (3, 4)$

$|\mathbf{v}| = 1 \Rightarrow a^2 + b^2 = 1$

$\measuredangle(\mathbf{u}, \mathbf{v}) = 60^\circ \Rightarrow \cos\measuredangle(\mathbf{u}, \mathbf{v}) = \frac{1}{2} \Rightarrow \frac{\mathbf{u} \cdot \mathbf{v}}{|\mathbf{u}|\,|\mathbf{v}|} = \frac{1}{2} \Rightarrow \frac{(3\mathbf{i} + 4\mathbf{j}) \cdot (a\mathbf{i} + b\mathbf{j})}{\sqrt{3^2 + 4^2} \cdot 1} = \frac{1}{2}$

$\Rightarrow \frac{3a + 4b}{5} = \frac{1}{2} \Rightarrow 6a + 8b = 5$

We have to solve the system of equations:

$$\begin{cases} 6a + 8b = 5 \\ a^2 + b^2 = 1 \end{cases}$$

$a = \frac{5 - 8b}{6} \Rightarrow \left(\frac{5 - 8b}{6}\right)^2 + b^2 = 1 \Rightarrow 100b^2 - 80b - 11 = 0 \Rightarrow$

$b = \frac{4 \pm 3\sqrt{3}}{10}, a = \frac{3 \mp 4\sqrt{3}}{10}$

So, there are two solutions for vector $\mathbf{v}$:

$\left(\frac{3 - 4\sqrt{3}}{10}, \frac{4 + 3\sqrt{3}}{10}\right)$ and $\left(\frac{3 + 4\sqrt{3}}{10}, \frac{4 - 3\sqrt{3}}{10}\right)$.

12 $\cos\measuredangle(\mathbf{a}, \mathbf{b}) = \cos\left(\frac{3\pi}{4}\right) = -\frac{\sqrt{2}}{2} \Rightarrow \frac{\mathbf{a} \cdot \mathbf{b}}{|\mathbf{a}|\,|\mathbf{b}|} = -\frac{\sqrt{2}}{2} \Rightarrow \frac{(t\mathbf{i} - \mathbf{j}) \cdot (\mathbf{i} + \mathbf{j})}{\sqrt{t^2 + (-1)^2} \cdot \sqrt{1^2 + 1^2}} = -\frac{\sqrt{2}}{2} \Rightarrow$

$\frac{t - 1}{\sqrt{t^2 + 1} \cdot \sqrt{2}} = -\frac{\sqrt{2}}{2} \Rightarrow t - 1 = -\sqrt{t^2 + 1} \Rightarrow (t - 1)^2 = t^2 + 1 \Rightarrow t = 0$

13 Let $\vec{a}$ and $\vec{b}$ be the sides of a rhombus. Then the diagonals are $\vec{a} + \vec{b}$ and $\vec{a} - \vec{b}$, and we have:

$|\vec{a}| = |\vec{b}|$

$(\vec{a} + \vec{b}) \cdot (\vec{a} - \vec{b}) = \vec{a}\vec{a} + \vec{b}\,\vec{a} - \vec{b}\,\vec{a} - \vec{b}\,\vec{b} = |\vec{a}|^2 - |\vec{b}|^2 = 0$

Since their dot product is zero, the diagonals are perpendicular.

14 **a)** For $\mathbf{u} = (0, 7)$, $\mathbf{v} = (6, 8)$:

$|\mathbf{u}| \cos\angle(\mathbf{u}, \mathbf{v}) = |\mathbf{u}| \frac{\mathbf{u} \cdot \mathbf{v}}{|\mathbf{u}| \cdot |\mathbf{v}|} = \frac{\mathbf{u} \cdot \mathbf{v}}{|\mathbf{v}|} = \frac{0 \times 6 + 7 \times 8}{\sqrt{6^2 + 8^2}} = \frac{56}{10} = 5.6$

b) For $\mathbf{u} = \begin{pmatrix} -\frac{1}{2} \\ 2 \end{pmatrix}$, $\mathbf{v} = \begin{pmatrix} -2 \\ \frac{1}{2} \end{pmatrix}$:

$|\mathbf{u}| \cos\angle(\mathbf{u}, \mathbf{v}) = \frac{\mathbf{u} \cdot \mathbf{v}}{|\mathbf{v}|} = \frac{\left(-\frac{1}{2}\right) \times (-2) + 2 \times \frac{1}{2}}{\sqrt{(-2)^2 + \left(\frac{1}{2}\right)^2}} = \frac{2}{\sqrt{\frac{17}{4}}} = \frac{4}{\sqrt{17}}$

15 The force vector is: $\mathbf{f} = 16(\cos 45^\circ \mathbf{i} + \sin 45^\circ \mathbf{j}) = 16\sqrt{2}\mathbf{i} + \sin 16\sqrt{2}\mathbf{j}$.

The route vector is: $\mathbf{s} = 55\mathbf{i}$.

The work done is: $\mathbf{s} \cdot \mathbf{f} = 55 \cdot 16\sqrt{2} + 0 \cdot 16\sqrt{2} = 440\sqrt{2}$.

16 **a)** For $P(0, 0), l : 3x - 4y + 5 = 0 \Rightarrow d = \dfrac{|3 \cdot 0 - 4 \cdot 0 + 5|}{\sqrt{3^2 + 4^2}} = \dfrac{5}{5} = 1$

b) For $P(2, 2), l : 3x - 2y - 2 = 0 \Rightarrow d = \dfrac{|3 \cdot 2 - 2 \cdot 2 - 2|}{\sqrt{3^2 + 2^2}} = 0$

c) For $P(1, 5), l : 5x - 3y - 11 = 0 \Rightarrow d = \dfrac{|5 \cdot 1 - 3 \cdot 5 - 11|}{\sqrt{5^2 + 3^2}} = \dfrac{21}{\sqrt{34}}$

17 Since $\overrightarrow{OP} \perp \overrightarrow{QR}, \overrightarrow{OQ} \perp \overrightarrow{PR} \Rightarrow \overrightarrow{OP} \cdot \overrightarrow{QR} = 0, \overrightarrow{OQ} \cdot \overrightarrow{PR} = 0$, then:

$$\overrightarrow{PQ} + \overrightarrow{QR} + \overrightarrow{RP} = \vec{0} \,/ \cdot \overrightarrow{OR}$$

$$\overrightarrow{PQ} \cdot \overrightarrow{OR} + \overrightarrow{QR} \cdot \overrightarrow{OR} + \overrightarrow{RP} \cdot \overrightarrow{OR} = 0$$

$$\overrightarrow{PQ} \cdot \overrightarrow{OR} + \overrightarrow{QR}(\overrightarrow{OP} + \overrightarrow{PR}) + \overrightarrow{RP}(\overrightarrow{OQ} + \overrightarrow{QR}) = 0$$

$$\overrightarrow{PQ} \cdot \overrightarrow{OR} + \underbrace{\overrightarrow{QR} \cdot \overrightarrow{OP}}_{0} + \overrightarrow{QR} \cdot \overrightarrow{PR} + \underbrace{\overrightarrow{RP} \cdot \overrightarrow{OQ}}_{0} + \overrightarrow{RP} \cdot \overrightarrow{QR} = 0$$

$$\overrightarrow{PQ} \cdot \overrightarrow{OR} + \underbrace{\overrightarrow{QR} \cdot \overrightarrow{PR} - \overrightarrow{PR} \cdot \overrightarrow{QR}}_{0} = 0$$

$$\overrightarrow{PQ} \cdot \overrightarrow{OR} = 0 \Rightarrow \overrightarrow{PQ} \perp \overrightarrow{OR}$$

18 For $\mathbf{u} = \begin{pmatrix} 3 \\ 4 \end{pmatrix}, \mathbf{v} = \begin{pmatrix} x \\ 1 \end{pmatrix}$, we have:

$$\cos 30^\circ = \frac{\mathbf{u} \cdot \mathbf{v}}{|\mathbf{u}| \cdot |\mathbf{v}|} = \frac{3x + 4}{\sqrt{3^2 + 4^2}\sqrt{x^2 + 1^2}} = \frac{3x + 4}{5\sqrt{x^2 + 1}} = \frac{\sqrt{3}}{2} \Rightarrow$$

$$\left(\frac{3x + 4}{5\sqrt{x^2 + 1}}\right)^2 = \left(\frac{\sqrt{3}}{2}\right)^2 \Rightarrow$$

$$\frac{9x^2 + 24x + 16}{25(x^2 + 1)} = \frac{3}{4} \Rightarrow 36x^2 + 96x + 64 = 75x^2 + 75 \Rightarrow$$

$$39x^2 - 96x + 11 = 0 \Rightarrow x_{1,2} = \frac{96 \pm \sqrt{96^2 - 4 \cdot 39 \cdot 11}}{2 \cdot 39} = \frac{96 \pm 50\sqrt{3}}{78} = \frac{48 \pm 25\sqrt{3}}{39}$$

19 For force vectors $\mathbf{a} = (-200, 400)$ and $\mathbf{b} = (200, 600)$:

The cosine of the angle between vector $\mathbf{a}$ and the horizontal $\mathbf{h} = \mathbf{i}$ is:

$$\cos(180^\circ - \alpha) = \frac{-200 \cdot 1 + 400 \cdot 0}{\sqrt{(-200)^2 + 400^2}\sqrt{1^2}} = \frac{-200}{200\sqrt{5}} = -\frac{1}{\sqrt{5}} \Rightarrow$$

$$180^\circ - \alpha = 116.6^\circ \Rightarrow \alpha = 63.4^\circ$$

The cosine of the angle between vector $\mathbf{b}$ and the horizontal $\mathbf{h} = \mathbf{i}$ is:

$$\cos \beta = \frac{200 \cdot 1 + 600 \cdot 0}{\sqrt{200^2 + 600^2}\sqrt{1^2}} = \frac{200}{200\sqrt{10}} = \frac{1}{\sqrt{10}} \Rightarrow$$

$$\beta = 116.6^\circ \Rightarrow \beta = 71.6^\circ$$

Then: $\theta = 180^\circ - (\alpha + \beta) = 45^\circ$.

20 Let $\mathbf{r} = |\mathbf{a}|\,\mathbf{b} + |\mathbf{b}|\,\mathbf{a}$, α be the angle between vectors **a** and **b**, and β the angle between vectors **a** and **r**. Then $\alpha - \beta$ is the angle between vectors **b** and **r**. So, we have:

$$\mathbf{r} = |\mathbf{a}|\,\mathbf{b} + |\mathbf{b}|\,\mathbf{a} \Rightarrow \mathbf{r} \cdot \mathbf{a} = (|\mathbf{a}|\,\mathbf{b} + |\mathbf{b}|\,\mathbf{a}) \cdot \mathbf{a} \Rightarrow$$

$$|\mathbf{r}|\,|\mathbf{a}| \cos \beta = |\mathbf{a}|\,\mathbf{ba} + |\mathbf{b}|\,\mathbf{a} \cdot \mathbf{a} \Rightarrow |\mathbf{r}|\,|\mathbf{a}| \cos \beta = |\mathbf{a}|\,\mathbf{ab} + |\mathbf{b}|\,|\mathbf{a}|^2 \Rightarrow$$

$$|\mathbf{r}| \cos \beta = \mathbf{ab} + |\mathbf{b}|\,|\mathbf{a}|$$

Also:

$$\mathbf{r} = |\mathbf{a}|\,\mathbf{b} + |\mathbf{b}|\,\mathbf{a} \Rightarrow \mathbf{r} \cdot \mathbf{b} = (|\mathbf{a}|\,\mathbf{b} + |\mathbf{b}|\,\mathbf{a}) \cdot \mathbf{b} \Rightarrow$$

$$|\mathbf{r}|\,|\mathbf{b}| \cos(\alpha - \beta) = |\mathbf{a}|\,\mathbf{b} \cdot \mathbf{b} + |\mathbf{b}|\,\mathbf{a} \cdot \mathbf{b} \Rightarrow |\mathbf{r}|\,|\mathbf{b}| \cos(\alpha - \beta) = |\mathbf{a}|\,|\mathbf{b}|^2 + |\mathbf{b}|\,\mathbf{ab} \Rightarrow$$

$$|\mathbf{r}| \cos(\alpha - \beta) = \mathbf{ab} + |\mathbf{a}|\,|\mathbf{b}|$$

And we can see that:

$$|\mathbf{r}| \cos(\alpha - \beta) = |\mathbf{r}| \cos \beta \Rightarrow \cos(\alpha - \beta) = \cos \beta \Rightarrow \alpha - \beta = \beta \Rightarrow \beta = \frac{\alpha}{2}$$

Chapter 10

Exercise 10.1

1 Since $\sqrt{-4}=2i,\ 5+\sqrt{-4}=5+2i.$

2 Since $\sqrt{-7}=\sqrt{7}i,\ 7-\sqrt{-7}=7-\sqrt{7}i.$

3 $-6=-6+0\cdot i$

4 $-\sqrt{49}=-7=-7+0\cdot i$

5 $\sqrt{-81}=9i=0+9i$

6 $-\sqrt{\dfrac{-25}{16}}=-\dfrac{5}{4}i=0-\dfrac{5}{4}i$

7 $-1-i$

8 $-3+4i-2+5i=-5+9i$

9 $(-3+4i)(2-5i)=-6+8i+15i+20=14+23i$

10 $3i-2+4i=-2+7i$

11 $(2-7i)(3+4i)=6-21i+8i+28=34-13i$

12 $(1+i)(2-3i)=2+2i-3i+3=5-i$

13 $\dfrac{3+2i}{2+5i}\cdot\dfrac{2-5i}{2-5i}=\dfrac{16-11i}{4+25}=\dfrac{16-11i}{29}=\dfrac{16}{29}-\dfrac{11}{29}i$

14 $\dfrac{2-i}{3+2i}\cdot\dfrac{3-2i}{3-2i}=\dfrac{4-7i}{9+4}=\dfrac{4-7i}{13}=\dfrac{4}{3}-\dfrac{7}{13}i$

15 1

16 $\dfrac{4}{9}+\dfrac{1}{4}=\dfrac{25}{36}$

17 $\dfrac{\dfrac{2}{3}-\dfrac{1}{2}i}{\dfrac{1}{3}+\dfrac{1}{2}i}=\dfrac{\dfrac{4-3i}{6}}{\dfrac{2+3i}{6}}=\dfrac{-1-18i}{4+9}=\dfrac{-1-18i}{13}=\dfrac{-1}{13}-\dfrac{18}{13}i$

18 $8-i$

19 $\dfrac{-i(3-7i)}{1^2}=-7-3i$

20 $4+10i$

21 $\dfrac{13(5+12i)}{25+144}=\dfrac{65+156i}{169}=\dfrac{5+12i}{13}=\dfrac{5}{13}+\dfrac{12}{13}i$

22 $\dfrac{12i(3-4i)}{9+16}=\dfrac{48+36i}{25}=\dfrac{48}{25}+\dfrac{36}{25}i$

23 $9i+2$

24 68

25 $\dfrac{(39-52i)(24-10i)}{24^2+10^2}=\dfrac{416-1638i}{676}=\dfrac{8}{13}-\dfrac{63}{26}i$

26 $\dfrac{1}{7-4i}=\dfrac{7+4i}{49+16}=\dfrac{7+4i}{65}=\dfrac{7}{65}+\dfrac{4}{65}i$

27 $\dfrac{1}{5-12i}=\dfrac{5+12i}{25+144}=\dfrac{5+12i}{169}=\dfrac{5}{169}+\dfrac{12}{169}i$

28 $\dfrac{3}{3-4i}+\dfrac{2}{6+8i}=\dfrac{3(3+4i)}{9+16}+\dfrac{2(6-8i)}{36+64}=\dfrac{9+12i}{25}+\dfrac{12-16i}{100}=\dfrac{9+12i}{25}+\dfrac{3-4i}{25}=\dfrac{12+8i}{25}=\dfrac{12}{25}+\dfrac{8}{25}i$

29 $\dfrac{54-19i}{5-12i}=\dfrac{(54-19i)(5+12i)}{25+144}=\dfrac{498+553i}{169}=\dfrac{498}{169}+\dfrac{553}{169}i$

30 $\dfrac{5-12i}{3+4i}=\dfrac{(5-12i)(3-4i)}{9+16}=\dfrac{-33-56i}{25}=\dfrac{-33}{25}-\dfrac{56}{25}i$

31 From $(2+3i)z=7+i\Rightarrow z=\dfrac{7+i}{2+3i}=\dfrac{17-19i}{13}$

Note: We can find z by solving a system of equations:

$(2+3i)(a+bi) = 7+i \Rightarrow 2a-3b+(3a+2b)i = 7+i \Rightarrow$

$$\begin{cases} 2a-3b=7 \\ 3a+3b=1 \end{cases} \Rightarrow \begin{cases} a=\dfrac{17}{13} \\ b=-\dfrac{19}{13} \end{cases}$$

However, it is easier to find z using division.

32 $2x-y+(xy+2)i = 1+3i \Rightarrow$

$2x-y=1$

$xy+2=3$

We have to solve the system of equations. From the first equation, $y = 2x-1$; hence:

$x(2x-1)+2=3 \Rightarrow 2x^2-x-1=0 \Rightarrow x_1=-\dfrac{1}{2}, x_2=1$ and $y_1=-2, y_2=1$. The solutions are: $x_1=-\dfrac{1}{2}, y_1=-2$ and $x_2=1, y_2=1$.

33 **a)** $\left(1+\sqrt{3}i\right)^3 = 1^3+3\cdot 1^2\cdot\sqrt{3}i+3\cdot 1\cdot\left(\sqrt{3}i\right)^2+\left(\sqrt{3}i\right)^3$

$= 1+3\sqrt{3}i+3\cdot 3(-1)+3\sqrt{3}(-i)$

$= 1-9+3\sqrt{3}i-3\sqrt{3}i=-8$

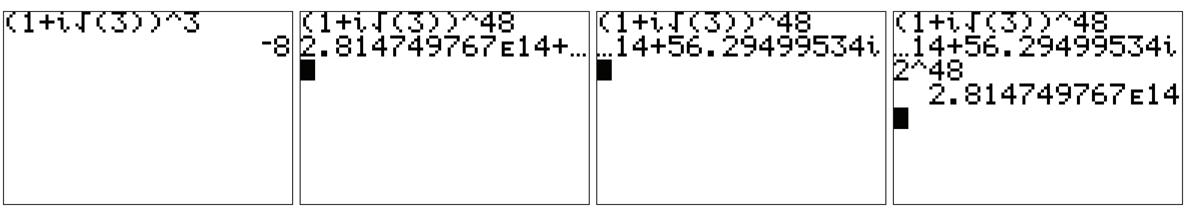
b) Firstly, we will write the number in the form: $\left(1+\sqrt{3}i\right)^{6n} = \left(\left(\left(1+\sqrt{3}i\right)^3\right)^2\right)^n$. Then, we will use the fact, from **a**, that $\left(1+\sqrt{3}i\right)^3=-8$. Now, we have:

$\left(1+\sqrt{3}i\right)^{6n} = \left(\left(\left(1+\sqrt{3}i\right)^3\right)^2\right)^n = \left((-8)^2\right)^n = (64)^n$

c) We will use the result from **b**: $\left(1+\sqrt{3}i\right)^{48} = \left(1+\sqrt{3}i\right)^{6\cdot 8} = 64^8$

Note (1): If we write $\left(1+\sqrt{3}i\right)^{6n} = (64)^n = \left(2^6\right)^n = 2^{6n}$, we have: $\left(1+\sqrt{3}i\right)^{48} = 2^{48}$. The result is the same, since $64^8 = 2^{48}$.

Note (2): It is not recommended to use a GDC for high powers of complex numbers. We have:



So, the third power is correct, but the 48th power has the same real part but its imaginary part is completely wrong!

34 **a)** $\left(-\sqrt{2}+i\sqrt{2}\right)^2 = \left(-\sqrt{2}\right)^2+2\cdot\left(-\sqrt{2}\right)\sqrt{2}i+\left(\sqrt{2}i\right)^2 = 2-4i+2(-1)=-4i$

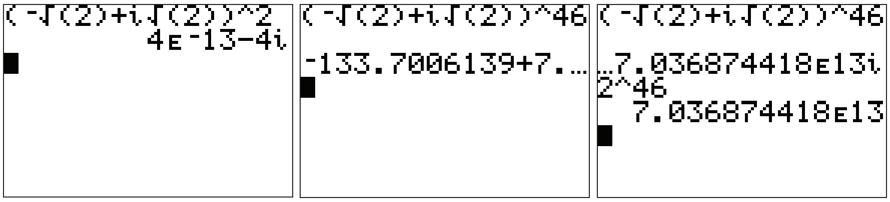
b) Firstly, we will write the number in the form: $\left(-\sqrt{2}+i\sqrt{2}\right)^{4k} = \left(\left(\left(-\sqrt{2}+i\sqrt{2}\right)^2\right)^2\right)^n$. Then, we will use the fact, from **a**, that $\left(-\sqrt{2}+i\sqrt{2}\right)^2=-4i$. Now, we have:

$\left(-\sqrt{2}+i\sqrt{2}\right)^{4k} = \left(\left(\left(-\sqrt{2}+i\sqrt{2}\right)^2\right)^2\right)^k = \left((-4i)^2\right)^k = (-16)^k$

c) We will use the results from **b** and **a**:

$$\left(-\sqrt{2}+i\sqrt{2}\right)^{46}=\left(-\sqrt{2}+i\sqrt{2}\right)^{44+2}=\left(-\sqrt{2}+i\sqrt{2}\right)^{4\cdot 11}\left(-\sqrt{2}+i\sqrt{2}\right)^{2}$$
$$=(-16)^{11}(-4i)=16^{11}\cdot 4i=4^{23}i=2^{46}i$$

Note: It is not recommended to use a GDC for high powers of complex numbers.
We have:



So, the second power is almost correct, but the 46th power has the same imaginary part but its real part is completely wrong!

35 Take $z=x+yi$, then $z+4i=x+(y+4)i$ and $z+i=x+(y+1)i$. Hence, we have to solve:

$$\sqrt{x^2+(y+4)^2}=2\sqrt{x^2+(y+1)^2}$$
$$x^2+(y+4)^2=4\left(x^2+(y+1)^2\right)\Rightarrow x^2+y^2+8y+16=4x^2+4y^2+8y+4$$
$$3x^2+3y^2=12\Rightarrow x^2+y^2=4$$

Therefore, $\sqrt{x^2+y^2}=2\Rightarrow |z|=2$.

36 $z=3+\dfrac{2i}{2-i\sqrt{2}}\cdot\dfrac{2+i\sqrt{2}}{2+i\sqrt{2}}=3+\dfrac{4i-2\sqrt{2}}{4+2}=\dfrac{9-\sqrt{2}+2i}{3}=\dfrac{9-\sqrt{2}}{3}+\dfrac{2}{3}i$

Note: It is easier to simplify the fraction (divide) first, and then add 3, as we did. The alternative is to add first and then simplify the fraction, as shown below.

$$z=3+\frac{2i}{2-i\sqrt{2}}=\frac{6-i3\sqrt{2}+2i}{2-i\sqrt{2}}=\frac{6+\left(2-3\sqrt{2}\right)i}{2-i\sqrt{2}}\cdot\frac{2+i\sqrt{2}}{2+i\sqrt{2}}$$
$$=\frac{\left(6+\left(2-3\sqrt{2}\right)i\right)\left(2-i\sqrt{2}\right)}{4+2}=\frac{9-\sqrt{2}}{3}+\frac{2}{3}i$$

37 From $(x+iy)(4-7i)=3+2i$, and dividing by $4-7i$, we have:

$x+iy=\dfrac{3+2i}{4-7i}\cdot\dfrac{4+7i}{4+7i}=\dfrac{12+21i+8i-14}{16+49}=\dfrac{-2+29i}{65}$. Hence, $x=-\dfrac{2}{65}$, $y=\dfrac{29}{65}$.

Note: It is easier to divide first and then solve the system, as we did. The alternative is to multiply first, as shown below.

$$(x+iy)(4-7i)=3+2i\Rightarrow 4x+7y+i(-7x+4y)=3+2i$$
$$\Rightarrow\begin{cases}4x+7y=3\\-7x+4y=2\end{cases}\Rightarrow\begin{cases}x=-\dfrac{2}{65}\\[2ex] y=\dfrac{29}{65}\end{cases}$$

38 From $i(z+1)=3z-2\Rightarrow -3z+iz=-2-i\Rightarrow(-3+i)z=-2-i$. Hence, dividing by $-3+i$, we have:

$z=\dfrac{-2-i}{-3+i}=\dfrac{1}{2}+\dfrac{1}{2}i$.

39 From $\dfrac{2-i}{1+2i}\sqrt{z}=2-3i$, we have: $\sqrt{z}=\dfrac{(2-3i)(1+2i)}{2-i}=\dfrac{8+i}{2-i}=3+2i$. Hence,

$z=(3+2i)^2=5+12i$.

40 From $(x+iy)^2 = 3-4i \Rightarrow x^2 - y^2 + 2xy = 3-4i$. Hence:

$$\Rightarrow \begin{cases} x^2 - y^2 = 3 \\ 2xy = -4 \end{cases} \Rightarrow y = -\frac{2}{x},\ x^2 - \left(-\frac{2}{x}\right)^2 = 3,\ x^2 - \frac{4}{x^2} = 3 \Rightarrow x^4 - 3x^2 - 4 = 0 \Rightarrow x_1^2 = 4,\ \cancel{x_2^2 = -1}$$

Therefore, we have two solutions: $(x, y) = (2, -1)$ and $(x, y) = (-2, 1)$.

41 a) From $(x^2 - y^2) + 2xyi = -8 + 6i \Rightarrow \begin{cases} x^2 - y^2 = -8 \\ 2xy = 6 \end{cases}$. Hence, using $y = \frac{3}{x}$, we have:

$x^2 - \left(\frac{3}{x}\right)^2 = -8 \Rightarrow x^4 + 8x^2 - 9 = 0 \Rightarrow x_1^2 = 1,\ \cancel{x_2^2 = -9}$. So, either $x = 1$ and $y = 3$, or $x = -1$ and $y = -3$.

b) Solving a quadratic equation with complex coefficients:

$$z_{1,2} = \frac{-1+i \pm \sqrt{(1-i)^2 - 4(2-2i)}}{2} = \frac{-1+i \pm \sqrt{1-2i-1-8+8i}}{2} = \frac{-1+i \pm \sqrt{-8+6i}}{2}$$

Since $\sqrt{-8+6i} = \pm(1+3i)$, then z equals either $z_1 = \frac{-1+i+(1+3i)}{2} = 2i$, or

$z_2 = \frac{-1+i-(1+3i)}{2} = -1-i$.

42 Using the notation $z = x + iy$, we have:

$$z^3 = 27i \Rightarrow x^3 + 3x^2yi + 3xy^2(-1) + y^3(-i) = 27i \Rightarrow (x^3 - 3xy^2) + i(3x^2y - y^3) = 27i$$

Hence: $\begin{cases} x^3 - 3xy^2 = 0 \\ 3x^2y - y^3 = 27 \end{cases} \Rightarrow \begin{cases} x(x^2 - 3y^2) = 0 \\ 3x^2y - y^3 = 27 \end{cases}$

If we let $x = 0$, from the second equation, we have: $-y^3 = 27 \Rightarrow y = -3$ and $\boxed{z_1 = -3i}$

If we let $x^2 = 3y^2$, from the second equation, we have:

$$3(3y^2)y - y^3 = 27 \Rightarrow 8y^3 = 27 \Rightarrow y^3 = \frac{27}{8} \Rightarrow y = \frac{3}{2};\ \text{hence: } x^2 = 3\left(\frac{3}{2}\right)^2 \Rightarrow x = \pm\frac{3}{2}\sqrt{3}.$$

So, the solutions are $\boxed{z_2 = \frac{3\sqrt{3}}{2} + \frac{3}{2}i}$ and $\boxed{z_3 = -\frac{3\sqrt{3}}{2} + \frac{3}{2}i}$.

43 Since the polynomial has real coefficients, then $\frac{1}{2} - 2i$ is also a zero. Hence,

$f(x) = 4\left(x - \left(\frac{1}{2} - 2i\right)\right)\left(x - \left(\frac{1}{2} + 2i\right)\right)(x - c)$. Since

$$\left(x - \left(\frac{1}{2} - 2i\right)\right)\left(x - \left(\frac{1}{2} + 2i\right)\right) = \left(\left(x - \frac{1}{2}\right) + 2i\right)\left(\left(x - \frac{1}{2}\right) - 2i\right)$$

$$= \left(x - \frac{1}{2}\right)^2 - (2i)^2 = x^2 - x + \frac{1}{4} + 4 = x^2 - x + \frac{17}{4},\ \text{the polynomial is:}$$

$f(x) = 4\left(x^2 - x + \frac{17}{4}\right)(x - c)$. To determine c, we can check for the free coefficient:

$$-51 = 4 \cdot \frac{17}{4}(-c) \Rightarrow c = 3$$

Hence, the other zeros are $\frac{1}{2} - 2i$ and 3.

44 Since a polynomial has real coefficients, $3-i\sqrt{2}$ is also a zero. Hence, using the factor theorem, we have:
$a\left(x-\frac{1}{2}\right)(x+1)\left(x-3-i\sqrt{2}\right)\left(x-3+i\sqrt{2}\right)=a\left(x^2+\frac{1}{2}x-\frac{1}{2}\right)\left(x^2-6x+9+2\right)$

After multiplication we have:

$$\left(x^2+\frac{1}{2}x-\frac{1}{2}\right)\left(x^2-6x+\underbrace{9+2}_{11}\right)=x^4+\frac{1}{2}x^3-\frac{1}{2}x^2$$
$$-6x^3-3x^2+3x$$
$$11x^2+\frac{11}{2}x-\frac{11}{2}$$
$$\overline{x^4-\frac{11}{2}x^3+\frac{15}{2}x^2+\frac{17}{2}x-\frac{11}{2}}$$

Since we need integer coefficients, we can use any even number for a; so, if we let $a=2$, the polynomial will be: $f(x)=2x^4-11x^3+15x^2+17x-11$.

45 Since a polynomial has real coefficients, $1-i\sqrt{3}$ is also a zero. Hence, using the factor theorem, we have: $a(x+2)^2\left(x-1-i\sqrt{3}\right)\left(x-1+i\sqrt{3}\right)=a\left(x^2+4x+4\right)\left(x^2-2x+1+3\right)$. After multiplication we have: $\left(x^2+4x+4\right)\left(x^2-2x+4\right)=x^4+2x^3+8x+16$.

So, we let $a=1$ and the polynomial is: $f(x)=x^4+2x^3+8x+16$.

46 Since the polynomial has real coefficients, then $5-2i$ is also a zero. Hence, $f(x)=(x-(5+2i))(x-(5-2i))(x-c)=\left(x^2-10x+25+4\right)(x-c)$. To determine c, we can check for the free coefficient: $87=29\cdot(-c)\Rightarrow c=-3$. Hence, the other zeros are $5-2i$ and -3.

47 Since the polynomial has real coefficients, then $1+i\sqrt{3}$ is also a zero. Hence, $f(x)=3\left(x-\left(1+i\sqrt{3}\right)\right)\left(x-\left(1-i\sqrt{3}\right)\right)(x-c)=3\left(x^2-2x+1+3\right)(x-c)$. To determine c, we can check for the free coefficient: $8=3\cdot4\cdot(-c)\Rightarrow c=-\frac{2}{3}$. Hence, the other zeros are $1+i\sqrt{3}$ and $-\frac{2}{3}$.

48 For $z=x+iy$, we have: $|a+bi|=\frac{|z|}{|z^*|}=\frac{\sqrt{x^2+y^2}}{\sqrt{x^2+\left(-y^2\right)}}=1$.

Note: The property of the number and its conjugate have the same modulus.

49 **a)** Using the binomial theorem, we have:

$(k+i)^4=k^4+4\cdot k^3i+6k^2(-1)+4k(-i)+1=k^4-6k^2+1+i\left(4k^3-4k\right)$. Therefore, the number is real if $4k^3-4k=0\Rightarrow 4k\left(k^2-1\right)=0\Rightarrow k=0$, or $k=\pm1$.

b) Using the calculation from **a**, we have: $k^4-6k^2+1=0\Rightarrow k_{1,2}^2=\frac{6\pm\sqrt{36-4}}{2}=3\pm2\sqrt{2}$. Since both numbers are positive, $k=\pm\sqrt{3\pm2\sqrt{2}}$.

50 $\begin{cases} iz_1+2z_2=3-i \\ 2z_1+(2+i)z_2=7+2i \end{cases}$ Multiply the first equation by –2, and the second by i.

$\begin{cases} -i2z_1-4z_2=-6+2i \\ 2iz_1+(2i-1)z_2=7i-2 \end{cases}$ Add the equations.

$(-5+2i)z_2=-8+9i\Rightarrow z_2=\frac{-8+9i}{-5+2i}=2-i$

Substituting z_2 in the first equation, we have:

$iz_1 + 2(2-i) = 3 - i \Rightarrow iz_1 = -1 + i \Rightarrow z_1 = 1 + i$

Hence, the solutions are $z_1 = 1 + i$ and $z_2 = 2 - i$.

51 $\begin{cases} iz_1 - (1+i)z_2 = 3 \\ (2+i)z_1 + iz_2 = 4 \end{cases}$ Multiply the first equation by i, and the second by $1 + i$.

$\begin{cases} -z_1 - i(1+i)z_2 = 3i \\ (2+i)(1+i)z_1 + i(1+i)z_2 = 4(1+i) \end{cases}$ Add the equations.

$-z_1 + (1+3i)z_1 = 3i + 4 + 4i \Rightarrow 3iz_1 = 4 + 7i \Rightarrow z_1 = \dfrac{7-4i}{3}$

Substituting z_1 in the second equation, we have:

$(2+i)\dfrac{7-4i}{3} + iz_2 = 4 \Rightarrow iz_2 = 4 - 6 + \dfrac{1}{3}i \Rightarrow z_2 = \dfrac{1}{3} + 2i$

Hence, the solutions are $z_1 = \dfrac{7-4i}{3}$ and $z_2 = \dfrac{1}{3} + 2i$.

Exercise 10.2

1 $r = \sqrt{2^2 + 2^2} = 2\sqrt{2}$, $\tan\theta = \dfrac{2}{2} = 1$, and, since the number is in the first quadrant, $\theta = \dfrac{\pi}{4}$. Hence, $2\sqrt{2}\operatorname{cis}\dfrac{\pi}{4}$.

2 $r = \sqrt{\sqrt{3}^2 + 1^2} = 2$, $\tan\theta = \dfrac{1}{\sqrt{3}}$, and, since the number is in the first quadrant, $\theta = \dfrac{\pi}{6}$. Hence, $2\operatorname{cis}\dfrac{\pi}{6}$.

3 $r = \sqrt{2^2 + (-2)^2} = 2\sqrt{2}$, $\tan\theta = \dfrac{-2}{2} = -1$, and, since the number is in the fourth quadrant, $\theta = \dfrac{7\pi}{4}$. Hence, $2\sqrt{2}\operatorname{cis}\dfrac{7\pi}{4}$.

4 $r = \sqrt{\sqrt{6}^2 + \left(-\sqrt{2}\right)^2} = 2\sqrt{2}$, $\tan\theta = \dfrac{-\sqrt{2}}{\sqrt{6}} = -\dfrac{1}{\sqrt{3}}$, and, since the number is in the fourth quadrant, $\theta = \dfrac{11\pi}{6}$. Hence, $2\sqrt{2}\operatorname{cis}\dfrac{11\pi}{6}$.

5 $r = \sqrt{2^2 + \left(-2\sqrt{3}\right)^2} = 4$, $\tan\theta = \dfrac{-2\sqrt{3}}{2} = -\sqrt{3}$, and, since the number is in the fourth quadrant, $\theta = \dfrac{5\pi}{6}$. Hence, $4\operatorname{cis}\dfrac{5\pi}{6}$.

6 $r = \sqrt{(-3)^2 + 3^2} = 3\sqrt{2}$, $\tan\theta = \dfrac{3}{-3} = -1$, and, since the number is in the second quadrant, $\theta = \dfrac{3\pi}{4}$. Hence, $3\sqrt{2}\operatorname{cis}\dfrac{3\pi}{4}$.

7 $r = \sqrt{4^2 + 0^2} = 4$, $\tan\theta = \dfrac{4}{0}$ is not defined, and, since the number is on the positive y-axis, $\theta = \dfrac{\pi}{2}$. Hence, $4\operatorname{cis}\dfrac{\pi}{2}$.

Note: From the geometric interpretation, we can see that the distance from the origin is 4 and the angle is $\dfrac{\pi}{2}$.

8 $r=\sqrt{(-3\sqrt{3})^2+(-3)^2}=6$, $\tan\theta=\frac{-3}{-3\sqrt{3}}=\frac{1}{\sqrt{3}}$, and, since the number is in the third quadrant, $\theta=\frac{7\pi}{6}$. Hence, $6\text{ cis }\frac{7\pi}{6}$.

9 $r=\sqrt{1^2+1^2}=\sqrt{2}$, $\tan\theta=\frac{1}{1}=1$, and, since the number is in the first quadrant, $\theta=\frac{\pi}{4}$. Hence, $\sqrt{2}\text{ cis }\frac{\pi}{4}$.

10 $r=\sqrt{(-15)^2+0^2}=15$, $\tan\theta=\frac{0}{15}=0$, and, since the number is on the negative x-axis, $\theta=\pi$. Hence, $15\text{ cis }\pi$.

Note: From the geometric interpretation, we can see that the distance from the origin is 15 and the angle is π.

11 $(4+3i)^{-1}=\frac{1}{4+3i}=\frac{4-3i}{25}$

$$r=\sqrt{\left(\frac{4}{25}\right)^2+\left(-\frac{3}{25}\right)^2}=\frac{1}{5},\ \tan\theta=\frac{-\frac{3}{25}}{\frac{4}{25}}=-\frac{3}{4}\text{, and, since the number is in the fourth quadrant,}$$

$\theta=\tan^{-1}\left(-\frac{3}{4}\right)+2\pi\approx 5.64$. Hence, $\frac{1}{5}\text{ cis }5.64$.

12 $i(3+3i)=-3+3i$

$r=\sqrt{(-3)^2+3^2}=3\sqrt{2}$, $\tan\theta=\frac{3}{-3}=-1$, and, since the number is in the second quadrant, $\theta=\frac{3\pi}{4}$.

Hence, $3\sqrt{2}\text{ cis }\frac{3\pi}{4}$.

13 $r=\sqrt{\pi^2+0^2}=\pi$, $\tan\theta=\frac{0}{\pi}=0$, and, since the number is on the positive x-axis, $\theta=0$. Hence, $\pi\text{ cis }0$.

Note: From the geometric interpretation, we can see that the distance from the origin is π and the angle is 0.

14 $r=\sqrt{0^2+e^2}=e$, $\tan\theta$ is not defined, and, since the number is on the positive y-axis, $\theta=\frac{\pi}{2}$. Hence, $e\text{ cis }\frac{\pi}{2}$.

Note: From the geometric interpretation, we can see that the distance from the origin is e and the angle is $\frac{\pi}{2}$.

15 $z_1=1\text{ cis }\frac{\pi}{2}$, $z_2=1\text{ cis }\frac{\pi}{3}$, and hence:

$$z_1z_2=\text{cis}\left(\frac{\pi}{2}+\frac{\pi}{3}\right)=\text{cis }\frac{5\pi}{6}=\cos\frac{5\pi}{6}+i\sin\frac{5\pi}{6}=-\frac{\sqrt{3}}{2}+\frac{1}{2}i$$

$$\frac{z_1}{z_2}=\text{cis}\left(\frac{\pi}{2}-\frac{\pi}{3}\right)=\text{cis }\frac{\pi}{6}=\cos\frac{\pi}{6}+i\sin\frac{\pi}{6}=\frac{\sqrt{3}}{2}+\frac{1}{2}i$$

16 $z_1=1\text{ cis }\frac{5\pi}{6}$, $z_2=1\text{ cis }\frac{7\pi}{6}$, and hence:

$$z_1z_2=\text{cis}\left(\frac{5\pi}{6}+\frac{7\pi}{6}\right)=\text{cis}(2\pi)=\cos 2\pi+i\sin 2\pi=1+0i=1$$

$$\frac{z_1}{z_2} = \text{cis}\left(\frac{5\pi}{6} - \frac{7\pi}{6}\right) = \text{cis}\left(-\frac{\pi}{3}\right) = \cos\frac{\pi}{3} - i\sin\frac{\pi}{3} = \frac{1}{2} - \frac{\sqrt{3}}{2}i$$

17 $z_1 = 1\,\text{cis}\,\frac{\pi}{6},\ z_2 = 1\,\text{cis}\,\frac{2\pi}{3}$, and hence:

$$z_1 z_2 = \text{cis}\left(\frac{\pi}{6} + \frac{2\pi}{3}\right) = \text{cis}\left(\frac{5\pi}{6}\right) = \cos\left(\frac{5\pi}{6}\right) + i\sin\left(\frac{5\pi}{6}\right) = -\frac{\sqrt{3}}{2} + \frac{1}{2}i$$

$$\frac{z_1}{z_2} = \text{cis}\left(\frac{\pi}{6} - \frac{2\pi}{3}\right) = \text{cis}\left(-\frac{\pi}{2}\right) = \cos\frac{\pi}{2} - i\sin\frac{\pi}{2} = 0 - 1i = -i$$

18 $z_1 = 1\,\text{cis}\,\frac{13\pi}{12},\ z_2 = 1\,\text{cis}\,\frac{5\pi}{12}$, and hence:

$$z_1 z_2 = \text{cis}\left(\frac{13\pi}{12} + \frac{5\pi}{12}\right) = \text{cis}\left(\frac{3\pi}{2}\right) = \cos\left(\frac{3\pi}{2}\right) + i\sin\left(\frac{3\pi}{2}\right) = 0 - 1\cdot i = -i$$

$$\frac{z_1}{z_2} = \text{cis}\left(\frac{13\pi}{12} - \frac{5\pi}{12}\right) = \text{cis}\left(\frac{2\pi}{3}\right) = \cos\frac{2\pi}{3} + i\sin\frac{2\pi}{3} = -\frac{1}{2} + \frac{\sqrt{3}}{2}i$$

19 $z_1 = 3\,\text{cis}\,\frac{3\pi}{4},\ z_2 = \frac{2}{3}\,\text{cis}\,\frac{4\pi}{3}$, and hence:

$$z_1 z_2 = 3\cdot\frac{2}{3}\,\text{cis}\left(\frac{3\pi}{4} + \frac{4\pi}{3}\right) = 2\,\text{cis}\left(\frac{25\pi}{12}\right) = 2\,\text{cis}\left(\frac{\pi}{12}\right) = 2\cos\left(\frac{\pi}{12}\right) + 2i\sin\left(\frac{\pi}{12}\right)$$

$$z_1 z_2 = \frac{\sqrt{6} + \sqrt{2}}{2} + \frac{\sqrt{6} - \sqrt{2}}{2}i$$

$$\frac{z_1}{z_2} = \frac{3}{\frac{2}{3}}\,\text{cis}\left(\frac{3\pi}{4} - \frac{4\pi}{3}\right) = \frac{9}{2}\,\text{cis}\left(-\frac{7\pi}{12}\right) = \frac{9}{2}\,\text{cis}\left(\frac{17\pi}{12}\right)$$

$$\frac{z_1}{z_2} = \frac{9}{2}\cos\left(\frac{17\pi}{12}\right) + \frac{9}{2}i\sin\left(\frac{17\pi}{12}\right) = \frac{9}{8}\left(-\sqrt{6} + \sqrt{2}\right) - \frac{9}{8}\left(\sqrt{6} + \sqrt{2}\right)i$$

20 $z_1 = 3\sqrt{2}\,\text{cis}\,\frac{5\pi}{4},\ z_2 = 2\,\text{cis}\,\frac{5\pi}{3}$, and hence:

$$z_1 z_2 = 3\sqrt{2}\cdot 2\,\text{cis}\left(\frac{5\pi}{4} + \frac{5\pi}{3}\right) = 6\sqrt{2}\,\text{cis}\left(\frac{35\pi}{12}\right) = 6\sqrt{2}\,\text{cis}\left(\frac{11\pi}{12}\right) = 6\sqrt{2}\cos\left(\frac{11\pi}{12}\right) + 6\sqrt{2}i\sin\left(\frac{11\pi}{12}\right)$$

$$= -\frac{6\sqrt{3} + 6}{2} + \frac{6\sqrt{3} - 6}{2}i = -3\sqrt{3} - 3 + \left(3\sqrt{3} - 3\right)i$$

$$\frac{z_1}{z_2} = \frac{3\sqrt{2}}{2}\,\text{cis}\left(\frac{5\pi}{4} - \frac{5\pi}{3}\right) = \frac{3\sqrt{2}}{2}\,\text{cis}\left(-\frac{5\pi}{12}\right) = \frac{3\sqrt{2}}{2}\,\text{cis}\left(\frac{19\pi}{12}\right)$$

$$= \frac{3\sqrt{2}}{2}\cos\left(\frac{19\pi}{12}\right) + \frac{3\sqrt{2}}{2}i\sin\left(\frac{19\pi}{12}\right) = \frac{3\sqrt{2}}{8}\left(\sqrt{6} - \sqrt{2}\right) - \frac{3\sqrt{2}}{8}\left(\sqrt{6} + \sqrt{2}\right)i$$

$$= \frac{3\sqrt{3} - 3}{4} - \frac{3\sqrt{3} + 3}{4}i$$

21 $z_1 = 1\,\text{cis}\,135^\circ,\ z_2 = 1\,\text{cis}\,90^\circ$, and hence:

$$z_1 z_2 = \text{cis}\left(135^\circ + 90^\circ\right) = \text{cis}\left(225^\circ\right) = \cos\left(225^\circ\right) + i\sin\left(225^\circ\right) = -\frac{\sqrt{2}}{2} - \frac{\sqrt{2}}{2}i = -\frac{\sqrt{2}}{2}(1 + i)$$

$$\frac{z_1}{z_2} = \text{cis}\left(135^\circ - 90^\circ\right) = \text{cis}\left(45^\circ\right) = \cos 45^\circ + i\sin 45^\circ = \frac{\sqrt{2}}{2} + \frac{\sqrt{2}}{2}i = \frac{\sqrt{2}}{2}(1 + i)$$

22 $z_1 = 3 \text{ cis } 120^\circ$, $z_2 = 2 \text{ cis } 240^\circ$, and hence:

$$z_1 z_2 = 3 \cdot 2 \text{ cis}\left(120^\circ + 240^\circ\right) = 6 \text{ cis}\left(360^\circ\right) = 6\cos\left(0^\circ\right) + 6i\sin\left(0^\circ\right) = 6 - 0 \cdot i = 6$$

$$\frac{z_1}{z_2} = \frac{3}{2}\text{cis}\left(120^\circ - 240^\circ\right) = \frac{3}{2}\text{cis}\left(-120^\circ\right) = \frac{3}{2}\text{cis}\left(240^\circ\right) = \frac{3}{2}\cos 240^\circ + \frac{3}{2}i\sin 240^\circ = -\frac{3}{4} - \frac{3\sqrt{3}}{4}i$$

23 $z_1 = \frac{5}{8}\text{cis } 225^\circ$, $z_2 = \frac{\sqrt{3}}{2}\text{cis } 330^\circ$, and hence:

$$z_1 z_2 = \frac{5}{8} \cdot \frac{\sqrt{3}}{2}\text{cis}\left(225^\circ + 330^\circ\right) = \frac{5\sqrt{3}}{16}\text{cis}\left(555^\circ\right) = \frac{5\sqrt{3}}{16}\text{cis}\left(195^\circ\right) = \frac{5\sqrt{3}}{16}\cos\left(195^\circ\right) + \frac{5\sqrt{3}}{16}i\sin\left(195^\circ\right)$$

$$= \frac{-5\sqrt{6} - 15\sqrt{2}}{64} + i\,\frac{5\sqrt{6} - 15\sqrt{2}}{64}$$

$$\frac{z_1}{z_2} = \frac{\frac{5}{8}}{\frac{\sqrt{3}}{2}}\text{cis}\left(225^\circ - 330^\circ\right) = \frac{5\sqrt{3}}{12}\text{cis}\left(-105^\circ\right) = \frac{5\sqrt{3}}{12}\text{cis}\left(255^\circ\right)$$

$$= \frac{5\sqrt{6} - 15\sqrt{2}}{48} - i\,\frac{5\sqrt{6} + 15\sqrt{2}}{48}$$

24 $z_1 = 3\sqrt{2}\text{ cis } 315^\circ$, $z_2 = 2 \text{ cis } 300^\circ$, and hence:

$$z_1 z_2 = 3\sqrt{2} \cdot 2 \text{ cis}\left(315^\circ + 300^\circ\right) = 6\sqrt{2}\text{ cis}\left(615^\circ\right) = 6\sqrt{2}\text{ cis}\left(255^\circ\right) = 6\sqrt{2}\cos\left(255^\circ\right) + 6\sqrt{2}i\sin\left(255^\circ\right)$$

$$= -3\sqrt{3} + 3 + i\left(3\sqrt{3} + 3\right)$$

$$\frac{z_1}{z_2} = \frac{3\sqrt{2}}{2}\text{cis}(315° - 300°) = \frac{3\sqrt{2}}{2}\text{cis}(15°)$$

$$= \frac{3\sqrt{3} + 3}{4} + \frac{i\left(3\sqrt{3} - 3\right)}{4}$$

25 For z_1: $r = \sqrt{\sqrt{3}^2 + 1^2} = 2$, $\tan\theta = \frac{1}{\sqrt{3}}$, and, since the number is in the first quadrant, $\theta = \frac{\pi}{6}$. Hence, $z_1 = 2\text{ cis}\frac{\pi}{6}$.

For z_2: $r = \sqrt{2^2 + \left(2\sqrt{3}\right)^2} = 4$, $\tan\theta = \frac{-2\sqrt{3}}{2} = -\sqrt{3}$, and, since the number is in the fourth quadrant, $\theta = \frac{5\pi}{3}$. Hence, $z_2 = 4\text{ cis}\frac{5\pi}{3}\left(\text{or } 4\text{ cis}\frac{-\pi}{3}\right)$.

$$\frac{1}{z_1} = \frac{1}{2}\text{cis}\left(-\frac{\pi}{6}\right)$$

$$\frac{1}{z_2} = \frac{1}{4}\text{cis}\left(-\frac{5\pi}{3}\right) = \frac{1}{4}\text{cis}\left(\frac{\pi}{3}\right)$$

$$z_1 z_2 = 2 \cdot 4\text{ cis}\left(\frac{\pi}{6} + \frac{5\pi}{3}\right) = 8\text{ cis}\left(\frac{11\pi}{6}\right)\left(\text{or } 8\text{ cis}\frac{-\pi}{6}\right)$$

$$\frac{z_1}{z_2} = \frac{2}{4}\text{cis}\left(\frac{\pi}{6} - \frac{5\pi}{3}\right) = \frac{1}{2}\text{cis}\left(-\frac{3\pi}{2}\right) = \frac{1}{2}\text{cis}\frac{\pi}{2}$$

26 For z_1: $r = \sqrt{8} = 2\sqrt{2}$, $\tan\theta = \dfrac{1}{\sqrt{3}}$, $\theta = \dfrac{\pi}{6}$; hence, $z_1 = 2\sqrt{2}\text{ cis }\dfrac{\pi}{6}$.

For z_2: $r = \sqrt{48} = 4\sqrt{3}$, $\tan\theta = -\sqrt{3}$, $\theta = \dfrac{5\pi}{3}$; hence, $z_2 = 4\sqrt{3}\text{ cis }\dfrac{5\pi}{3}\left(\text{or } 4\sqrt{3}\text{ cis }\dfrac{-\pi}{3}\right)$.

$$\frac{1}{z_1} = \frac{1}{2\sqrt{2}}\text{cis}\left(-\frac{\pi}{6}\right) = \frac{\sqrt{2}}{4}\text{cis}\left(-\frac{\pi}{6}\right)$$

$$\frac{1}{z_2} = \frac{1}{4\sqrt{3}}\text{cis}\left(-\frac{5\pi}{3}\right) = \frac{\sqrt{3}}{12}\text{cis}\left(\frac{\pi}{3}\right)$$

$$z_1z_2 = 8\sqrt{6}\text{ cis}\left(\frac{\pi}{6}+\frac{5\pi}{3}\right) = 8\sqrt{6}\text{ cis}\left(\frac{11\pi}{6}\right)\left(\text{or } 8\sqrt{6}\text{ cis }\frac{-\pi}{6}\right)$$

$$\frac{z_1}{z_2} = \frac{\sqrt{2}}{2\sqrt{3}}\text{cis}\left(\frac{\pi}{6}-\frac{5\pi}{3}\right) = \frac{\sqrt{6}}{6}\text{cis}\left(-\frac{3\pi}{2}\right) = \frac{\sqrt{6}}{6}\text{cis }\frac{\pi}{2}$$

27 For z_1: $r = 8$, $\tan\theta = \dfrac{1}{\sqrt{3}}$, $\theta = \dfrac{\pi}{6}$; hence, $z_1 = 8\text{ cis }\dfrac{\pi}{6}$.

For z_2: $r = 3\sqrt{2}$, $\tan\theta = 1$, $\theta = \dfrac{5\pi}{4}$; hence, $z_2 = 3\sqrt{2}\text{ cis }\dfrac{5\pi}{4}\left(\text{or } 3\sqrt{2}\text{ cis }\dfrac{-3\pi}{4}\right)$.

$$\frac{1}{z_1} = \frac{1}{8}\text{cis}\left(-\frac{\pi}{6}\right)$$

$$\frac{1}{z_2} = \frac{1}{3\sqrt{2}}\text{cis}\left(-\frac{5\pi}{4}\right) = \frac{\sqrt{2}}{6}\text{cis}\left(\frac{3\pi}{4}\right)$$

$$z_1z_2 = 24\sqrt{2}\text{ cis}\left(\frac{\pi}{6}+\frac{5\pi}{4}\right) = 24\sqrt{2}\text{ cis}\left(\frac{17\pi}{12}\right)\left(\text{or } 24\sqrt{2}\text{ cis }\frac{-7\pi}{12}\right)$$

$$\frac{z_1}{z_2} = \frac{8}{3\sqrt{2}}\text{cis}\left(\frac{\pi}{6}-\frac{5\pi}{4}\right) = \frac{4\sqrt{2}}{3}\text{cis}\left(-\frac{13\pi}{12}\right) = \frac{4\sqrt{2}}{3}\text{cis}\left(\frac{11\pi}{12}\right)$$

28 For z_1: $r = \sqrt{3}$, $\tan\theta$ is not defined, $\theta = \dfrac{\pi}{2}$; hence, $z_1 = \sqrt{3}\text{ cis }\dfrac{\pi}{2}$.

For z_2: $r = 2\sqrt{2}$, $\tan\theta = \sqrt{3}$, $\theta = \dfrac{4\pi}{3}$; hence, $z_2 = 2\sqrt{2}\text{ cis }\dfrac{4\pi}{3}\left(\text{or } 2\sqrt{2}\text{ cis }\dfrac{-2\pi}{3}\right)$.

$$\frac{1}{z_1} = \frac{1}{\sqrt{3}}\text{cis}\left(-\frac{\pi}{2}\right) = \frac{\sqrt{3}}{3}\text{cis}\left(-\frac{\pi}{2}\right)$$

$$\frac{1}{z_2} = \frac{1}{2\sqrt{2}}\text{cis}\left(-\frac{4\pi}{3}\right) = \frac{\sqrt{2}}{4}\text{cis}\left(\frac{2\pi}{3}\right)$$

$$z_1z_2 = 2\sqrt{6}\text{ cis}\left(\frac{\pi}{2}+\frac{4\pi}{3}\right) = 2\sqrt{6}\text{ cis}\left(\frac{11\pi}{6}\right)\left(\text{or } 2\sqrt{6}\text{ cis }\frac{-\pi}{6}\right)$$

$$\frac{z_1}{z_2} = \frac{\sqrt{3}}{2\sqrt{2}}\text{cis}\left(\frac{\pi}{2}-\frac{4\pi}{3}\right) = \frac{\sqrt{6}}{4}\text{cis}\left(-\frac{5\pi}{6}\right) = \frac{\sqrt{6}}{4}\text{cis}\left(\frac{7\pi}{6}\right)$$

29 For z_1: $r = \sqrt{10}$, $\tan\theta = 1$, $\theta = \dfrac{\pi}{4}$; hence, $z_1 = \sqrt{10}\text{ cis }\dfrac{\pi}{4}$.

For z_2: $r = 2\sqrt{2}$, $\tan\theta$ is not defined, $\theta = \dfrac{\pi}{2}$; hence, $z_2 = 2\sqrt{2}\text{ cis }\dfrac{\pi}{2}$.

$$\frac{1}{z_1} = \frac{1}{\sqrt{10}}\text{cis}\left(-\frac{\pi}{4}\right) = \frac{\sqrt{10}}{10}\text{cis}\left(-\frac{\pi}{4}\right)$$

$$\frac{1}{z_2} = \frac{1}{2\sqrt{2}} \operatorname{cis}\left(-\frac{\pi}{2}\right) = \frac{\sqrt{2}}{4} \operatorname{cis}\left(-\frac{\pi}{2}\right)$$

$$z_1 z_2 = 4\sqrt{5} \operatorname{cis}\left(\frac{\pi}{4} + \frac{\pi}{2}\right) = 4\sqrt{5} \operatorname{cis}\left(\frac{3\pi}{4}\right)$$

$$\frac{z_1}{z_2} = \frac{\sqrt{10}}{2\sqrt{2}} \operatorname{cis}\left(\frac{\pi}{4} - \frac{\pi}{2}\right) = \frac{\sqrt{5}}{2} \operatorname{cis}\left(-\frac{\pi}{4}\right)$$

30 For z_1 : $r = 2$, $\tan\theta = \sqrt{3}$, $\theta = \frac{\pi}{3}$; hence, $z_1 = 2 \operatorname{cis}\frac{\pi}{3}$.

For z_2: $r = 2\sqrt{3}$, $\tan\theta = 0$, $\theta = 0$; hence, $z_2 = 2\sqrt{3} \operatorname{cis} 0$.

$$\frac{1}{z_1} = \frac{1}{2} \operatorname{cis}\left(-\frac{\pi}{3}\right)$$

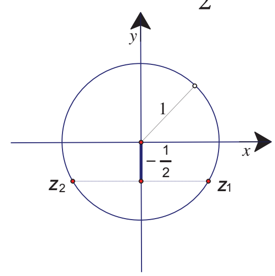
$$\frac{1}{z_2} = \frac{1}{2\sqrt{3}} \operatorname{cis}(-0) = \frac{\sqrt{3}}{6} \operatorname{cis}(0)$$

$$z_1 z_2 = 4\sqrt{3} \operatorname{cis}\left(\frac{\pi}{3} + 0\right) = 4\sqrt{3} \operatorname{cis}\left(\frac{\pi}{3}\right)$$

$$\frac{z_1}{z_2} = \frac{2}{2\sqrt{3}} \operatorname{cis}\left(\frac{\pi}{3} - 0\right) = \frac{\sqrt{3}}{3} \operatorname{cis}\left(\frac{\pi}{3}\right)$$

31 a) Let $z = x + yi$, hence: $\sqrt{x^2 + (y-1)^2} = \sqrt{x^2 + (y+2)^2} \Rightarrow y^2 - 2y + 1 = y^2 + 4y + 4 \Rightarrow y = -\frac{1}{2}$

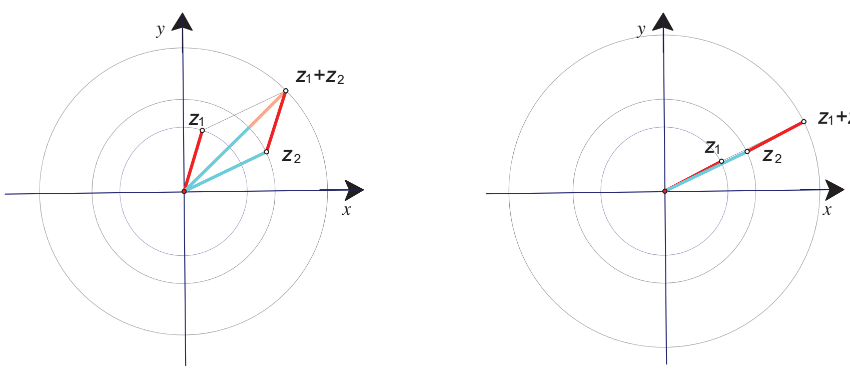
b) i) The points are on the unit circle and their y-coordinates are $-\frac{1}{2}$.



ii) For z_1 : $\sin\theta = -\frac{1}{2}$, and, since the number is in the fourth quadrant, $\theta = \frac{11\pi}{6}$ (or $-\frac{\pi}{6}$). Hence, $\arg(z_1) = \frac{11\pi}{6}$.

For z_2: $\sin\theta = -\frac{1}{2}$, and, since the number is in the third quadrant, $\theta = \frac{7\pi}{6}$ (or $-\frac{5\pi}{6}$). Hence, $\arg(z_2) = \frac{7\pi}{6}$.

32



On the above diagrams, $|z_1|$ is represented by the red line segment(s), and $|z_2|$ by the blue line segment(s). It is obvious that the line segment which represents $|z_1 + z_2|$ (blue + pink) is shorter than the 'blue + red' segment. They are the same if z_1 and z_2 are on the same line (on the same side of the origin), as shown on the second diagram.

33 $z=\sqrt{3}\left(-\frac{1}{2}+i\frac{\sqrt{3}}{2}\right)=\frac{-\sqrt{3}+3i}{2}$ and $z^2=3\left(\cos\frac{4\pi}{3}+i\sin\frac{4\pi}{3}\right)=\frac{-3-3\sqrt{3}i}{2}$

a) $$\frac{3}{\sqrt{3}+z}=\frac{3}{\sqrt{3}+\frac{-\sqrt{3}+3i}{2}}=\frac{3}{\frac{\sqrt{3}+3i}{2}}=\frac{6\left(\sqrt{3}-3i\right)}{3+9}=\frac{\sqrt{3}}{2}-\frac{3i}{2}$$

b) $$\frac{2z}{3+z^2}=\frac{2\,\frac{-\sqrt{3}+3i}{2}}{3+\frac{-3-3\sqrt{3}i}{2}}=\frac{-\sqrt{3}+3i}{\frac{3-3\sqrt{3}i}{2}}=\frac{-\sqrt{3}\,\cancel{\left(1-\sqrt{3}i\right)}}{\frac{3\cancel{\left(1-\sqrt{3}i\right)}}{2}}=\frac{-2\sqrt{3}}{3}$$

c) $$\frac{3-z^2}{3+z^2}=\frac{3-\frac{-3-3\sqrt{3}i}{2}}{3+\frac{-3-3\sqrt{3}i}{2}}=\frac{\frac{9+3\sqrt{3}i}{2}}{\frac{3-3\sqrt{3}i}{2}}=\frac{3\sqrt{3}\left(\sqrt{3}+i\right)}{3\left(1-\sqrt{3}i\right)}=\frac{\sqrt{3}\cdot 4i}{1+3}=\sqrt{3}i$$

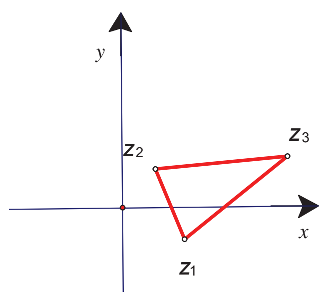
34 For $z_1: |z_1|=\sqrt{12+4}=4$, $\tan\theta=-\frac{1}{\sqrt{3}}$; hence, $\arg z_1=-\frac{\pi}{6}$.

For $z_2: |z_2|=\sqrt{4+4}=2\sqrt{2}$, $\tan\theta=1$; hence, $\arg z_2=\frac{\pi}{4}$.

Since $z_3=z_1z_2$, $|z_3|=8\sqrt{2}$, and $\arg z_3=-\frac{\pi}{6}+\frac{\pi}{4}=\frac{\pi}{12}$.

35 Method I:

We can find the side lengths and the angles of the triangle:



$$|z_1z_2|=\sqrt{\left(2-2\sqrt{3}\right)^2+4^2}=\sqrt{32-8\sqrt{3}}$$

$$|z_1z_3|=\sqrt{\left(2\sqrt{3}+4\right)^2+\left(4\sqrt{3}-2\right)^2}=\sqrt{80}$$

$$|z_2z_3|=\sqrt{\left(4\sqrt{3}+2\right)^2+\left(4\sqrt{3}-6\right)^2}=\sqrt{136-32\sqrt{3}}$$

Angle θ between sides $|z_1z_2|$ and $|z_1z_3|$ is:

$$\cos\theta=\frac{|z_1z_2|^2+|z_1z_3|^2-|z_2z_3|^2}{2|z_1z_2||z_1z_3|}\Rightarrow\theta=76.6689...^\circ$$

Hence, the area is: $A=\frac{1}{2}|z_1z_2||z_1z_3|\sin\theta\approx 18.5$.

```
√((2-2√(3))²+4²)
→A
          4.259529732
√((2√(3)+4)²+(4√
(3)-2)²)→B
           8.94427191
```

```
          4.259529732
√((2√(3)+4)²+(4√
(3)-2)²)→B
           8.94427191
√((4√(3)+2)²+(4√
(3)-6)²)→C
          8.976322975
█
```

```
(3)-6)²)→C
          8.976322975
(A²+B²-C²)/(2AB)
          .2305769141
cos⁻¹(Ans
          76.66896054
```

```
          .2305769141
cos⁻¹(Ans
          76.66896054
1/2*A*B*sin(Ans)
          18.53589838
```

Method II:

We can find the side lengths of the triangle and use Hero's formula for the area:

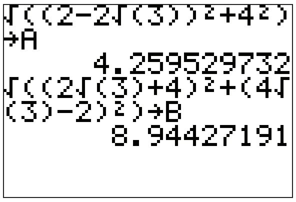
$$|z_1z_2|=\sqrt{\left(2-2\sqrt{3}\right)^2+4^2}=\sqrt{32-8\sqrt{3}}=a$$

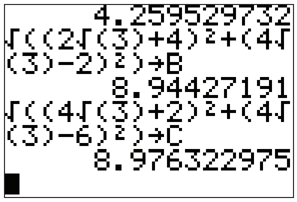
$$|z_1z_3| = \sqrt{\left(2\sqrt{3}+4\right)^2 + \left(4\sqrt{3}-2\right)^2} = \sqrt{80} = b$$

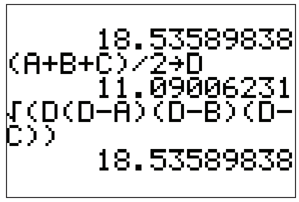
$$|z_2z_3| = \sqrt{\left(4\sqrt{3}+2\right)^2 + \left(4\sqrt{3}-6\right)^2} = \sqrt{136-32\sqrt{3}} = c$$

$A = \sqrt{s\,(s-a)\,(s-b)\,(s-c)}$, where $s = \dfrac{a+b+c}{2}$

Hence, $A \approx 18.5$.







Method III:

There is a formula for the area of a triangle using the coordinates of the vertices. If the vertices are $A\left(x_1, y_1\right), B\left(x_2, y_2\right), C\left(x_3, y_3\right)$, then the area is:

$$A = \frac{1}{2}\left|x_1\left(y_2-y_3\right) + x_2\left(y_3-y_1\right) + x_3\left(y_1-y_2\right)\right|$$

Applying the formula, we have:

$$A = \frac{1}{2}\left|2\sqrt{3}\left(2-4\sqrt{3}+4\right) + 2\left(4\sqrt{3}-4+2\right) + \left(4\sqrt{3}+4\right)\left(-2-2\right)\right| = \frac{1}{2}\left|4\sqrt{3}-44\right| = 22-2\sqrt{3}$$

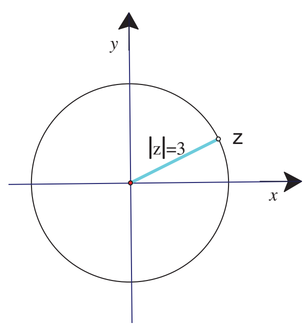
Method IV:

After completing Chapter 14 of the textbook, we will know the formula for the area of a triangle using a vector product.

$$\overrightarrow{z_1z_2} \times \overrightarrow{z_1z_3} = \begin{pmatrix} 2-2\sqrt{3} \\ 4 \\ 0 \end{pmatrix} \times \begin{pmatrix} 2\sqrt{3}+4 \\ 4\sqrt{3}-2 \\ 0 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 4\sqrt{3}-44 \end{pmatrix}$$

$$A = \frac{1}{2}\left|\overrightarrow{z_1z_2} \times \overrightarrow{z_1z_3}\right| = 22-2\sqrt{3}$$

36 **a)** The set of points is the circle with centre (0, 0) and radius 3.



b) Let $z = x + yi$; hence: $x + yi = x - yi \Rightarrow y = 0$. The set of points is the y-axis.

c) Let $z = x + yi$; hence: $x + yi + x - yi = 8 \Rightarrow 2x = 8 \Rightarrow x = 4$. The set of points is the line $x = 4$.

d) Let $z = x + yi$; hence: $\sqrt{(x-3)^2 + y^2} = 2 \Rightarrow (x-3)^2 + y^2 = 4$. The set of points is the circle with centre (3, 0) and radius 2.

e) Let $z = x + yi$; hence:

$$\sqrt{(x-1)^2 + y^2} + \sqrt{(x-3)^2 + y^2} = 2 \Rightarrow (x-1)^2 + y^2 = 4 - 4\sqrt{(x-3)^2 + y^2} + (x-3)^2 + y^2$$

$$x^2 - 2x + 1 + y^2 = 4 - 4\sqrt{(x-3)^2 + y^2} + x^2 - 6x + 9 + y^2$$

$$4x - 12 = -4\sqrt{(x-3)^2 + y^2}$$

$x - 3 = -\sqrt{(x-3)^2 + y^2}$ (Since the right side of the equation is negative, then $x - 3 < 0 \Rightarrow x < 3$.)

$$(x-3)^2 = (x-3)^2 + y^2 \Rightarrow y = 0$$

Therefore: $\sqrt{(x-1)^2}+\sqrt{(x-3)^2}=2 \Rightarrow |x-1|+|x-3|=2 \Rightarrow 1 \leqslant x \leqslant 3$

The set of points is $\{(x, y), 1 \leqslant x \leqslant 3, y=0\}$; hence, the line segment between (1, 0) and (3, 0).

37 Let $z = x + yi$:

a) $\sqrt{x^2+y^2} \leqslant 3$

The set of points is the disk with centre (0, 0) and radius 3.

b) $\sqrt{x^2+(y-3)^2} \geqslant 2$

The solution is the set of points outside the disk with centre (0, 3) and radius 2.

Exercise 10.3

1 $z=4e^{-i\frac{2\pi}{3}}=4\left(\cos\left(-\frac{2\pi}{3}\right)+i\sin\left(-\frac{2\pi}{3}\right)\right)=4\left(-\frac{1}{2}-i\frac{\sqrt{3}}{2}\right)=-2-i2\sqrt{3}$

2 $z=3e^{2\pi i}=3(\cos(2\pi)+i\sin(2\pi))=3(1+i\cdot 0)=3$

3 $z=3e^{0.5\pi i}=3(\cos(0.5\pi)+i\sin(0.5\pi))=3(0+i\cdot 1)=3i$

4 $z=4\operatorname{cis}\left(\frac{7\pi}{12}\right)=4\left(\cos\left(\frac{7\pi}{12}\right)+i\sin\left(\frac{7\pi}{12}\right)\right)=4\left(\frac{\sqrt{2}-\sqrt{6}}{4}+i\frac{\sqrt{2}+\sqrt{6}}{4}\right)=\sqrt{2}-\sqrt{6}+i\left(\sqrt{2}+\sqrt{6}\right)$

5 $z=13e^{\frac{\pi i}{3}}=13\left(\cos\left(\frac{\pi}{3}\right)+i\sin\left(\frac{\pi}{3}\right)\right)=13\left(\frac{1}{2}+i\frac{\sqrt{3}}{2}\right)=\frac{13}{2}+\frac{13\sqrt{3}}{2}i$

6 $z=3e^{1+\frac{\pi}{3}i}=3ee^{\frac{\pi}{3}i}=3e\left(\cos\left(\frac{\pi}{3}\right)+i\sin\left(\frac{\pi}{3}\right)\right)=3e\left(\frac{1}{2}+i\frac{\sqrt{3}}{2}\right)=\frac{3e}{2}+\frac{3e\sqrt{3}}{2}i$

7 $r=\sqrt{8}=2\sqrt{2}$, $\tan\theta=1$, the first quadrant, $\theta=\frac{\pi}{4}$; hence, $z=2\sqrt{2}e^{\frac{\pi}{4}i}$

8 $r=2$, $\tan\theta=\frac{1}{\sqrt{3}}$, the first quadrant, $\theta=\frac{\pi}{6}$; hence, $z=2e^{\frac{\pi}{6}i}$

9 $r=\sqrt{8}=2\sqrt{2}$, $\tan\theta=-\frac{1}{\sqrt{3}}$, the fourth quadrant, $\theta=-\frac{\pi}{6}$; hence, $z=2\sqrt{2}e^{-\frac{\pi}{6}i}$

10 $r=4$, $\tan\theta=-\sqrt{3}$, the fourth quadrant, $\theta=-\frac{\pi}{3}$; hence, $z=4e^{-\frac{\pi}{3}i}$

11 $r=\sqrt{18}=3\sqrt{2}$, $\tan\theta=-1$, the second quadrant, $\theta=\frac{3\pi}{4}$; hence, $z=3\sqrt{2}e^{\frac{3\pi}{4}i}$

12 $r=4$, $\tan\theta$ is not defined, positive y-axis, $\theta=\frac{\pi}{2}$; hence, $z=4e^{\frac{\pi}{2}i}$

13 $r=6$, $\tan\theta=\frac{1}{\sqrt{3}}$, the third quadrant, $\theta=\frac{7\pi}{6}$; hence, $z=6e^{\frac{7\pi}{6}i}$

14 $z=-3+3i$, $r=\sqrt{18}=3\sqrt{2}$, $\tan\theta=-1$, the second quadrant, $\theta=\frac{3\pi}{4}$; hence, $z=3\sqrt{2}e^{\frac{3\pi}{4}i}$

15 $r=\pi$, $\tan\theta=0$, positive x-axis, $\theta=0$; hence, $z=\pi e^{0\cdot i}$ $(=\pi e^{2\pi i})$

16 $r=e$, $\tan\theta$ is not defined, positive y-axis, $\theta=\frac{\pi}{2}$; hence, $z=e\cdot e^{\frac{\pi}{2}i}=e^{1+\frac{\pi}{2}i}$

17 $r=\sqrt{2}$, $\tan\theta=1$, the first quadrant, $\theta=\frac{\pi}{4}$; hence, $1+i=\sqrt{2}\operatorname{cis}\left(\frac{\pi}{4}\right)$

$(1+i)^{10}=\left(\sqrt{2}\right)^{10}\operatorname{cis}\left(10\cdot\frac{\pi}{4}\right)=2^5\operatorname{cis}\left(\frac{\pi}{2}\right)=32\left(\cos\frac{\pi}{2}+i\sin\frac{\pi}{2}\right)=32i$

18 $r = 2$, $\tan\theta = -\dfrac{1}{\sqrt{3}}$, the fourth quadrant, $\theta = -\dfrac{\pi}{6}$; hence, $\sqrt{3} - i = 2\operatorname{cis}\left(-\dfrac{\pi}{6}\right)$

$$\left(\sqrt{3}-i\right)^6 = (2)^6 \operatorname{cis}\left(6\cdot\frac{-\pi}{6}\right) = 64\operatorname{cis}(-\pi) = 64\left(\cos(-\pi) + i\sin(-\pi)\right) = -64$$

19 $r = 6$, $\tan\theta = \sqrt{3}$, the first quadrant, $\theta = \dfrac{\pi}{3}$; hence, $3 + 3i\sqrt{3} = 6\operatorname{cis}\left(\dfrac{\pi}{3}\right)$

$$\left(3+3i\sqrt{3}\right)^9 = (6)^9 \operatorname{cis}\left(9\cdot\frac{\pi}{3}\right) = 10\,077\,696\operatorname{cis}(3\pi) = 10\,077\,696\left(\cos(\pi) + i\sin(\pi)\right) = -10\,077\,696$$

20 $r = 2\sqrt{2}$, $\tan\theta = -1$, the fourth quadrant, $\theta = -\dfrac{\pi}{4}$; hence, $2 - 2i = 2\sqrt{2}\operatorname{cis}\left(-\dfrac{\pi}{4}\right)$

$$(2-2i)^{12} = \left(2\sqrt{2}\right)^{12}\operatorname{cis}\left(12\cdot\frac{-\pi}{4}\right) = 262\,144\operatorname{cis}(-3\pi) = 262\,144\left(\cos(\pi) + i\sin(\pi)\right) = -262\,144$$

21 $r = \sqrt{6}$, $\tan\theta = -1$, the fourth quadrant, $\theta = -\dfrac{\pi}{4}$; hence, $\sqrt{3} - i\sqrt{3} = \sqrt{6}\operatorname{cis}\left(-\dfrac{\pi}{4}\right)$

$$\left(\sqrt{3}-i\sqrt{3}\right)^8 = \sqrt{6}^8\operatorname{cis}\left(8\cdot\frac{-\pi}{4}\right) = 1296\operatorname{cis}(-2\pi) = 1296\left(\cos(0) + i\sin(0)\right) = 1296$$

22 $r = \sqrt{18} = 3\sqrt{2}$, $\tan\theta = -1$, the second quadrant, $\theta = \dfrac{3\pi}{4}$; hence, $z = 3\sqrt{2}\operatorname{cis}\left(\dfrac{3\pi}{4}\right)$

$$(-3+3i)^7 = \left(3\sqrt{2}\right)^7\operatorname{cis}\left(7\cdot\frac{3\pi}{4}\right) = 17\,496\sqrt{2}\operatorname{cis}\left(\frac{5\pi}{4}\right) = 17\,496\sqrt{2}\left(\cos\left(\frac{5\pi}{4}\right) + i\sin\left(\frac{5\pi}{4}\right)\right)$$

$$= 17\,496\sqrt{2}\left(-\frac{1}{\sqrt{2}} - i\frac{1}{\sqrt{2}}\right) = 17\,496(-1-i)$$

23 $r = \sqrt{6}$, $\tan\theta = -1$, the fourth quadrant, $\theta = -\dfrac{\pi}{4}$; hence, $\sqrt{3} - i\sqrt{3} = \sqrt{6}\operatorname{cis}\left(-\dfrac{\pi}{4}\right)$

$$\left(\sqrt{3}-i\sqrt{3}\right)^{-8} = \sqrt{6}^{-8}\operatorname{cis}\left(-8\cdot\frac{-\pi}{4}\right) = \frac{1}{1296}\operatorname{cis}(2\pi) = \frac{1}{1296}\left(\cos(0) + i\sin(0)\right) = \frac{1}{1296}$$

24 $r = 6$, $\tan\theta = \dfrac{1}{\sqrt{3}}$, the third quadrant, $\theta = \dfrac{7\pi}{6}$; hence, $-3\sqrt{3} - 3i = 6\operatorname{cis}\left(\dfrac{7\pi}{6}\right)$

$$\left(-3\sqrt{3}-3i\right)^{-7} = (6)^{-7}\operatorname{cis}\left(-7\cdot\frac{7\pi}{6}\right) = \frac{1}{279\,936}\operatorname{cis}\left(-\frac{\pi}{6}\right) = \frac{1}{279\,936}\left(\frac{\sqrt{3}}{2} - i\frac{1}{2}\right) = \frac{1}{559\,872}\left(\sqrt{3}-i\right)$$

25 $r = 2$, $\tan\theta = \dfrac{1}{\sqrt{3}}$, the first quadrant, $\theta = \dfrac{\pi}{6}$; hence, $\sqrt{3} + i = 2\operatorname{cis}\left(\dfrac{\pi}{6}\right)$

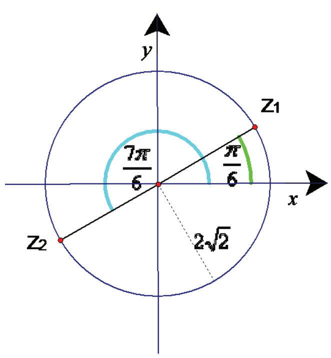
$$2\left(\sqrt{3}+i\right)^7 = 2(2)^7\operatorname{cis}\left(7\cdot\frac{\pi}{6}\right) = 256\operatorname{cis}\left(\frac{7\pi}{6}\right) = 256\left(-\frac{\sqrt{3}}{2} - i\frac{1}{2}\right) = -128\sqrt{3} - 128i$$

26 $r = 8$, $\tan\theta = \sqrt{3}$, the first quadrant, $\theta = \dfrac{\pi}{3}$; hence, $4 + 4i\sqrt{3} = 8\operatorname{cis}\left(\dfrac{\pi}{3}\right)$

$$\left(\sqrt{4+4i\sqrt{3}}\right)_{1,2} = \sqrt{8}\operatorname{cis}\left(\frac{\frac{\pi}{3}}{2} + \frac{2k\pi}{2}\right) = 2\sqrt{2}\operatorname{cis}\left(\frac{\pi}{6} + k\pi\right);\ k = 0, 1$$

$$\left(\sqrt{4+4i\sqrt{3}}\right)_1 = 2\sqrt{2}\left(\cos\frac{\pi}{6} + i\sin\frac{\pi}{6}\right) = 2\sqrt{2}\left(\frac{\sqrt{3}}{2} + \frac{1}{2}\right) = \sqrt{6} + i\sqrt{2}$$

$$\left(\sqrt{4+4i\sqrt{3}}\right)_2 = 2\sqrt{2}\left(\cos\frac{7\pi}{6} + i\sin\frac{7\pi}{6}\right) = 2\sqrt{2}\left(-\frac{\sqrt{3}}{2} - \frac{1}{2}\right) = -\sqrt{6} - i\sqrt{2}$$



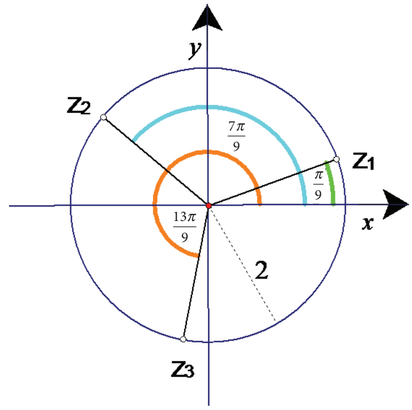
27 $r = 8$, $\tan\theta = \sqrt{3}$, the first quadrant, $\theta = \frac{\pi}{3}$; hence, $4 + 4i\sqrt{3} = 8\text{ cis}\left(\frac{\pi}{3}\right)$

$$\left(\sqrt[3]{4+4i\sqrt{3}}\right)_{1,2,3} = \sqrt[3]{8}\text{ cis}\left(\frac{\frac{\pi}{3}}{3} + \frac{2k\pi}{3}\right) = 2\text{ cis}\left(\frac{\pi}{9} + \frac{2k\pi}{3}\right);\ k = 0, 1, 2$$

$$\left(\sqrt[3]{4+4i\sqrt{3}}\right)_{1} = 2\text{ cis}\left(\frac{\pi}{9}\right) = 2e^{i\frac{\pi}{9}}$$

$$\left(\sqrt[3]{4+4i\sqrt{3}}\right)_{2} = 2\text{ cis}\left(\frac{\pi}{9} + \frac{2\pi}{3}\right) = 2\text{ cis}\left(\frac{7\pi}{9}\right) = 2e^{i\frac{7\pi}{9}}$$

$$\left(\sqrt[3]{4+4i\sqrt{3}}\right)_{3} = 2\text{ cis}\left(\frac{\pi}{9} + \frac{4\pi}{3}\right) = 2\text{ cis}\left(\frac{13\pi}{9}\right) = 2e^{i\frac{13\pi}{9}}$$



28 $r = 1$, $\tan\theta = 0$, on the negative x-axis, $\theta = \pi$; hence, $-1 = 1\text{ cis}(\pi)$

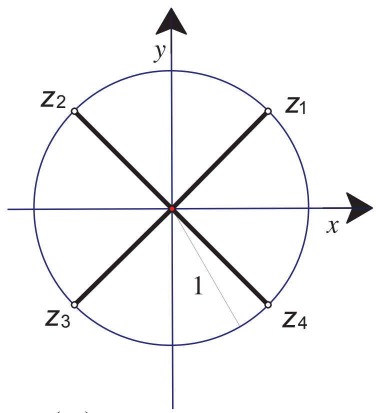
$$\left(\sqrt[4]{-1}\right)_{1,2,3,4} = \sqrt[4]{1}\text{ cis}\left(\frac{\pi}{4} + \frac{2k\pi}{4}\right) = \text{cis}\left(\frac{\pi}{4} + \frac{k\pi}{2}\right);\ k = 0, 1, 2, 3$$

$$\left(\sqrt[4]{-1}\right)_{1} = \text{cis}\left(\frac{\pi}{4}\right) = \cos\frac{\pi}{4} + i\sin\frac{\pi}{4} = \frac{\sqrt{2}}{2} + i\frac{\sqrt{2}}{2}$$

$$\left(\sqrt[4]{-1}\right)_{2} = \text{cis}\left(\frac{3\pi}{4}\right) = \cos\frac{3\pi}{4} + i\sin\frac{3\pi}{4} = -\frac{\sqrt{2}}{2} + i\frac{\sqrt{2}}{2}$$

$$\left(\sqrt[4]{-1}\right)_{3} = \text{cis}\left(\frac{5\pi}{4}\right) = \cos\frac{5\pi}{4} + i\sin\frac{5\pi}{4} = -\frac{\sqrt{2}}{2} - i\frac{\sqrt{2}}{2}$$

$$\left(\sqrt[4]{-1}\right)_{4} = \text{cis}\left(\frac{7\pi}{4}\right) = \cos\frac{7\pi}{4} + i\sin\frac{7\pi}{4} = \frac{\sqrt{2}}{2} - i\frac{\sqrt{2}}{2}$$



29 $r = 1$, $\tan\theta$ is not defined, on the positive y-axis, $\theta = \frac{\pi}{2}$; hence, $i = 1\text{ cis}\left(\frac{\pi}{2}\right)$

$$\left(\sqrt[6]{i}\right)_{1,2,3,4,5,6} = \sqrt[4]{1}\text{ cis}\left(\frac{\frac{\pi}{2}}{6} + \frac{2k\pi}{6}\right) = \text{cis}\left(\frac{\pi}{12} + \frac{k\pi}{3}\right);\ k = 0, 1, 2, 3, 4, 5$$

$$\left(\sqrt[6]{i}\right)_{1} = \text{cis}\left(\frac{\pi}{12}\right) = \frac{\sqrt{2}+\sqrt{6}}{4} + i\frac{\sqrt{6}-\sqrt{2}}{4}$$

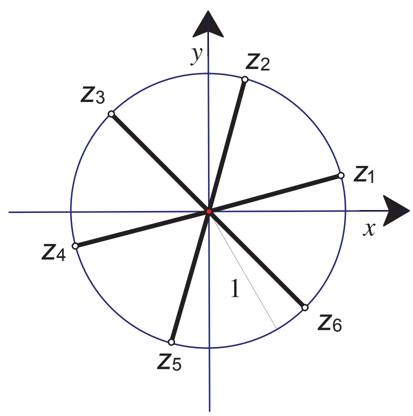
$$\left(\sqrt[6]{i}\right)_{2} = \text{cis}\left(\frac{5\pi}{12}\right) = \frac{-\sqrt{2}+\sqrt{6}}{4} + i\frac{\sqrt{6}+\sqrt{2}}{4}$$

$$\left(\sqrt[6]{i}\right)_{3} = \text{cis}\left(\frac{3\pi}{4}\right) = -\frac{\sqrt{2}}{2} + i\frac{\sqrt{2}}{2}$$

$$\left(\sqrt[6]{i}\right)_{4} = \text{cis}\left(\frac{13\pi}{12}\right) = -\frac{\sqrt{2}+\sqrt{6}}{4} + i\frac{-\sqrt{6}+\sqrt{2}}{4}$$

$$\left(\sqrt[6]{i}\right)_{5} = \text{cis}\left(\frac{17\pi}{12}\right) = \frac{\sqrt{2}-\sqrt{6}}{4} - i\frac{\sqrt{6}+\sqrt{2}}{4}$$

$$\left(\sqrt[6]{i}\right)_{6} = \text{cis}\left(\frac{7\pi}{4}\right) = \frac{\sqrt{2}}{2} - i\frac{\sqrt{2}}{2}$$



30 $r = \sqrt{243} = 9\sqrt{3}$, $\tan\theta = \sqrt{2}$, the third quadrant, $\theta = \arctan\sqrt{2} + \pi \approx 4.0969$; hence, $-9 - 9i\sqrt{2} = 9\sqrt{3}\,\text{cis}\,(4.0969)$ (to 4 d.p.)

$$\left(\sqrt[5]{-9-9i\sqrt{2}}\right)_{1,2,3,4,5} = \sqrt[5]{\sqrt{3^5}}\ \text{cis}\left(\frac{4.0969}{5} + \frac{2\pi}{5}\right) = \sqrt{3}\ \text{cis}\left(\frac{4.0969}{5} + \frac{2\pi}{5}\right); k = 0, 1, 2, 3, 4$$

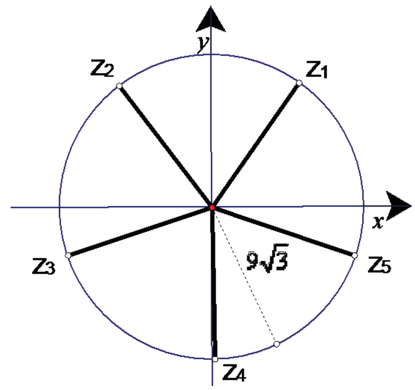
$\left(\sqrt[5]{-9-9i\sqrt{2}}\right)_1 = \sqrt{3}\,\text{cis}\,(0.8194)$ (to 4 d.p.)

$\left(\sqrt[5]{-9-9i\sqrt{2}}\right)_2 = \sqrt{3}\,\text{cis}\,(2.0760)$ (to 4 d.p.)

$\left(\sqrt[5]{-9-9i\sqrt{2}}\right)_3 = \sqrt{3}\,\text{cis}\,(3.3327)$ (to 4 d.p.)

$\left(\sqrt[5]{-9-9i\sqrt{2}}\right)_4 = \sqrt{3}\,\text{cis}\,(4.5893)$ (to 4 d.p.)

$\left(\sqrt[5]{-9-9i\sqrt{2}}\right)_5 = \sqrt{3}\,\text{cis}\,(5.8459)$ (to 4 d.p.)



31 From $z^5 - 32 = 0 \Rightarrow z^5 = 32$; hence, we have to find the fifth roots of 32.

$r = 32$, $\tan\theta = 0$, the positive x-axis, $\theta = 0$; hence, $32 = 32e^{i\cdot 0}$

$$\left(\sqrt[5]{32}\right)_{1,2,3,4,5} = \sqrt[5]{32}e^{i\left(\frac{0}{5}+\frac{2k\pi}{5}\right)} = 2e^{i\frac{2k\pi}{5}}; k = 0, 1, 2, 3, 4$$

$\left(\sqrt[5]{32}\right)_1 = 2e^{i\cdot 0} = 2$, $\left(\sqrt[5]{32}\right)_2 = 2e^{i\frac{2\pi}{5}}$, $\left(\sqrt[5]{32}\right)_3 = 2e^{i\frac{4\pi}{5}}$, $\left(\sqrt[5]{32}\right)_4 = 2e^{i\frac{6\pi}{5}}$, $\left(\sqrt[5]{32}\right)_5 = 2e^{i\frac{8\pi}{5}}$

32 From $z^8 + i = 0 \Rightarrow z^8 = -i$; hence, we have to find the eighth roots of $-i$.

$r = 1$, $\tan\theta$ is not defined, the negative y-axis, $\theta = \frac{3\pi}{2}$; hence, $-i = e^{i\frac{3\pi}{2}}$

$$\left(\sqrt[8]{-i}\right)_{1,2,3,4,5,6,7,8} = \sqrt[8]{1}e^{i\left(\frac{\frac{3\pi}{2}}{8}+\frac{2k\pi}{8}\right)} = e^{i\left(\frac{3\pi}{16}+\frac{k\pi}{4}\right)}; k = 0, 1, 2, 3, 4, 5, 6, 7$$

$\left(\sqrt[8]{-i}\right)_1 = e^{i\frac{3\pi}{16}}$, $\left(\sqrt[8]{-i}\right)_2 = e^{i\frac{7\pi}{16}}$, $\left(\sqrt[8]{-i}\right)_3 = e^{i\frac{11\pi}{16}}$, $\left(\sqrt[8]{-i}\right)_4 = e^{i\frac{15\pi}{16}}$, $\left(\sqrt[8]{-i}\right)_5 = e^{i\frac{19\pi}{16}}$, $\left(\sqrt[8]{-i}\right)_6 = e^{i\frac{23\pi}{16}}$,

$\left(\sqrt[8]{-i}\right)_7 = e^{i\frac{27\pi}{16}}$, $\left(\sqrt[8]{-i}\right)_8 = e^{i\frac{31\pi}{16}}$

33 From $z^3 + 4\sqrt{3} - 4i = 0 \Rightarrow z^3 = -4\sqrt{3} + 4i$; hence, we have to find the third roots of $4\sqrt{3} - 4i$.

$r = 8$, $\tan\theta = -\frac{1}{\sqrt{3}}$, the second quadrant, $\theta = \frac{5\pi}{6}$; hence, $4\sqrt{3} - 4i = 8e^{i\frac{5\pi}{6}}$

$$\left(\sqrt[3]{4\sqrt{3}-4i}\right)_{1,2,3} = \sqrt[3]{8}e^{i\left(\frac{\frac{5\pi}{6}}{3}+\frac{2k\pi}{3}\right)} = 2e^{i\left(\frac{5\pi}{18}+\frac{2k\pi}{3}\right)}; k = 0, 1, 2$$

$\left(\sqrt[3]{4\sqrt{3}-4i}\right)_1 = 2e^{i\frac{5\pi}{18}}$, $\left(\sqrt[3]{4\sqrt{3}-4i}\right)_2 = 2e^{i\frac{17\pi}{18}}$, $\left(\sqrt[3]{4\sqrt{3}-4i}\right)_3 = 2e^{i\frac{29\pi}{18}}$

34 From $z^4 - 16 = 0 \Rightarrow z^4 = 16$; hence we have to find the fourth roots of 16.

$r = 16$, $\tan\theta = 0$, the positive x-axis, $\theta = 0$; hence, $16 = 16e^{i\cdot 0}$

$$\left(\sqrt[4]{16}\right)_{1,2,3,4} = \sqrt[4]{16}e^{i\left(\frac{0}{4}+\frac{2k\pi}{4}\right)} = 2e^{i\frac{k\pi}{2}}; k = 0, 1, 2, 3$$

$\left(\sqrt[4]{16}\right)_1 = 2e^{i0} = 2$, $\left(\sqrt[4]{16}\right)_2 = 2e^{i\frac{\pi}{2}} = 2i$, $\left(\sqrt[4]{16}\right)_3 = 2e^{i\pi} = -2$, $\left(\sqrt[4]{16}\right)_2 = 2e^{i\frac{3\pi}{2}} = -2i$

35 From $z^5+128=128i \Rightarrow z^5=-128+128i$; hence, we have to find the fifth roots of $-128+128i$.

$r=128\sqrt{2}=2^{\frac{15}{2}}$, $\tan\theta=-1$, the second quadrant, $\theta=\frac{3\pi}{4}$; hence, $-128+128i=2^{\frac{15}{2}}e^{i\frac{3\pi}{4}}$

$$\left(\sqrt[5]{-128+128i}\right)_{1,2,3,4,5}=\sqrt[5]{2^{\frac{15}{2}}}e^{i\left(\frac{\frac{3\pi}{4}}{5}+\frac{2k\pi}{5}\right)}=2^{\frac{3}{2}}e^{i\left(\frac{3\pi}{20}+\frac{2k\pi}{5}\right)}=\sqrt{8}e^{i\left(\frac{3\pi}{20}+\frac{2k\pi}{5}\right)};\ k=0,1,2,3,4$$

$$\left(\sqrt[5]{-128+128i}\right)_1=\sqrt{8}e^{i\frac{3\pi}{20}},\ \left(\sqrt[5]{-128+128i}\right)_2=\sqrt{8}e^{i\frac{11\pi}{20}},\ \left(\sqrt[5]{-128+128i}\right)_3=\sqrt{8}e^{i\frac{19\pi}{20}},$$

$$\left(\sqrt[5]{-128+128i}\right)_4=\sqrt{8}e^{i\frac{27\pi}{20}},\ \left(\sqrt[5]{-128+128i}\right)_5=\sqrt{8}e^{i\frac{7\pi}{4}}=2\sqrt{2}\left(\frac{1}{\sqrt{2}}-\frac{1}{\sqrt{2}}i\right)=2-2i$$

36 From $z^6-64i=0 \Rightarrow z^6=64i$; hence, we have to find the sixth roots of $64i$.

$r=64$, $\tan\theta$ is not defined, the positive y-axis, $\theta=\frac{\pi}{2}$; hence, $64i=64e^{i\frac{\pi}{2}}$

$$\left(\sqrt[6]{64i}\right)_{1,2,3,4,5,6}=\sqrt[6]{64}e^{i\left(\frac{\frac{\pi}{2}}{6}+\frac{2k\pi}{6}\right)}=2e^{i\left(\frac{\pi}{12}+\frac{k\pi}{3}\right)};\ k=0,1,2,3,4,5$$

$$\left(\sqrt[6]{64i}\right)_1=2e^{i\frac{\pi}{12}},\ \left(\sqrt[6]{64i}\right)_2=2e^{i\frac{5\pi}{12}},\ \left(\sqrt[6]{64i}\right)_3=2e^{i\frac{3\pi}{4}}=2\left(-\frac{1}{\sqrt{2}}+\frac{1}{\sqrt{2}}i\right)=-\sqrt{2}+\sqrt{2}i,$$

$$\left(\sqrt[6]{64i}\right)_4=2e^{i\frac{13\pi}{12}},\ \left(\sqrt[6]{64i}\right)_5=2e^{i\frac{17\pi}{12}},\ \left(\sqrt[6]{64i}\right)_6=2e^{i\frac{7\pi}{4}}=2\left(\frac{1}{\sqrt{2}}-\frac{1}{\sqrt{2}}i\right)=\sqrt{2}-\sqrt{2}i$$

37 $\text{cis}(9\beta)\,\text{cis}(-5\beta)=\text{cis}(9\beta-5\beta)=\text{cis}(4\beta)=\cos(4\beta)+i\sin(4\beta)$

38 $\frac{\text{cis}(6\beta)\,\text{cis}(4\beta)}{\text{cis}(3\beta)}=\text{cis}(6\beta+4\beta-3\beta)=\text{cis}(7\beta)=\cos(7\beta)+i\sin(7\beta)$

39 $(\text{cis}(9\beta))^{\frac{1}{3}}=\text{cis}\left(\frac{9\beta}{3}\right)=\text{cis}(3\beta)=\cos(3\beta)+i\sin(3\beta)$

40 $\sqrt[n]{\text{cis}(2n\beta)}=\text{cis}\left(\frac{2n\beta}{n}\right)=\text{cis}(2\beta)=\cos(2\beta)+i\sin(2\beta)$

41 $\cos(\alpha+\beta)=\text{Re}\left(e^{(\alpha+\beta)i}\right)=\text{Re}\left(e^{\alpha i}e^{\beta i}\right)$

Hence, we have to find $e^{\alpha i}e^{\beta i}$:

$$\begin{aligned}e^{\alpha i}e^{\beta i}&=(\cos\alpha+i\sin\alpha)(\cos\beta+i\sin\beta)\\&=\cos\alpha\cos\beta-\sin\alpha\sin\beta+i(\cos\alpha\sin\beta+\sin\alpha\cos\beta)\end{aligned}$$

The real part of $e^{\alpha i}e^{\beta i}$ is $\cos\alpha\cos\beta-\sin\alpha\sin\beta$, so $\cos(\alpha+\beta)=\cos\alpha\cos\beta-\sin\alpha\sin\beta$.

42 $\cos(4\alpha)=\text{Re}\left(e^{(4\alpha)i}\right)=\text{Re}\left(\left(e^{\alpha i}\right)^4\right)$

Hence, we have to find $\left(e^{\alpha i}\right)^4$:

$$\begin{aligned}\left(e^{\alpha i}\right)^4&=(\cos(\alpha)+i\sin(\alpha))^4\\&=\cos^4\alpha+4\cos^3\alpha\sin\alpha i+6\cos^2\alpha\sin^2\alpha i^2+4\cos\alpha\sin^3\alpha i^3+\sin^4\alpha i^4\end{aligned}$$

Use the binomial theorem.

The real part of the number is:

$$\begin{aligned}\cos^4\alpha+6\cos^2\alpha\sin^2\alpha i^2+\sin^4\alpha i^4&=\cos^4\alpha+6\cos^2\alpha\left(1-\cos^2\alpha\right)(-1)+\left(1-\cos^2\alpha\right)^2(1)\\&=\cos^4\alpha-6\cos^2\alpha+6\cos^4\alpha+1-2\cos^2\alpha+\cos^4\alpha\\&=8\cos^4\alpha-8\cos^2\alpha+1;\text{ therefore:}\end{aligned}$$

$\cos(4\alpha)=8\cos^4\alpha-8\cos^2\alpha+1$

43 $\cos(5\alpha) = \text{Re}\left(e^{(5\alpha)i}\right) = \text{Re}\left(\left(e^{\alpha i}\right)^5\right)$

Hence, we have to find $\left(e^{\alpha i}\right)^5$:

$\left(e^{\alpha i}\right)^5 = (\cos(\alpha) + i\sin(\alpha))^5$ Use the binomial theorem.

$= \cos^5\alpha + 5\cos^4\alpha\sin\alpha i + 10\cos^3\alpha\sin^2\alpha i^2 + 10\cos^2\alpha\sin^3\alpha i^3 + 5\cos\alpha\sin^4\alpha i^4 + \sin^5\alpha i^5$

The real part of the number is:

$\cos^5\alpha + 10\cos^3\alpha\sin^2\alpha(-1) + 5\cos\alpha\sin^4\alpha(1)$

$= \cos^5\alpha - 10\cos^3\alpha(1-\cos^2\alpha) + 5\cos\alpha(1-\cos^2\alpha)^2$

$= \cos^5\alpha - 10\cos^3\alpha + 10\cos^5\alpha + 5\cos\alpha - 10\cos^3\alpha + 5\cos^4\alpha$

$= 16\cos^5\alpha - 20\cos^3\alpha + 5\cos\alpha$; therefore:

$\cos(5\alpha) = 16\cos^5\alpha - 20\cos^3\alpha + 5\cos\alpha$

44 Using the formula from question 41, $\cos(4\alpha) = 8\cos^4\alpha - 8\cos^2\alpha + 1$, and the double angle formula, $\cos 2\alpha = 2\cos^2\alpha - 1 \Rightarrow 8\cos^2\alpha = 4(\cos 2\alpha - 1)$, we have:

$\cos(4\alpha) = 8\cos^4\alpha - 4(\cos 2\alpha - 1) + 1 \Rightarrow \cos(4\alpha) + 4\cos 2\alpha + 3 = 8\cos^4\alpha$; hence,

$\cos^4\alpha = \frac{1}{8}(\cos(4\alpha) + 4\cos 2\alpha + 3)$

$\cos(4\alpha) = 8\cos^4\alpha - 8\cos^2\alpha + 1$

45 a) Since $\frac{1}{z} = \cos(-2\alpha) + i\sin(-2\alpha)$, we have: $z + \frac{1}{z} = (\cos(2\alpha) + i\sin(2\alpha)) + (\cos(-2\alpha) + i\sin(-2\alpha))$.

Using the even/odd property, we have: $\cos(-2\alpha) = \cos(2\alpha)$ and $\sin(-2\alpha) = -\sin(2\alpha)$. Hence,

$z + \frac{1}{z} = \cos(2\alpha) + i\sin(2\alpha) + \cos(2\alpha) - i\sin(2\alpha) = 2\cos(2\alpha)$

$z - \frac{1}{z} = (\cos(2\alpha) + i\sin(2\alpha)) - (\cos(-2\alpha) + i\sin(-2\alpha))$

$= \cos(2\alpha) + i\sin(2\alpha) - \cos(2\alpha) + i\sin(2\alpha) = 2i\sin(2\alpha)$

b) $z^n + \frac{1}{z^n} = (\cos(2n\alpha) + i\sin(2n\alpha)) + (\cos(-2n\alpha) + i\sin(-2n\alpha))$

$= \cos(2n\alpha) + i\sin(2n\alpha) + \cos(2n\alpha) - i\sin(2n\alpha) = 2\cos(2n\alpha)$

Hence, $z^n + \frac{1}{z^n} = 2\cos(2n\alpha) \Rightarrow \cos(2n\alpha) = \frac{1}{2}\left(z^n + \frac{1}{z^n}\right)$

$z^n - \frac{1}{z^n} = (\cos(2n\alpha) + i\sin(2n\alpha)) - (\cos(-2n\alpha) + i\sin(-2n\alpha))$

$= \cos(2n\alpha) + i\sin(2n\alpha) - \cos(2n\alpha) + i\sin(2n\alpha) = 2i\sin(2n\alpha)$

So, $z^n - \frac{1}{z^n} = 2i\sin(2n\alpha) \Rightarrow \sin(2n\alpha) = \frac{1}{2i}\left(z^n - \frac{1}{z^n}\right)$.

46 $(1+3w)\left(1+3w^2\right) = 1 + 3w + 3w^2 + 9\underbrace{w^3}_{=1} = 10 + 3w(1+w)$

Since $w(1+w) = \frac{w(1+w)(1-w)}{1-w} = \frac{w - \overbrace{w^3}^{=1}}{1-w} = \frac{w-1}{1-w} = -1$, we have: $(1+3w)\left(1+3w^2\right) = 10 + 3(-1) = 7$.

Note: We can establish the formula $w + w^2 = -1$ using the values of the cube roots of 1:

$1, w = \cos\frac{2\pi}{3} + i\sin\frac{2\pi}{3} = -\frac{1}{2} + \frac{\sqrt{3}}{2}i, w^2 = \cos\frac{4\pi}{3} + i\sin\frac{4\pi}{3} = -\frac{1}{2} - \frac{\sqrt{3}}{2}i$.

Hence, $w + w^2 = \left(-\frac{1}{2} + \frac{\sqrt{3}}{2}i\right) + \left(-\frac{1}{2} - \frac{\sqrt{3}}{2}i\right) = -1$.

47 a) For the fourth roots of $1 = 1\ \text{cis}\,(0)$:

$$\sqrt[4]{1} = \text{cis}\left(\frac{2k\pi}{4}\right) = \text{cis}\left(\frac{k\pi}{2}\right); k = 0, 1, 2, 3$$

$$\left(\sqrt[4]{1}\right)_1 = \text{cis}\,(0) = 1, \left(\sqrt[4]{1}\right)_2 = \text{cis}\left(\frac{\pi}{2}\right), \left(\sqrt[4]{1}\right)_3 = \text{cis}\,(\pi) = \left(\text{cis}\left(\frac{\pi}{2}\right)\right)^2, \left(\sqrt[4]{1}\right)_4 = \text{cis}\left(\frac{3\pi}{2}\right) = \left(\text{cis}\left(\frac{\pi}{2}\right)\right)^3$$

Therefore, we can denote the fourth roots as $1, \beta, \beta^2, \beta^3$.

b) $(1+\beta)\left(1+\beta^2+\beta^3\right) = 1+\beta+\beta^2+2\beta^3+\beta^4 = 2+i-1-2i = 1-i$

Note: We can solve the task using the values: $\beta = i, \beta^2 = -1, \beta^3 = -i$, so:
$(1+\beta)\left(1+\beta^2+\beta^3\right) = (1+i)(1-1-i) = (1+i)(-i) = 1-i$.

c) $\beta+\beta^2+\beta^3 = \underbrace{1+\beta+\beta^2+\beta^3}_{=0} - 1 = -1$

Note: We have used the formula $1+\beta+\beta^2+\beta^3 = 0$. This formula holds because $\left(1+\beta+\beta^2+\beta^3\right)(1-\beta) = 1-\beta^4 = 0$ and $\beta \neq 1 \Rightarrow 1-\beta \neq 0$.

Although, notice that we can solve the task using the values: $\beta = i, \beta^2 = -1, \beta^3 = -i$, so:

$\beta+\beta^2+\beta^3 = i+(-1)+(-i) = -1$.

48 a) For the fifth roots of $1 = 1\ \text{cis}\,(0)$:

$$\sqrt[5]{1} = \text{cis}\left(\frac{2k\pi}{5}\right); k = 0, 1, 2, 3, 4$$

$$\sqrt[5]{1} = \text{cis}\,(0) = 1,\ \sqrt[5]{1} = \text{cis}\left(\frac{2\pi}{5}\right),\ \sqrt[5]{1} = \text{cis}\left(\frac{4\pi}{5}\right) = \left(\text{cis}\left(\frac{2\pi}{5}\right)\right)^2,\ \sqrt[5]{1} = \text{cis}\left(\frac{6\pi}{5}\right) = \left(\text{cis}\left(\frac{2\pi}{5}\right)\right)^3,$$

$$\sqrt[5]{1} = \text{cis}\left(\frac{8\pi}{5}\right) = \left(\text{cis}\left(\frac{2\pi}{5}\right)\right)^4$$

Therefore, we can denote the fifth roots as $1, \alpha, \alpha^2, \alpha^3, \alpha^4$.

b)
$$\begin{aligned}(1+\alpha)\left(1+\alpha^4\right) &= 1+\alpha+\alpha^4+\alpha^5 = 2+\alpha+\alpha^4\\ &= 2+\cos\frac{2\pi}{5}+i\sin\frac{2\pi}{5}+\cos\left(-\frac{2\pi}{5}\right)+i\sin\left(-\frac{2\pi}{5}\right)\\ &= 2+2\cos\frac{2\pi}{5} = 2+\frac{-1+\sqrt{5}}{2} = \frac{3+\sqrt{5}}{2}\end{aligned}$$

c) $$1+\alpha+\alpha^2+\alpha^3+\alpha^4 = \frac{\left(1+\alpha+\alpha^2+\alpha^3+\alpha^4\right)(1-\alpha)}{1-\alpha} = \frac{1-\alpha^5}{1-\alpha} = \frac{1-1}{1-\alpha} = 0$$

49 $$1+i\sqrt{3} = 2\ \text{cis}\left(\frac{\pi}{3}\right) \Rightarrow \left(1+i\sqrt{3}\right)^n = 2^n\ \text{cis}\left(\frac{n\pi}{3}\right)$$

$$1-i\sqrt{3} = 2\ \text{cis}\left(-\frac{\pi}{3}\right) \Rightarrow \left(1-i\sqrt{3}\right)^n = 2^n\ \text{cis}\left(-\frac{n\pi}{3}\right)$$

Hence:

$$\begin{aligned}\left(1+i\sqrt{3}\right)^n+\left(1-i\sqrt{3}\right)^n &= 2^n\left(\cos\left(\frac{n\pi}{3}\right)+i\sin\left(\frac{n\pi}{3}\right)+\cos\left(-\frac{n\pi}{3}\right)+i\sin\left(-\frac{n\pi}{3}\right)\right)\\ &= 2^n\left(\cos\left(\frac{n\pi}{3}\right)+i\sin\left(\frac{n\pi}{3}\right)+\cos\left(\frac{n\pi}{3}\right)-i\sin\left(\frac{n\pi}{3}\right)\right)\\ &= 2^{n+1}\cos\left(\frac{n\pi}{3}\right)\end{aligned}$$

Use the even/odd properties.

Therefore, the number is real.

For $n = 18$, the value is: $2^{19} \cos\left(\frac{18\pi}{3}\right) = 2^{19} = 524\,288$.

50 Since $\arg(2a + 3i)^3 = 135^\circ$, then $\arg(2a + 3i) = \frac{135^\circ}{3} = 45^\circ$. Therefore, $\tan\theta = 1 = \frac{3}{2a} \Rightarrow a = \frac{3}{2}$.

Chapter 11

Exercise 11.1

1 **a)** The experimental unit would be the gender of a student. A sensible population would be all the students in a certain school (a large one), city, district, or country. The sample could be students from one class, or even a smaller group of students. The variable is qualitative as it describes a characteristic of a student (female or male) rather than a numerical quantity.

b) The experimental unit would be a final exam taken by a 10th-grade student. A sensible population would be all the exams done by 10th-grade students in a certain school (a large one), city, district, or country. The sample could be the exams of students from one class, or even a smaller group of students. The variable is quantitative since we need to count the number of errors.

c) The experimental unit would be a newborn child. A sensible population would be all the newborn children in a certain city, district, or country. The sample could be newborn children born in the same hospital, or born on the same day. The variable is quantitative since we measure the height.

d) The experimental unit would be a child aged less than 14. A sensible population would be all the children aged less than 14 who live in a certain city, district, and country. The sample could be all the children aged less than 14 who live in the same building, block, or street. The variable is qualitative as it describes a characteristic of a child (blue eyes, brown eyes, green eyes, and so on) rather than a numerical quantity.

e) The experimental unit would be a working person. A sensible population would be all the working people in a certain city, district, or country. The sample could be people working in the same company, or people living in the same part of the city. The variable is quantitative since we measure the time it takes them to travel to work.

f) The experimental unit would be a country leader. A sensible population would be all the country leaders worldwide. The sample could be all the country leaders within a certain geographical region, or a continent, or even the leaders of the same country throughout history. The variable is qualitative as it describes a characteristic of a leader (excellent, good, fair, or poor) rather than a numerical quantity.

g) The experimental unit would be a student. A sensible population would be all the countries of origin of students at an international school, or a group of international schools within a certain country or a geographical region. The sample could be all the countries of origin of students from one grade of an international school. The variable is qualitative as it describes a characteristic of a student through their country of origin (Austria, Germany, Italy, Croatia, and so on) rather than a numerical quantity.

2 **Note:** Answers for question 2 are not unique.

a) Skewed to the left, since there are many players who don't score at all and there are a few who are top scorers.

b) It should be symmetric since the weights will be grouped around one particular weight. (Later on, we will find out that this weight is called the mean weight.)

c) Again skewed to the left, since there are a few students who travel a lot and visit many countries.

d) In this case we would again expect a distribution skewed to the left, because some students do receive a lot of emails (especially those who use social networking sites).

3 a) Quantitative because we can measure the time taken to finish the essay.

b) Quantitative because we can count the number of students in each section.

c) Qualitative since the rating has descriptors rather than a numerical quantity.

d) Qualitative since the country of origin is listed by its name rather than a numerical quantity.

4 a) Discrete since we can count the exact number of students from each country.

b) The weight of exam papers cannot be measured exactly and therefore it is continuous.

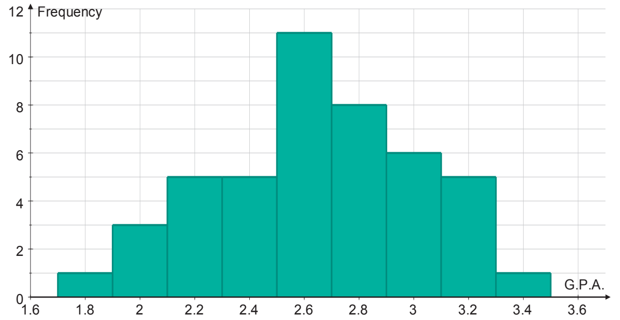
c) Time cannot be measured exactly and therefore it is continuous.

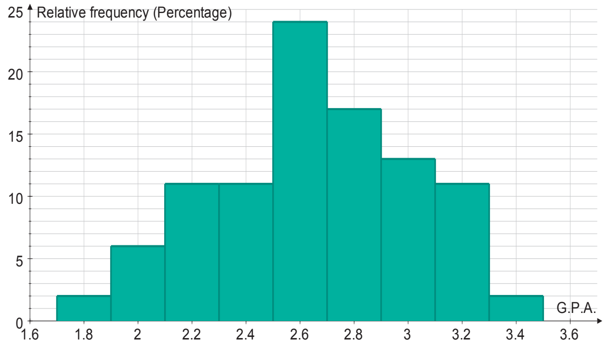
d) Discrete since the number of customers must be exact. (Customer satisfaction is a different experiment altogether.)

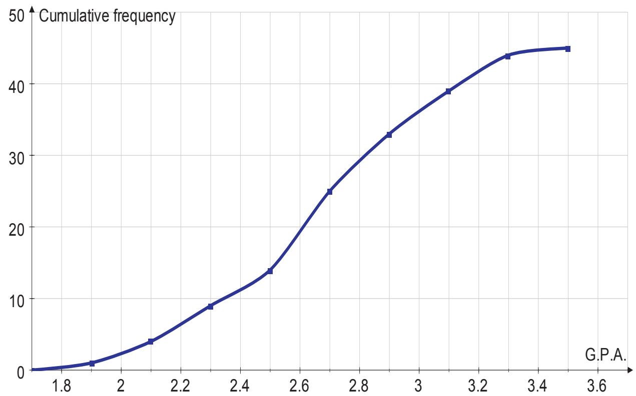
e) Again, time cannot be measured exactly and therefore it is continuous. We always measure time to a certain degree of accuracy, but never exactly.

f) The amount of sugar, as a mass, cannot be measured exactly and therefore it is continuous. On the other hand, if we take a microscope and count the grains of sugar without breaking them apart, then we can say that it is discrete; but quite honestly who would do that?

5 This analysis is done by grouping the grades into classes of length 2, since the range of the data is not very large.



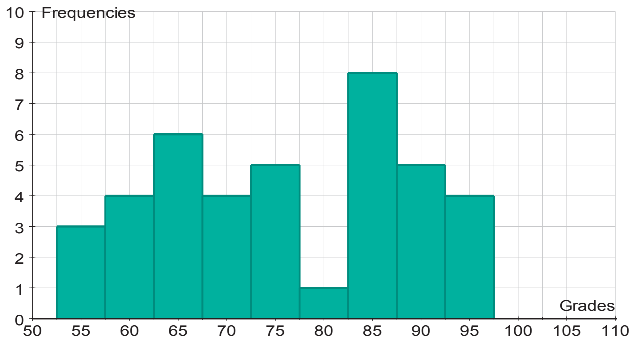




The data looks relatively symmetrical, with no apparent outliers.

6 We will group the data into classes of length 5, and count the frequencies for each interval.

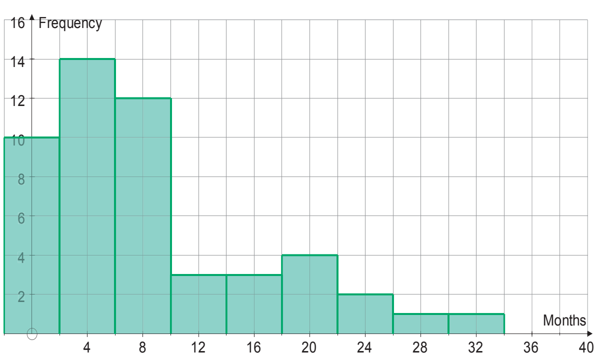
Interval	Midpoint	Frequency
52.5–57.5	55	3
57.5–62.5	60	4
62.5–67.5	65	6
67.5–72.5	70	4
72.5–77.5	75	5
77.5–82.5	80	1
82.5–87.5	85	8
87.5–92.5	90	5
92.5–97.5	95	4



We can say that this distribution is almost bimodal, where one group has a mode of 65 and the other group has a mode of 85.

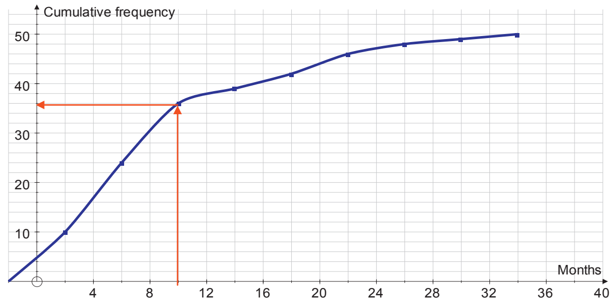
7 a) This set of data has a very large range and therefore we are going to group it into suitable intervals. The midpoint of each interval is shown in the table.

Midpoint	Frequency
0	10
4	14
8	12
12	3
16	3
20	4
24	2
28	1
32	1



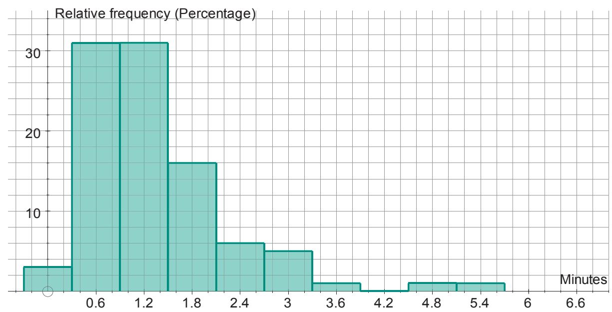
b) The data is not symmetric but skewed to the right.

c)

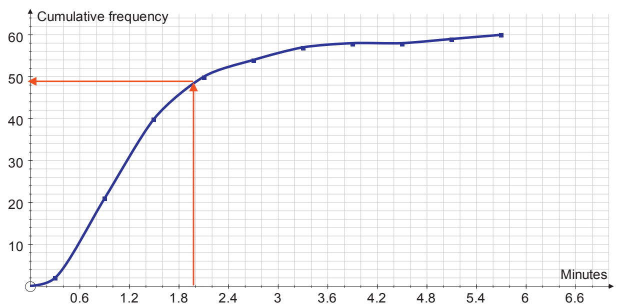


From the diagram, we notice that there are 36 young drivers who will lose their licence; therefore, 72% may lose their licence.

8 a) We will use classes of length 0.6, having 0 as the midpoint of the first interval.



b)

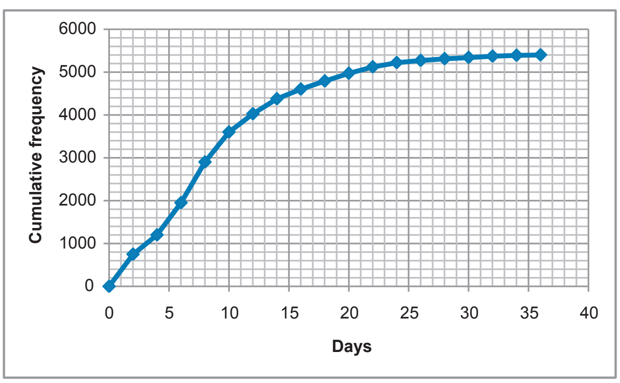


Using the graph, we can say that there are approximately 49 customers who need to wait up to 2 minutes; therefore, 11 customers will need to wait longer than that.

9 a) The data is skewed to the right, with the modal value as 6–8 days spent at hospital. A very few patients will stay longer than 20 days.

b) Due to the poor scale on the frequency diagram, it is going to be difficult to estimate the frequencies. Our estimates are as follows:

Interval	Frequency	Cumulative frequency
0–2	750	750
2–4	450	1200
4–6	750	1950
6–8	950	2900
8–10	700	3600
10–12	425	4025
12–14	350	4375
14–16	225	4600
16–18	190	4790
18–20	180	4970
20–22	150	5120
22–24	100	5220
24–26	50	5270
26–28	40	5310
28–30	30	5340
30–32	30	5370
32–34	20	5390
34–36	10	5400



c) Using the table, or graph, we can estimate the percentage of patients who stayed less than 6 days as:

$$p = \frac{1950}{5400} = 0.361\,11... \approx 36\%.$$

10 a) From the frequency graph, we can see that the longest time spent doing his exercise is between 38 and 40 minutes; therefore, the longest time is 40 minutes.

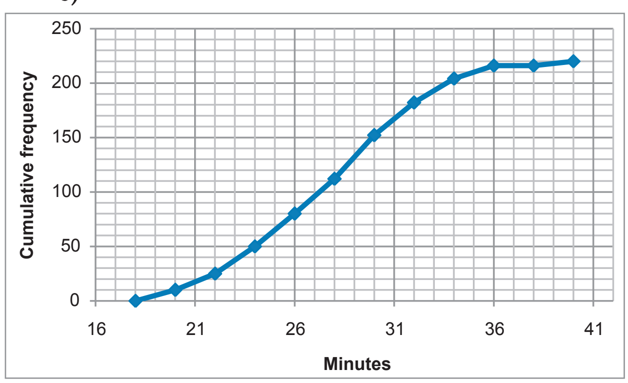
b) In order to solve this problem, we need the cumulative frequencies.

Interval	Frequency	Cumulative frequency
18–20	10	10
20–22	15	25
22–24	25	50
24–26	30	80
26–28	32	112
28–30	40	152
30–32	30	182
32–34	22	204
34–36	12	216
36–38	0	216
38–40	4	220

The percentage of time spent exercising more than 30 minutes is:

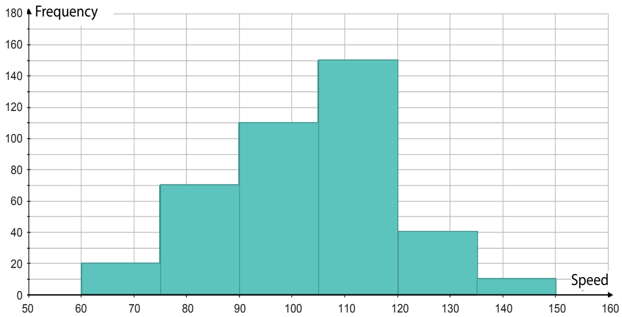
$$p = 1 - \frac{152}{220} = 0.309\,09... \approx 31\%$$

c)

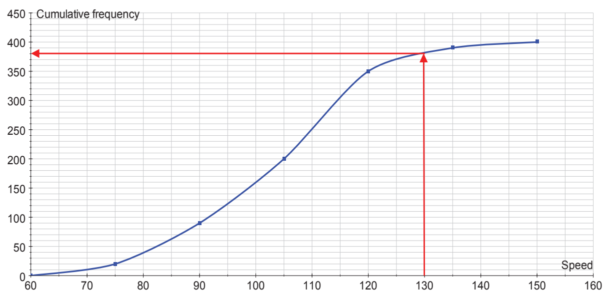


11 a) The frequency table is given in horizontal form in the question.

b)

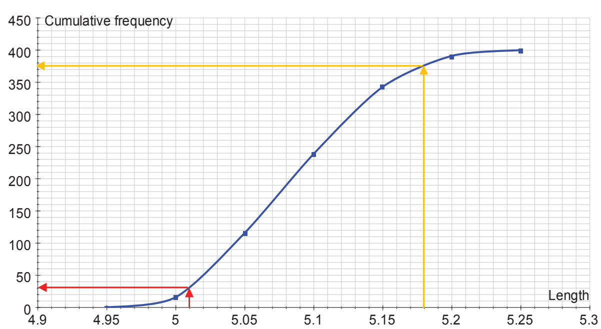


c) To draw the cumulative frequency graph, we take the endpoints of the intervals and calculate the corresponding cumulative frequencies, which are: 20, 90, 200, 350, 390 and 400.



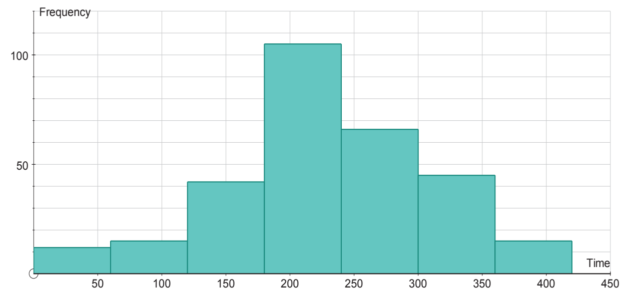
d) To estimate the number of drivers exceeding the speed limit, we draw a vertical line from 130 km/h on the diagram above. Our estimate is 380; therefore, since there are 400 cars, 20 cars, or 5% of the cars, were exceeding the speed limit.

12 a) To draw the cumulative frequency graph, we take the endpoints of the intervals and calculate the corresponding cumulative frequencies, which are: 16, 116, 239, 343, 391 and 400.

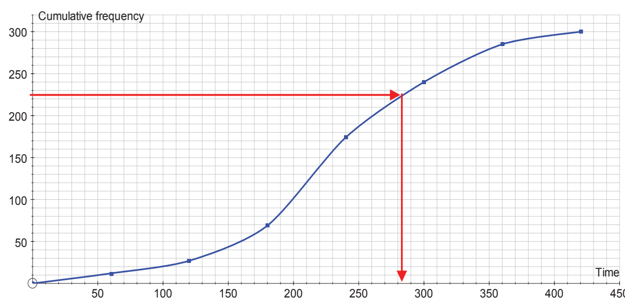


b) We need to draw two vertical lines, from 5.01 and 5.18, on the diagram in part **a**. Our estimate for the number of components of length 5.01 mm is 30 and of 5.18 mm is 376. Therefore, $30 + (400 - 376) = 54$ components will be scrapped, that is, 13.5% of the components.

13 a)



b) To draw the cumulative frequency graph, we take the endpoints of the intervals and calculate the corresponding cumulative frequencies, which are: 12, 27, 69, 174, 240, 285 and 300.



c) There are 300 customers and 25% of 300 is 75. To find the waiting time that is exceeded by 75 customers, we draw a horizontal line at the cumulative frequency of 225 (see diagram above). Our estimate is 285 seconds (4 minutes 45 seconds).

Exercise 11.2

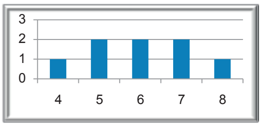
1 **a)** $\bar{x} = \dfrac{5+4+7+8+6+6+5+7}{8} = \dfrac{48}{8} = 6$

b) In order to find the median, we need to list the observations in order of magnitude. Since there are eight observations, we need to take the two middle ones, which are the fourth and fifth observations, and take their average.

4, 5, 5, 6, 6, 7, 7, 8

Since the middle observations are both 6, the median is 6.

c) The data is symmetric with respect to the mean value.



2 **a)** $\bar{x} = \dfrac{5+7+8+6+12+7+8+11+4+10}{10} = \dfrac{78}{10} = 7.8$

b) 4, 5, 6, 7, 7, 8, 8, 10, 11, 12

There are ten observations, so we need to identify the fifth and sixth observations. The fifth observation is 7 and the sixth is 8; therefore, the median is 7.5.

c) This set of data is bimodal, with 7 and 8 as the modes, since these two observations have the highest frequency.

3

Number of DVD players (x_i)	0	1	2	3	Σ
Number of households (f_i)	12	24	8	6	50
$x_i \times f_i$	0	24	16	18	58

We calculate the mean value by using the formula $\bar{x} = \dfrac{\sum x_i f_i}{\sum f_i} = \dfrac{58}{50} = 1.16$.

There are 50 pieces of data, so to find the median value we have to identify the 25th and 26th observations, when listed in order of magnitude, and then take their average. From the table, we notice that all the observations from the 13th until the 36th are 1; therefore, the median is 1.

The median is the measure that best describes this data since the data is skewed to the right and, as such, the mean is more influenced by it.

4 We are going to list the losses in order of magnitude: 3, 270, 393, 417, 814, 861, 1362, 10 567, 21 167, 39 093.

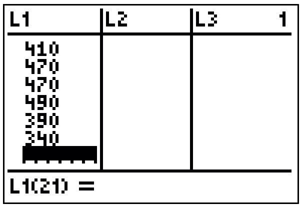
The median value is: $\tilde{x} = \dfrac{814+861}{2} = 837.5$ millions of dollars.

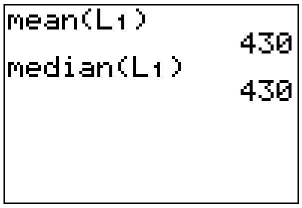
The mean is the total sum of losses divided by 10: $\bar{x} = \dfrac{74\,947}{10} = 7494.7$ millions of dollars.

In this case, the median is more appropriate as there are extreme values.

5 For this question we will use a calculator since there are many observations.

Firstly, we input all the observations in list L_1 and then, from the List menu, we can use the mean and median features of the GDC.





Since both measures are equal, and the data looks symmetrical, either measure looks good.

6 **a)** $\bar{x} = \dfrac{4460}{90} = 49.56$, correct to the nearest cent.

b) $\bar{x} = \dfrac{4460 + 74 + 60}{90 + 2} = 49.93$, correct to the nearest cent.

7 We need to consider all the bags, measure their total weight and divide it by the total number of bags.

$$\bar{x} = \frac{144 \cdot 2.15 + 56 \cdot 1.8}{144 + 56} = \frac{309.6 + 100.8}{200} = 2.052$$

So, the mean weight of a bag of potatoes is 2.052 g.

8 **a)** $\bar{x} = \dfrac{\sum x_i}{25} = \dfrac{749}{25} = 29.96$

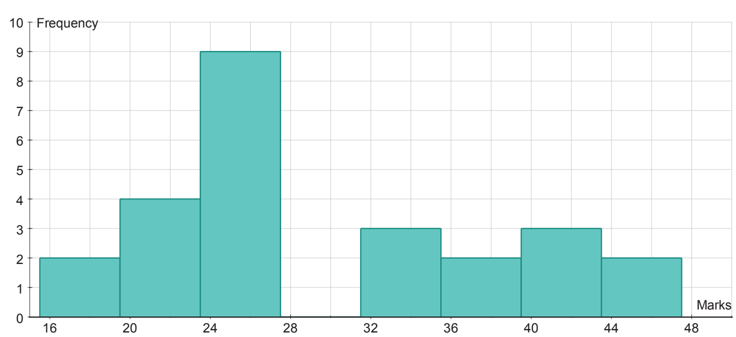
b)

1	8 9
2	0 2 3 3 4 4 5 6 6 6 7 7 7
3	3 4 5 6 8
4	0 2 2 6 6

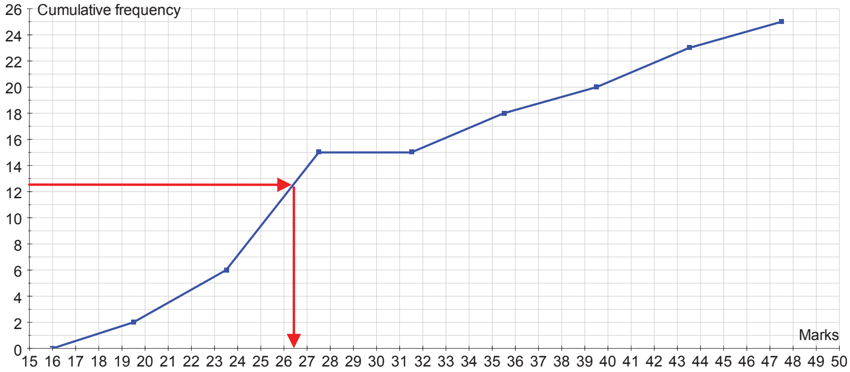
Since there are 25 observations, we have to find the 13th $\left(\dfrac{n+1}{2}\right)$ observation. Looking at the stem plot, the 13th observation is 27; therefore, the median is 27.

c) To draw a histogram we need to group the data into suitable intervals. We will use intervals of length 4, starting from 16.

Grades	Frequency
16–19	2
20–23	4
24–27	9
28–31	0
32–35	3
36–39	2
40–43	3
44–47	2



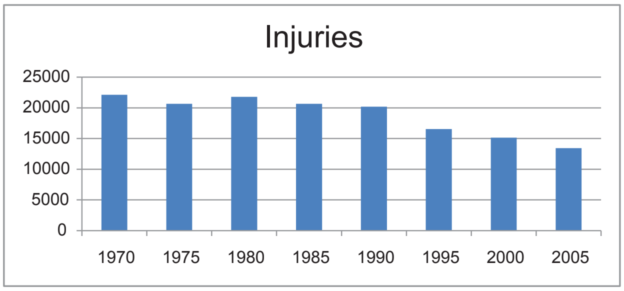
d)



An estimate for the median (which corresponds to a cumulative frequency of 12.5) is 26.5.

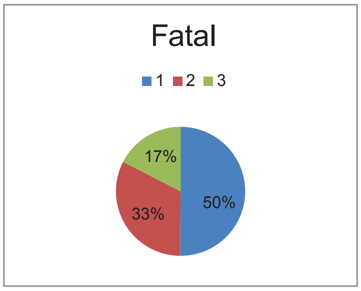
9 a)

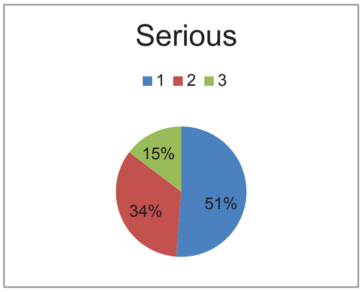
Year	Fatal	Serious	Slight	Total
1970	758	7860	13 515	22 133
1975	699	6912	13 041	20 652
1980	644	7218	13 926	21 788
1985	550	6507	13 587	20 644
1990	491	5237	14 443	20 171
1995	361	4071	12 102	16 534
2000	297	3007	11 825	15 129
2005	264	2250	10 922	13 436

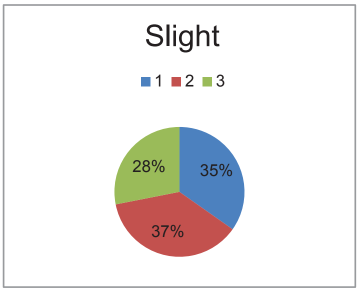


From the bar graph, we notice that the number of injuries are decreasing year on year.

b) Key: 1 denotes the year 1970, 2 the year 1990, and 3 the year 2005.

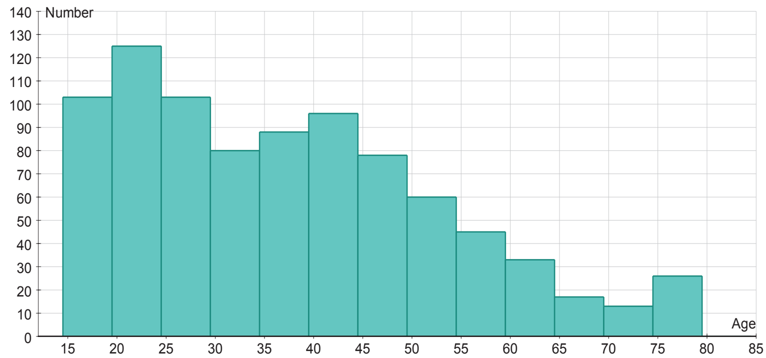






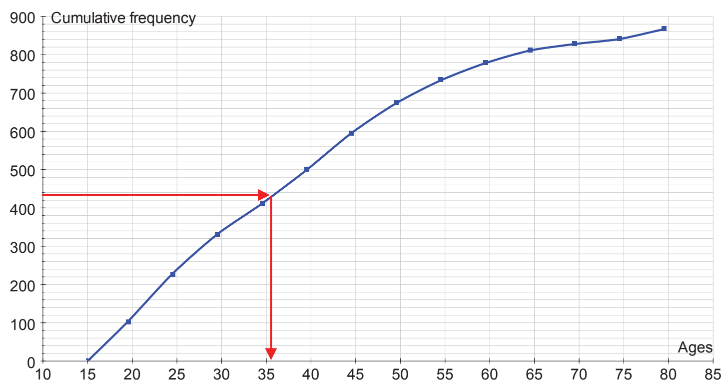
10 a)

Age	Number
15–19	103
20–24	125
25–29	103
30–34	80
35–39	88
40–44	96
45–49	78
50–54	60
55–59	45
60–64	33
65–69	17
70–74	13
75–79	26



b) To estimate the mean value, we need to use the midpoints of the intervals (17, 22, 27, ..., 77) and the corresponding frequencies: $\bar{x} = \dfrac{\sum x_i f_i}{\sum f_i} = 37.61.$

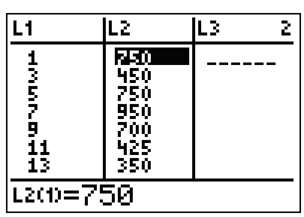
c)

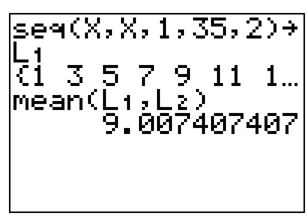


Since there were a total of 867 casualties, we need to draw a horizontal line at 433.5 to find an estimate of the median of the data. Our estimate is 36 years of age.

To answer questions 11–15, we are going to use our graphs from the previous exercise, together with calculator software.

11 Since there are 5400 patients, we need to draw a horizontal line at 2700 (on the cumulative frequency diagram) to estimate the median. Our estimate is 7.5 days.

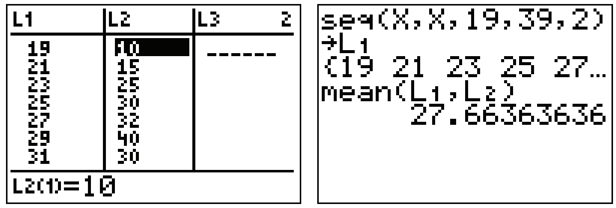




To estimate the mean, we use the midpoints of the intervals and the corresponding frequencies. So, the estimate of the mean is 9.01 days (correct to three significant figures).

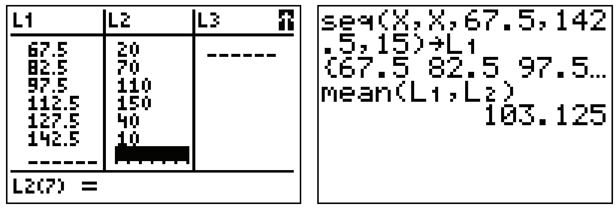
12 Since there are 220 recordings, we need to draw a horizontal line at 110 (on the cumulative frequency diagram) to estimate the median. Our estimate is 28 minutes.

To estimate the mean, we use the midpoints of the intervals and the corresponding frequencies. So, the estimate of the mean is 27.7 minutes (correct to three significant figures).



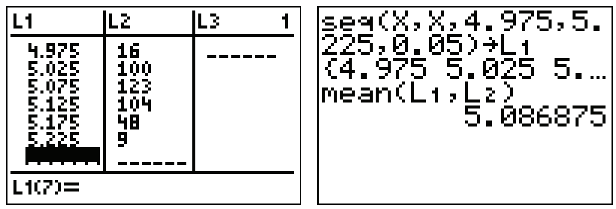
13 There are 400 cars and one of the cumulative frequencies is exactly 200; therefore, the median value is 105 km/h [(105, 200) is a point on the cumulative frequency diagram].

To estimate the mean, we use the midpoints of the intervals and the corresponding frequencies. So, the estimate of the mean is 103 km/h (correct to three significant figures).



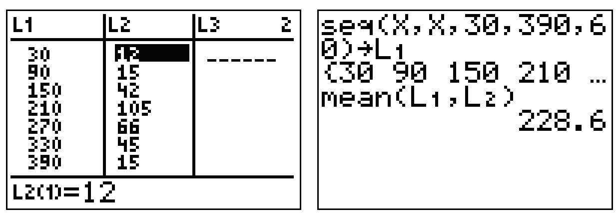
14 An estimate for the median (which corresponds to a cumulative frequency of 200) is 5.08.

To estimate the mean, we use the midpoints of the intervals and the corresponding frequencies. So, the estimate of the mean length is 5.09 (correct to three significant figures).



15 An estimate for the median (which corresponds to a cumulative frequency of 150) is 225 seconds.

To estimate the mean, we use the midpoints of the intervals and the corresponding frequencies. So, the estimate of the mean waiting time is 229 seconds (correct to three significant figures).



16 a) $\sum_{i=1}^{40} x_i = 1664 \Rightarrow \bar{x} = \dfrac{1664}{40} = 41.6$

b) $\sum_{i=1}^{20} (x_i - 20) = 1664 \Rightarrow \bar{x} = \dfrac{1664}{20} + 20 = 83.2 + 20 = 103.2$

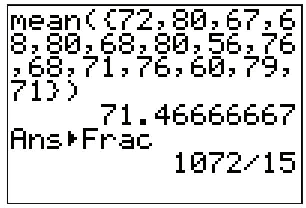
17 a) $\sum_{i=1}^{60} (x_i + 12) = 4404 \Rightarrow \bar{x} = \dfrac{4404}{60} - 12 = 73.4 - 12 = 61.4$

b) Average score of the whole group of 100 students $= \dfrac{61.4 \cdot 60 + 67.4 \cdot 40}{100} = 63.8$

Exercise 11.3

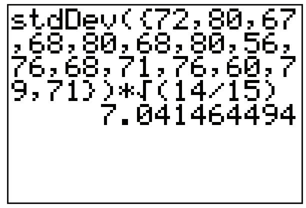
1 We are going to use a graphic display calculator. Since there are only 15 patients, we will use a simple list with 15 elements.

a) The mean pulse of the 15 patients is: $\bar{x} = \dfrac{1072}{15} \approx 71.5$ (to 3 s.f.).

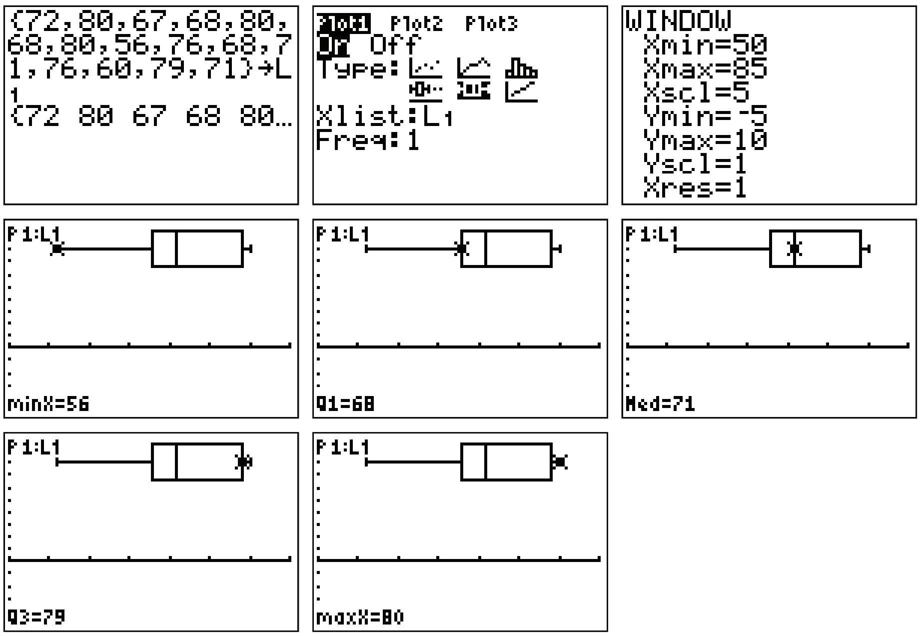


For the next part of the question, we will use the standard deviation feature directly from the List menu, but we need to adopt it since the calculator finds the unbiased estimation of the population standard deviation; therefore, we need to multiply it by $\sqrt{\dfrac{n-1}{n}}$.

The standard deviation is: $s_n = 7.04$ (to 3 s.f.).



b) We will input all the data into list L_1 and then plot the box-whisker diagram.



c) Since we can see that the data is skewed to the left, we need to check whether there are any outliers to the left. $IQR = Q_3 - Q_1 = 79 - 68 = 11$. The outliers lie 1.5IQR from the lower or upper quartile, so, in this problem, we calculate: $Q_1 - 1.5 \times IQR = 68 - 1.5 \times 11 = 51.5 < 56 = x_{min}$; therefore, there are no outliers.

2 **a)** For this question, we will use a spreadsheet. We input the data into a column and use the functions to find the following:

Mean value $\bar{x} = 162.6$.

Standard deviation $s_{n-1} = 23.35$.

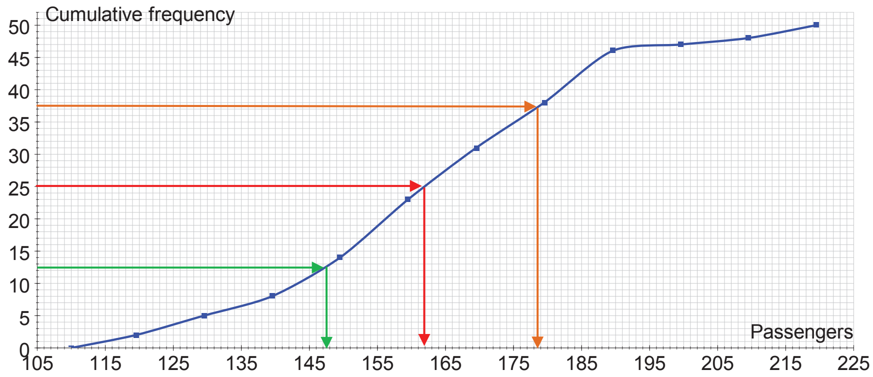
b)

11	7 9
12	5 6 7
13	0 8 9
14	1 2 3 6 7 9
15	0 3 3 4 4 5 6 8 9
16	0 2 3 3 4 5 6 8
17	1 3 4 4 7 8 9
18	0 2 2 5 5 7 7 9
19	8
20	9
21	0 8

Since there are 50 observations, we need to find the 25th and 26th observations and take their average. By counting the observations, we find the 25th and 26th observations are 162 and 163 respectively; therefore, the median is $\frac{162+163}{2} = 162.5$.

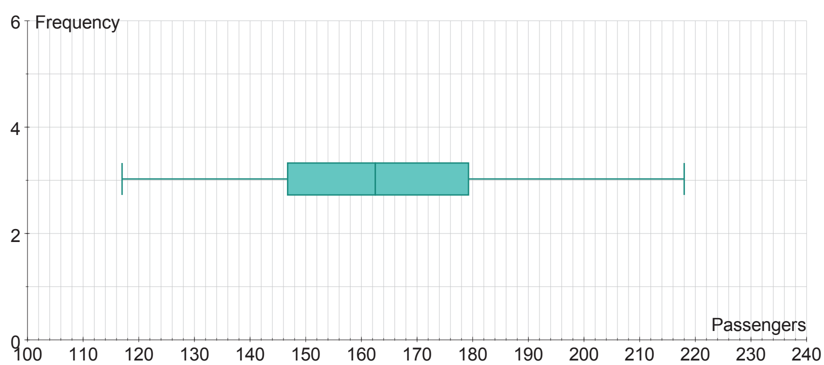
c) To draw the cumulative frequency curve, we need to group the data into suitable intervals.

Interval	Frequency
110–119	2
120–129	3
130–139	3
140–149	6
150–159	9
160–169	8
170–179	7
180–189	8
190–199	1
200–209	1
210–219	2



The first and third quartiles correspond to the cumulative frequencies of 12.5 and 37.5 respectively; therefore, an estimate for Q_1 is 147.5 and Q_3 is 178.5. An estimate for the median (which corresponds to a cumulative frequency of 25) is 162.

It is also possible to use the raw data that we entered into the spreadsheet to draw the box diagram. In that case, the measures are slightly more accurate.



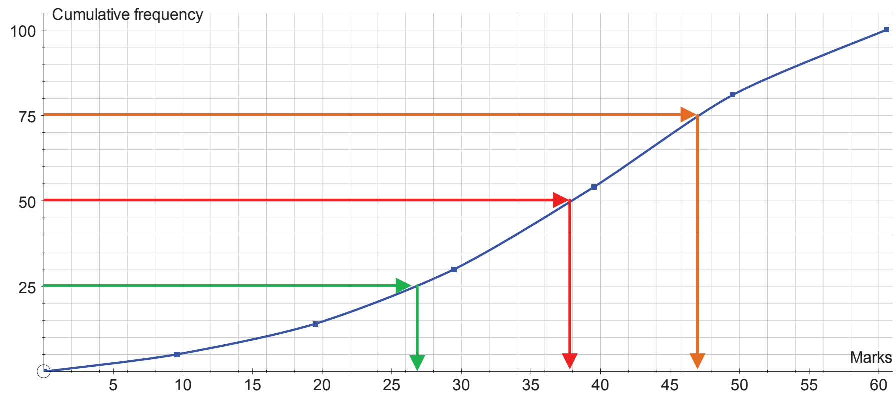
d) $IQR = Q_3 - Q_1 = 178.5 - 147.5 = 31$. If there are any outliers, they would be outside of the interval $[Q_1 - 1.5IQR, Q_3 + 1.5IQR] \Rightarrow [101, 225]$, which includes the whole range of the number of passengers; therefore, there are no outliers.

e) The empirical rule states that the whole range should lie within three standard deviations of the mean value. That will give an even larger segment $[92, 233]$; therefore, there are no outliers.

3

Marks	0–9	10–19	20–29	30–39	40–49	50–60
Number of students	5	9	16	24	27	19

a)

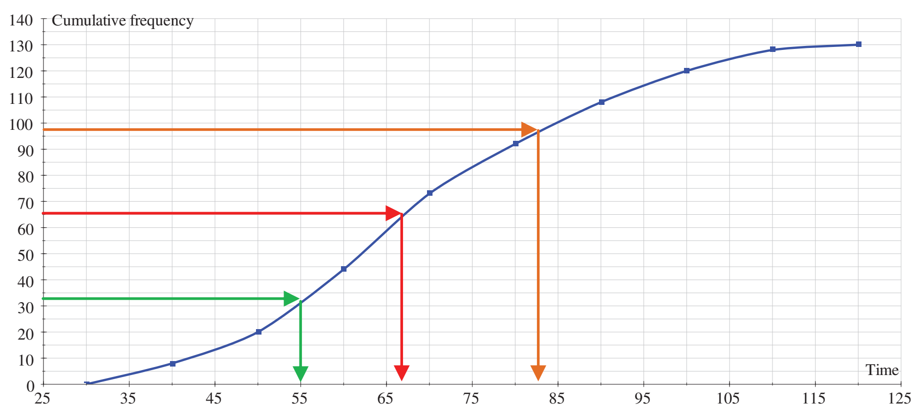


b) We draw horizontal lines at 50, 25 and 75 to find estimates for the median and quartiles. The estimates are as follows: the median is 38, the lower quartile is 27, and the upper quartile is 47.

4

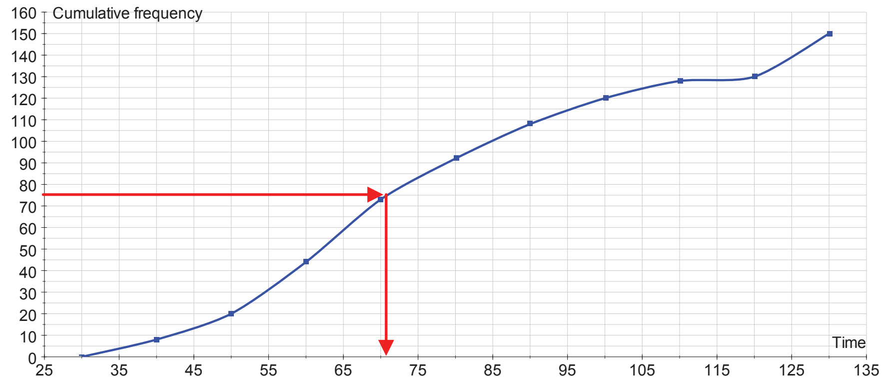
Time	30–40	40–50	50–60	60–70	70–80	80–90	90–100	100–110	110–120
Number of students	8	12	24	29	19	16	12	8	2

a)



b) We draw horizontal lines at 65, 32.5 and 97.5 to find estimates for the median and quartiles. The estimates are as follows: the median is 67, the lower quartile is 55, and the upper quartile is 83. Therefore, $IQR = Q_3 - Q_1 = 83 - 55 = 28$.

c)



The horizontal line at 75 gives us an estimate of 71 minutes for the median finishing time for all 150 students.

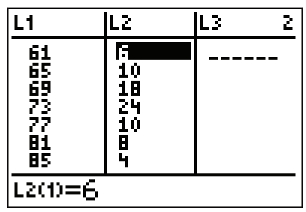
5 $\bar{x} = \dfrac{26 \times 22 + 84 \times 32}{110} = \dfrac{326}{11} \approx 29.6$

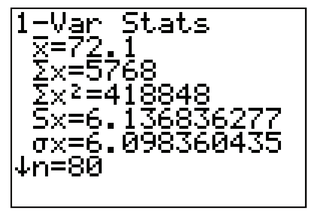
6

Score	59–63	63–67	67–71	71–75	75–79	79–83	83–87
Number of students	6	10	18	24	10	8	4

a) To find the mean and standard deviation of the given data, we need to use the midpoints of the intervals.

The mean value is 72.1 and the standard deviation is 6.10 (correct to three significant figures).

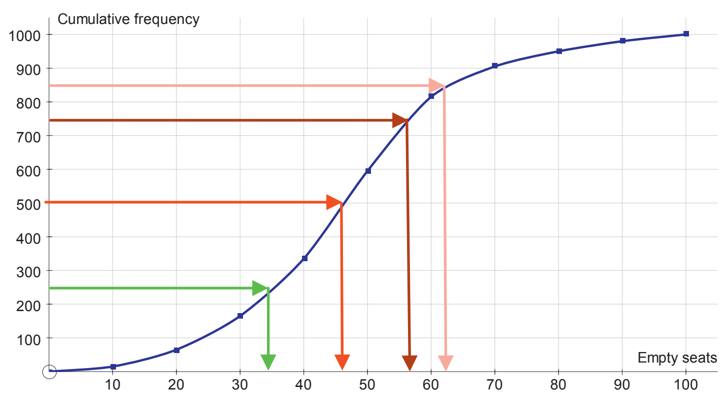




7 a)

Number of empty seats	$x \leqslant 10$	$x \leqslant 20$	$x \leqslant 30$	$x \leqslant 40$	$x \leqslant 50$	$x \leqslant 60$	$x \leqslant 70$	$x \leqslant 80$	$x \leqslant 90$	$x \leqslant 100$
Days	15	65	165	335	595	815	905	950	980	1000

b)



c) i) To find an estimate of the median number of empty seats, we need to look at the cumulative frequency of 500 (see diagram above). The estimate is 47.

ii) For the first and third quartiles, we look at the cumulative frequencies of 250 and 750 respectively. Hence, an estimate for the first quartile is 35, whilst an estimate for the third quartile is 58. The IQR is the difference between the third and first quartiles, which is 23.

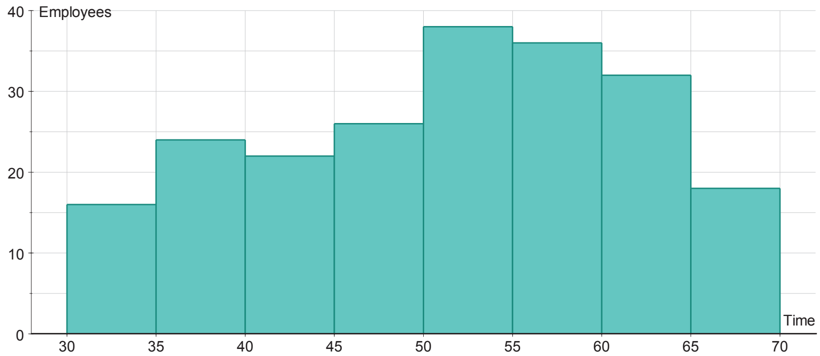
iii) From the previous estimates, we can see that the number of bumper days was about 250.

iv) The highest 15% corresponds to the cumulative frequency of 850. An estimate for the number of empty seats for that cumulative frequency is 63.

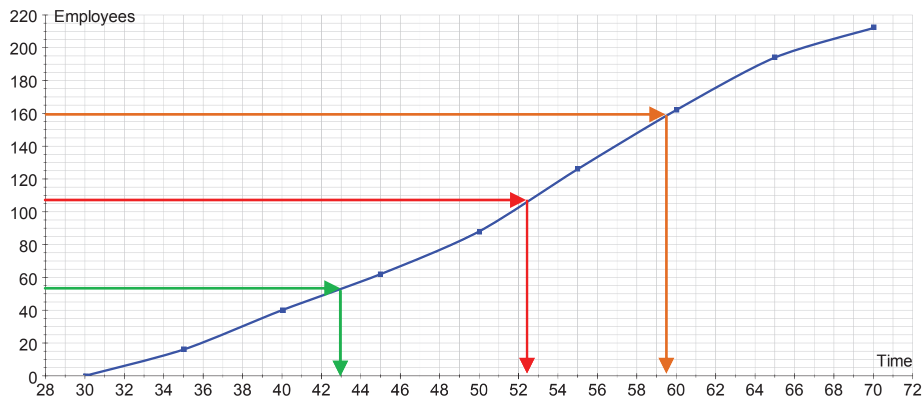
8

Time	Number of employees
30–35	16
35–40	24
40–45	22
45–50	26
50–55	38
55–60	36
60–65	32
65–70	18

a)



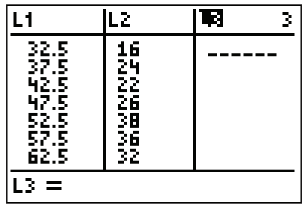
b)

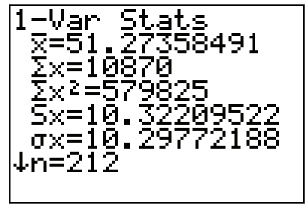


We draw horizontal lines at 106, 53 and 159 to find estimates for the median and quartiles. The estimates are as follows: the median is 52.5, the lower quartile is 43, and the upper quartile is 59.5. Therefore, IQR = 59.5 – 43 = 16.5.

c) To estimate the mean and standard deviation, we will use the midpoints of the intervals.

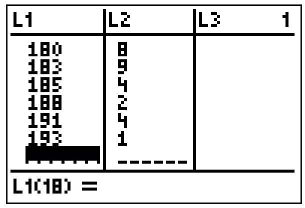
The mean value is 51.3 and the standard deviation is 10.3 (correct to three significant figures).

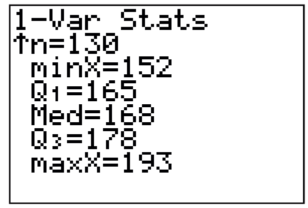




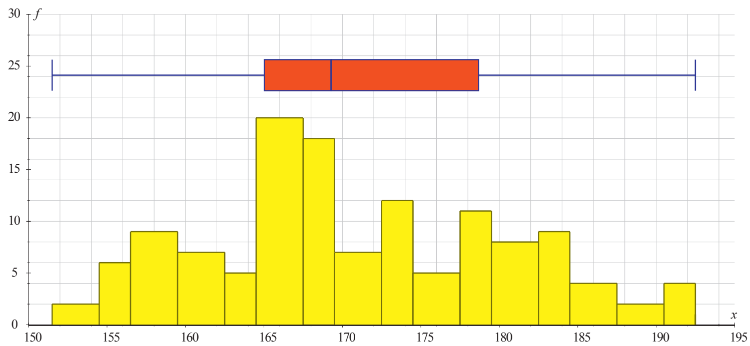
9 a) We can solve this problem by using a calculator.

The minimum value is 152 and the maximum value is 193. The lower and upper quartiles are 165 and 178 respectively, whilst the median value is 168.

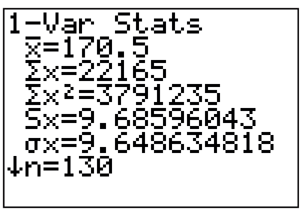




b)



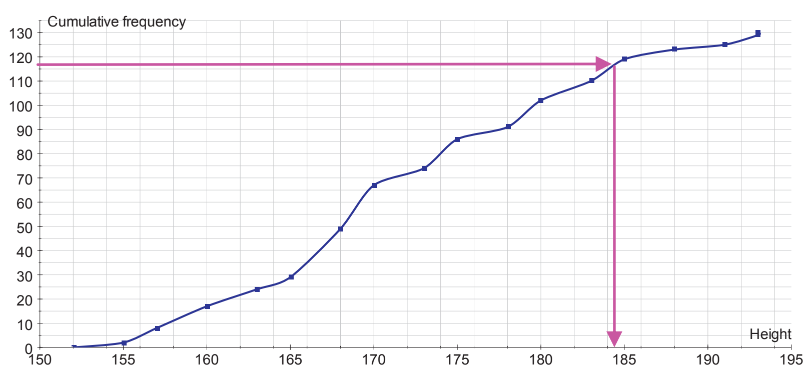
c)



The mean value is 170.5 and the standard deviation is 9.65, correct to three significant figures.

d) The players' heights have a very wide spread. There are a few short players and there are a few extremely tall players. The heights are skewed slightly to the right, with no apparent outliers. There is a very small range from the first quartile to the median. 25% of all the players have heights within those 3 cm, from 165–168 cm.

e)



The 90th percentile of 130 players corresponds to a cumulative frequency of 117. Using the graph, we estimate the 90th percentile as 184.5 cm.

f) $\bar{x} = \dfrac{22\,165 + 182 \times 10}{130 + 10} = \dfrac{23\,985}{140} \approx 171$, correct to the nearest cm.

10 a) The mean is not going to change since the added observation has the same value as the mean of the previous observations. Therefore, the new mean is also 12.

b) $\bar{x} = \dfrac{9 \cdot 11 + 21}{10} = 12$; the new mean is 12.

c) $21 = \dfrac{9 \times 11 + x}{10} \Rightarrow 210 = 99 + x \Rightarrow x = 111$; therefore, the last observation is 111.

11 If the mean of all 10 data points is 30, then $\sum x = 30 \times 10 = 300$.

a) If the value of 25 was incorrectly entered as 15, that means the total sum should increase by 10; therefore, the correct mean value is: $\bar{x} = \dfrac{310}{10} = 31$.

b) Since the added value is greater than the mean value, the mean is going to increase. The new mean is: $\bar{x} = \dfrac{310 + 32}{10 + 1} = \dfrac{342}{11} \approx 31.1$.

12 $\bar{x} = \dfrac{1}{2} \times 20 + \dfrac{1}{6} \times 40 + \dfrac{1}{3} \times 60 = \dfrac{110}{3} \approx 36.7$

13 $\dfrac{7 + 10 + 12 + 17 + 21 + x + y}{7} = 12 \Rightarrow 67 + x + y = 84 \Rightarrow x + y = 17$

$$\frac{7^2 + 10^2 + 12^2 + 17^2 + 21^2 + x^2 + y^2}{7} - 12^2 = \frac{172}{7} \Rightarrow \frac{15 + x^2 + y^2}{7} = \frac{172}{7} \Rightarrow x^2 + y^2 = 157$$

To solve the simultaneous equations, we will use the substitution method by expressing the variable y (from the first equation) in terms of x.

$$\begin{cases} x + y = 17 \\ x^2 + y^2 = 157 \end{cases} \Rightarrow \begin{cases} y = 17 - x \\ x^2 + (17 - x)^2 = 157 \end{cases} \Rightarrow \begin{cases} y = 17 - x \\ 2x^2 - 34x + 289 = 157 \end{cases} \Rightarrow \begin{cases} y = 17 - x \\ x^2 - 17x + 66 = 0 \end{cases} \Rightarrow$$

$$\begin{cases} y = 17 - x \\ x = 6 \text{ or } x = 11 \end{cases} \Rightarrow \begin{cases} y = 11 \text{ or } \cancel{y = 6} \\ x = 6 \text{ or } \cancel{x = 11} \end{cases}$$

Note that we have discarded one solution because of the condition that $x < y$.

14 $\displaystyle\sum_{i=1}^{25} x_i = 278 \Rightarrow \bar{x} = \frac{278}{25} = 11.12$

$\displaystyle\sum_{i=1}^{25} x_i^2 = 3682 \Rightarrow s_n^2 = \frac{3682}{25} - 11.12^2 = 23.6256$

To answer questions 15–19, we are going to use our tables and graphs from the previous exercises, together with calculator and computer software.

15 Estimates of the upper and lower quartiles are 12 and 4.5 respectively; therefore, IQR = 7.5. The standard deviation is 6.63 (to 3 s.f.).

16 Estimates of the upper and lower quartiles are 30.5 and 24.5 respectively; therefore, IQR = 6. The standard deviation is 4.46 (to 3 s.f.).

17 Estimates of the upper and lower quartiles are 91 and 114 respectively; therefore, IQR = 23. The standard deviation is 16.7 (to 3 s.f.).

18 Estimates of the upper and lower quartiles are 5.045 and 5.4 respectively; therefore, IQR = 0.355. The standard deviation is 0.0569 (to 3 s.f.).

19 Estimates of the upper and lower quartiles are 280 and 180 respectively; therefore, IQR = 100. The standard deviation is 82.1 (to 3 s.f.).

Chapter 12

Exercise 12.1 and 12.2

1 **a)** $S = \{\text{left-handed, right-handed}\}$

b) $S = \{h \in \mathbb{R} : 50 < h < 250\}$, where height ($h$) is in centimetres.

c) $S = \{t \in \mathbb{R} : 0 < t < 240\}$, if we decide that the night starts at 20:00 and we don't study after midnight.

2 $S = \{(1, \text{h}), (2, \text{h}), (3, \text{h}), (4, \text{h}), (5, \text{h}), (6, \text{h}), (1, \text{t}), (2, \text{t}), (3, \text{t}), (4, \text{t}), (5, \text{t}), (6, \text{t})\}$

3 **a)** $S = \{2\clubsuit, 3\clubsuit, 4\clubsuit, 5\clubsuit, 6\clubsuit, 7\clubsuit, 8\clubsuit, 9\clubsuit, 10\clubsuit, J\clubsuit, Q\clubsuit, K\clubsuit, A\clubsuit, 2\spadesuit, 3\spadesuit, 4\spadesuit, 5\spadesuit, 6\spadesuit, 7\spadesuit, 8\spadesuit, 9\spadesuit, 10\spadesuit, J\spadesuit, Q\spadesuit, K\spadesuit, A\spadesuit, 2\diamondsuit, 3\diamondsuit, 4\diamondsuit, 5\diamondsuit, 6\diamondsuit, 7\diamondsuit, 8\diamondsuit, 9\diamondsuit, 10\diamondsuit, J\diamondsuit, Q\diamondsuit, K\diamondsuit, A\diamondsuit, 2\heartsuit, 3\heartsuit, 4\heartsuit, 5\heartsuit, 6\heartsuit, 7\heartsuit, 8\heartsuit, 9\heartsuit, 10\heartsuit, J\heartsuit, Q\heartsuit, K\heartsuit, A\heartsuit\}$

b) We need to list all the possible pairs from a deck of 52 cards, which is many more than the listing in **a**, so we are just going to initiate a possible listing.

$S = \{(2\clubsuit, 3\clubsuit), (2\clubsuit, 4\clubsuit), \ldots(2\clubsuit, A\clubsuit), (2\clubsuit, 2\spadesuit), (2\clubsuit, 3\spadesuit), \ldots(2\clubsuit, A\spadesuit), (2\clubsuit, 2\diamondsuit), (2\clubsuit, 3\diamondsuit), \ldots (2\clubsuit, A\diamondsuit), (2\clubsuit, 2\heartsuit), (2\clubsuit, 3\heartsuit), \ldots(2\clubsuit, A\heartsuit), (3\clubsuit, 4\clubsuit), (3\clubsuit, 5\clubsuit), \ldots(3\clubsuit, A\clubsuit), (3\clubsuit, 2\spadesuit), (3\clubsuit, 3\spadesuit), \ldots(3\clubsuit, A\spadesuit), (3\clubsuit, 2\diamondsuit), (3\clubsuit, 3\diamondsuit), \ldots(3\clubsuit, A\diamondsuit), (3\clubsuit, 2\heartsuit), (3\clubsuit, 3\heartsuit), \ldots(3\clubsuit, A\heartsuit), \ldots(K\heartsuit, A\heartsuit), \ldots\}$

c) In the first experiment there are 52 outcomes, as there are 52 cards in the deck. In the second experiment there are $\frac{52 \cdot 51}{2} = \frac{2652}{2} = 1326$ outcomes, since the order of the pair doesn't matter.

4 **a)** Since Tim tossed 20 coins 10 times, there are 200 possible outcomes. The sum of all the number of heads that appeared in the experiment is the number of favourable outcomes.

$11 + 9 + 10 + 8 + 13 + 9 + 6 + 7 + 10 + 11 = 94 \Rightarrow \text{P}(H) = \frac{94}{200} = \frac{47}{100} = 0.47$

b) Tim should expect any number between 0 and 20.

c) If he tossed 20 coins 1000 times, we would expect heads to be obtained exactly half the time; therefore, 10 000 heads.

5 **a)** $S = \{(1, 1), (1, 2), (1, 3), (1, 4), (2, 1), (2, 2), (2, 3), (2, 4), (3, 1), (3, 2), (3, 3), (3, 4), (4, 1), (4, 2), (4, 3), (4, 4)\}$

b) We need to look at each pair from **a** and add 1 to the sum of the components. So, $S = \{3, 4, 5, 6, 7, 8, 9\}$.

6 **a)** Since we are replacing the first ball drawn, there are three different colours possible for the first ball drawn **and** three different colours possible for the second ball drawn.
$S = \{(\text{b}, \text{b}), (\text{b}, \text{g}), (\text{b}, \text{y}), (\text{g}, \text{b}), (\text{g}, \text{g}), (\text{g}, \text{y}), (\text{y}, \text{b}), (\text{y}, \text{g}), (\text{y}, \text{y})\}$

b) $A = \{(\text{y}, \text{b}), (\text{y}, \text{g}), (\text{y}, \text{y})\}$

c) $B = \{(\text{b}, \text{b}), (\text{g}, \text{g}), (\text{y}, \text{y})\}$

7 **a)** Since we do not replace the first ball drawn, there are only two colours possible for the second ball drawn.
$S = \{(\text{b}, \text{g}), (\text{b}, \text{y}), (\text{g}, \text{b}), (\text{g}, \text{y}), (\text{y}, \text{b}), (\text{y}, \text{g})\}$

b) $A = \{(y, b), (y, g)\}$

c) $B = \varnothing$

8 a) $S = \{(h, h, h), (h, h, t), (h, t, h), (t, h, h), (h, t, t), (t, h, t), (t, t, h), (t, t, t)\}$

b) $A = \{(h, h, h), (h, h, t), (h, t, h), (t, h, h)\}$

9 Let H = Hungary, I = Italy, b = boat, d = drive, f = fly, t = train.

Country and mode of travel: $S = \{(I, f), (I, d), (I, t), (H, d), (H, b)\}$

Fly to destination: $S = \{(I, f)\}$

10 a) $S = \{(0, g), (0, f), (0, s), (0, c), (1, g), (1, f), (1, s), (1, c)\}$

b) $A = \{(0, s), (0, c)\}$

c) $B = \{(0, g), (0, f), (1, g), (1, f)\}$

In this case we are not concerned as to whether the patient is insured or not.

d) $C = \{(1, g), (1, f), (1, s), (1, c)\}$

11 The study is investigating three different characteristics. There are 2 classifications for gender, 3 classifications for drinking habits, and 4 classifications for marital status; so, there are $2 \cdot 3 \cdot 4 = 24$ different classifications for a person in this study.

a) $$S = \{(G_1, K_1, M_1), (G_1, K_1, M_2), (G_1, K_1, M_3), (G_1, K_1, M_4), (G_1, K_2, M_1), (G_1, K_2, M_2), (G_1, K_2, M_3), (G_1, K_2, M_4), (G_1, K_3, M_1), (G_1, K_3, M_2), (G_1, K_3, M_3), (G_1, K_3, M_4), (G_2, K_1, M_1), (G_2, K_1, M_2), (G_2, K_1, M_3), (G_2, K_1, M_4), (G_2, K_2, M_1), (G_2, K_2, M_2), (G_2, K_2, M_3), (G_2, K_2, M_4), (G_2, K_3, M_1), (G_2, K_3, M_2), (G_2, K_3, M_3), (G_2, K_3, M_4)\}$$

b) Set A is defined as 'the person is a male'; therefore, it consists of all the triplets containing G_2.

$$A = \{(G_2, K_1, M_1), (G_2, K_1, M_2), (G_2, K_1, M_3), (G_2, K_1, M_4), (G_2, K_2, M_1), (G_2, K_2, M_2), (G_2, K_2, M_3), (G_2, K_2, M_4), (G_2, K_3, M_1), (G_2, K_3, M_2), (G_2, K_3, M_3), (G_2, K_3, M_4)\}$$

Set B is defined as 'the person drinks'; therefore, it consists of all the triplets containing K_2 or K_3.

$$B = \{(G_1, K_2, M_1), (G_1, K_2, M_2), (G_1, K_2, M_3), (G_1, K_2, M_4), (G_1, K_3, M_1), (G_1, K_3, M_2), (G_1, K_3, M_3), (G_1, K_3, M_4), (G_2, K_2, M_1), (G_2, K_2, M_2), (G_2, K_2, M_3), (G_2, K_2, M_4), (G_2, K_3, M_1), (G_2, K_3, M_2), (G_2, K_3, M_3), (G_2, K_3, M_4)\}$$

Set B can also be described as the set that consists of all the triplets not containing K_1.

Set C is defined as 'the person is single'; therefore, it consists of all the triplets containing M_2.

$$C = \{(G_1, K_1, M_2), (G_1, K_2, M_2), (G_1, K_3, M_2), (G_2, K_1, M_2), (G_2, K_2, M_2), (G_2, K_3, M_2)\}$$

c) Set $A \cup B$ can be described as the set that consists of male persons or persons who drink.

Set $A \cap C$ can be described as the set that consists of single male persons.

Set C' can be described as the set that consists of non-single persons.

Set $A \cap B \cap C$ can be described as the set that consists of single male persons who drink.

Set $A' \cap B$ can be described as the set that consists of female persons who drink.

12 a) Since we are taking four cars at a time, we will be recording quadruplets. For example, (L, L, L, L), (R, R, R, S), (L, S, R, S), and so on. Every car leaving the highway has three options; therefore, there are $3^4 = 81$ different quadruplets.

b) If all cars go in the same direction, we have (L, L, L, L), (R, R, R, R) and (S, S, S, S).

c) If only two cars turn right, the remaining two cars will either turn left or go straight: (R, R, L, L), (R, R, S, S), (R, R, L, S), and now we have to find the remaining permutations. For example, let us take the first quadruplet and its permutations: (R, L, R, L), (R, L, L, R), (L, R, L, R), (L, R, R, L), (L, L, R, R). The same pattern works for the second quadruplet. For these two quadruplets, there are 12 different permutations altogether. The last quadruplet has more permutations since we have three possible ways of leaving the highway. There are 12 possibilities. So, the remaining quadruplets are (R, R, S, L), (R, L, R, S), (R, S, R, L), (R, L, S, R), (R, S, L, R), (L, R, R, S), (S, R, R, L), (L, R, S, R), (S, R, L, R), (L, S, R, R), (S, L, R, R). Hence, there are a total of 24 outcomes where only two cars turn right.

d) Only two cars going in the same direction contains the previous part, and there are two further ways of the cars going in the same direction: L, L and S, S. So, altogether, there are $3 \cdot 24 = 72$ different outcomes.

13 Since we have to look at three different components, we will be recording triplets. The first component, size of the vehicle, has three different classifications, whilst the remaining two components have just two different outcomes. Therefore, there are $3 \cdot 2 \cdot 2 = 12$ different triplets.

a) $U = \{(T, SY, O), (T, SY, F), (T, SN, O), (T, SN, F), (B, SY, O), (B, SY, F), (B, SN, O), (B, SN, F), (C, SY, O), (C, SY, F), (C, SN, O), (C, SN, F)\}$

b) $SY = \{(T, SY, O), (T, SY, F), (B, SY, O), (B, SY, F), (C, SY, O), (C, SY, F)\}$

c) $C = \{(C, SY, O), (C, SY, F), (C, SN, O), (C, SN, F)\}$

d) $C \cap SY = \{(C, SY, O), (C, SY, F)\}$

$C' = \{(T, SY, O), (T, SY, F), (T, SN, O), (T, SN, F), (B, SY, O), (B, SY, F), (B, SN, O), (B, SN, F)\}$

$C \cup SY = \{(T, SY, O), (T, SY, F), (B, SY, O), (B, SY, F), (C, SY, O), (C, SY, F), (C, SN, O), (C, SN, F)\}$

14 a) Since there are three components and each can work or not, there are $2 \cdot 2 \cdot 2 = 8$ different outcomes.

$U = \{(0, 0, 0), (0, 0, 1), (0, 1, 0), (1, 0, 0), (0, 1, 1), (1, 0, 1), (1, 1, 0), (1, 1, 1)\}$

b) $X = \{(0, 1, 1), (1, 0, 1), (1, 1, 0)\}$

c) $Y = \{(0, 1, 1), (1, 0, 1), (1, 1, 0), (1, 1, 1)\}$

d) $Z = \{(1, 0, 1), (1, 1, 0), (1, 1, 1)\}$

e) $Z' = \{(0, 0, 0), (0, 0, 1), (0, 1, 0), (1, 0, 0), (0, 1, 1)\}$

$X \cup Z = \{(0, 1, 1), (1, 0, 1), (1, 1, 0), (1, 1, 1)\}$

$X \cap Z = \{(1, 0, 1), (1, 1, 0)\}$

$Y \cup Z = \{(0, 1, 1), (1, 0, 1), (1, 1, 0), (1, 1, 1)\} = Y$

$Y \cap Z = \{(1, 0, 1), (1, 1, 0), (1, 1, 1)\} = Z$

15 **a)** $U = \{1, 2, 31, 32, 41, 42, 51, 52, 341, 342, 431, 432, 351, 352, 531, 532, 451, 452, 541, 542, 3451, 3452, 3541, 3542, 4351, 4352, 4531, 4532, 5341, 5342, 5431, 5432\}$.
There are 32 different outcomes altogether.

b) $A = \{31, 32, 41, 42, 51, 52\}$. There are six possible outcomes.

c) $B = \{31, 32, 41, 42, 51, 52, 341, 342, 431, 432, 351, 352, 531, 532, 451, 452, 541, 542, 3451, 3452, 3541, 3542, 4351, 4352, 4531, 4532, 5341, 5342, 5431, 5432\}$.
There are 30 possible outcomes.

d) $C = \{1, 31, 41, 51, 341, 431, 351, 531, 451, 541, 3451, 3541, 4351, 4531, 5341, 5431\}$. There are 16 possible outcomes.

Exercise 12.3

1 **a)** There are six multiples of 3 from 1 to 20. So, $P(A) = \frac{6}{20} = \frac{3}{10}$.

Note: The number of multiples can be obtained by using the greatest integer function:
$\left\lfloor \frac{20}{3} \right\rfloor = \lfloor 6.67 \rfloor = 6$.

b) We will use the complementary event that the number is a multiple of 4. There are five multiples of 4 from 1 to 20. So, $P(B) = 1 - P(B') = 1 - \frac{5}{20} = 1 - \frac{1}{4} = \frac{3}{4}$.

2 **a)** $P(A') = 1 - P(A) = 1 - 0.37 = 0.63$

b) $P(A \cup A') = P(S) = 1$ or $P(A \cup A') = P(A) + P(A') = 0.37 + 0.63 = 1$

3 **a)** **i)** There is one ace of hearts in a deck of cards, so: $P(A_i) = \frac{1}{52}$.

ii) There is one ace of hearts and 13 spades in a deck, so: $P(A_{ii}) = \frac{1+13}{52} = \frac{14}{52} = \frac{7}{26}$.

iii) The ace of hearts is already included in the 13 hearts in a deck, so we only need to add the three remaining aces. So, $P(A_{iii}) = \frac{13+3}{52} = \frac{16}{52} = \frac{4}{13}$.

iv) There are 12 face cards. We will use the probability of the complementary event:
$P(A_{iv}) = 1 - P(A_{iv}') = 1 - \frac{12}{52} = 1 - \frac{3}{13} = \frac{10}{13}$

b) As the drawn card is not replaced, there are 51 cards remaining in the deck.

i) $P(B_i) = \frac{1}{51}$

ii) $P(B_{ii}) = 1 - P(B_{ii}') = 1 - \frac{12}{51} = 1 - \frac{4}{17} = \frac{13}{17}$

c) As the drawn card is replaced in the deck, there are no influences on the drawing of the next card. Therefore, the probability is the same as in **a**.

i) $P(C_i) = \frac{1}{52}$

ii) $P(C_{ii}) = 1 - P(C_{ii}') = 1 - \frac{12}{52} = 1 - \frac{3}{13} = \frac{10}{13}$

4 The total number of students is 30. Looking at the table, we obtain:

a) $\mathrm{P}(A) = \frac{4+12+8}{30} = \frac{24}{30} = \frac{4}{5}$

b) $\mathrm{P}(B) = \frac{8+3}{30} = \frac{11}{30}$

c) All of the students studied less than six hours; therefore, $\mathrm{P}(C) = 1$.

5 There are 12 different possible outcomes: 6 possible outcomes for the die and 2 possible outcomes for the coin, so, by the counting principle, we obtain 12.

a) There are three outcomes that are greater than 3, i.e. 4, 5 or 6, and since it doesn't matter whether a head or a tail is obtained we get: $\mathrm{P}(A) = \frac{6}{12} = \frac{1}{2}$.

b) Obtaining a head and a 6 is just one possible outcome out of 12; therefore, $\mathrm{P}(B) = \frac{1}{12}$.

6 Let the probability of any other number than 1 appear be x, so the probability of 1 is $2x$. The sum of all the probabilities is 1; therefore, $\sum p_i = 1 \Rightarrow 7x = 1 \Rightarrow x = \frac{1}{7}$.

a) $\mathrm{P}(A) = \frac{1}{7}$

b) The odd numbers are 1, 3 and 5, so $\mathrm{P}(B) = \frac{2+1+1}{7} = \frac{4}{7}$.

7 **a)** **i)** There are 6 possible outcomes for the first die and 6 possible outcomes for the second die; therefore, there are a total of 36 possible outcomes.

$$\begin{aligned} S = \{&(1,1),(1,2),(1,3),(1,4),(1,5),(1,6),(2,1),(2,2),(2,3),(2,4),(2,5),(2,6),\\ &(3,1),(3,2),(3,3),(3,4),(3,5),(3,6),(4,1),(4,2),(4,3),(4,4),(4,5),(4,6),\\ &(5,1),(5,2),(5,3),(5,4),(5,5),(5,6),(6,1),(6,2),(6,3),(6,4),(6,5),(6,6)\} \end{aligned}$$

ii) There are six possible pairs with equal numbers; therefore, $\mathrm{P}(A) = \frac{6}{36} = \frac{1}{6}$.

iii) Looking at the sample space, we notice that there are eight such outcomes:

$$B = \{(1,3),(2,4),(3,1),(3,5),(4,2),(4,6),(5,3),(6,4)\} \Rightarrow \mathrm{P}(B) = \frac{8}{36} = \frac{2}{9}$$

iv) This event is complementary to the event in **ii**, so $\mathrm{P}(C) = 1 - \frac{1}{6} = \frac{5}{6}$.

b) The probability distribution for the sum of the numbers that appear is shown in the table.

X (sum)	2	3	4	5	6	7	8	9	10	11	12
P(X)	$\frac{1}{36}$	$\frac{2}{36}$	$\frac{3}{36}$	$\frac{4}{36}$	$\frac{5}{36}$	$\frac{6}{36}$	$\frac{5}{36}$	$\frac{4}{36}$	$\frac{3}{36}$	$\frac{2}{36}$	$\frac{1}{36}$

i) There is no sum equal to 1 and therefore $\mathrm{P}(D) = 0$.

ii) Looking at the table, we can read that $\mathrm{P}(E) = \frac{4}{36} = \frac{1}{9}$.

iii) Looking at the table, we can read that $\mathrm{P}(F) = \frac{5}{36}$.

iv) The largest sum is 12 and therefore $\mathrm{P}(G) = 0$.

8 **a)** Since the sum of all the probabilities is equal to 1,
$P(AB|US) = 1-(0.43+0.41+0.12) = 1-0.96 = 0.04$.

b) Since the events are mutually exclusive, the probability of their union is the sum of their probabilities. Therefore, $P(O \cup B|US) = 0.43+0.12 = 0.55$.

c) Since we have to independently select two people, their probabilities should be multiplied; therefore, $P(USC) = P(O|US)\cdot P(O|C) = 0.43\cdot 0.36 = 0.1548 = 0.155$, correct to three significant figures.

d) Since we have to independently select three people, their probabilities should be multiplied; therefore, $P(O) = P(O|US)\cdot P(O|C)\cdot P(O|R) = 0.43\cdot 0.36\cdot 0.39 = 0.060\,372 = 0.0604$, correct to three significant figures.

e) Firstly, we need to find the probability of type B in Russia:
$P(B|R) = 1-(0.39+0.34+0.09) = 1-0.82 = 0.18$

Similarly, as in **d**, we need to calculate the probability of only one blood type:

$P(A) = P(A|US)\cdot P(A|C)\cdot P(A|R) = 0.41\cdot 0.27\cdot 0.34 = 0.037\,638 = 0.0376$

$P(B) = P(B|US)\cdot P(B|C)\cdot P(B|R) = 0.12\cdot 0.26\cdot 0.18 = 0.056\,16 = 0.005\,62$

$P(AB) = P(AB|US)\cdot P(AB|C)\cdot P(AB|R) = 0.04\cdot 0.11\cdot 0.09 = 0.000\,396$

$P(S) = P(O)+P(A)+P(B)+P(AB) = 0.104\,022 = 0.104$, correct to three significant figures.

9 **a)** Yes, since the sum of all the probabilities is 1.

b) No, since four mutually exclusive events are given and their sum exceeds 1.

c) No. There is the same number of cards in each suit, but, by looking at the probability distribution, we notice that one heart and one diamond are missing, whilst we have two extra spades.

10 **a)** Since the sum of all the probabilities is 1, $P(O) = 1-(0.58+0.24+0.12) = 1-0.94 = 0.06$.

b) We need to use the complementary event, so $P(G') = 1-0.58 = 0.42$.

c) $P(GG) = 0.58\cdot 0.58 = 0.3364 = 0.336$, correct to three significant figures.

d) The two Swiss that we select could have German as their mother tongue, French as their mother tongue, or Italian as their mother tongue. Therefore,
$P(GG)+P(FF)+P(II)+P(OO) = 0.58^2+0.24^2+0.12^2+0.06^2 = 0.412$.

11 **a)** We use the probability of the complementary event, so:
$P(A) = 1-(0.165+0.142+0.075+0.081+0.209+0.145) = 1-0.817 = 0.183$.

b) Again, the complementary event will be used: $P(B) = 1-(0.165+0.145) = 1-0.31 = 0.69$.

12 $$f(x) = \frac{\binom{n}{x+1}}{\binom{n}{x}} = \frac{\dfrac{\cancel{n!}}{(x+1)\cancel{!}\,\cancel{(n-x-1)!}}}{\dfrac{\cancel{n!}}{\cancel{x!}\,(n-x)\cancel{!}}} = \frac{n-x}{x+1} < 1 \Rightarrow n-x < x+1 \Rightarrow x > \frac{n-1}{2}$$

13 **a)** $\binom{n}{2} = 190 \Rightarrow \dfrac{n(n-1)}{2} = 190 \Rightarrow n^2-n = 380 \Rightarrow (n-20)(n+19) = 0$. Since n must be a positive integer, the only possible solution is $n = 20$.

b) We know the symmetrical property of binomial coefficients, $\binom{n}{r} = \binom{n}{n-r}$. Therefore, $\binom{n}{4} = \binom{n}{8} \Rightarrow n = 4 + 8 = 12$.

14 There are 36 different outcomes that we will present as ordered pairs. The first component will represent the outcome of the white die, whilst the second component will represent the red die.

$$U = \{(1,1),(1,2),(1,3),(1,4),(1,5),(1,6),(2,1),(2,2)(2,3),(2,4),(2,5),(2,6),$$
$$(3,1),(3,2),(3,3),(3,4),(3,5),(3,6),(4,1),(4,2),(4,3),(4,4),(4,5),(4,6),$$
$$(5,1),(5,2),(5,3),(5,4),(5,5),(5,6),(6,1),(6,2),(6,3),(6,4),(6,5),(6,6)\}$$

a) By inspection, we can see that the sum is greater than 8 on ten different occasions; therefore, $\mathrm{P}(A) = \frac{10}{36} = \frac{5}{18}$.

b) A number greater than 4 means that 5 or 6 will appear on the first die. There are 12 such possible outcomes; therefore, $\mathrm{P}(B) = \frac{12}{36} = \frac{1}{3}$.

c) At most a total of 5 means that the sum could be 2, 3, 4 or 5. By inspection, we can see that again there are 10 possible pairs; therefore, $\mathrm{P}(C) = \frac{10}{36} = \frac{5}{18}$. We can see that it is the same as the probability of obtaining a sum greater than 8 and at most 5, which is true due to the symmetrical property of the outcomes.

15 a) There are 9 books on the shelf altogether. We select 3 books and one of the books must be the only thesaurus on the shelf. Since we don't care which of the remaining books will be selected, we have to select 2 out of 8 books.

$$\mathrm{P}(A) = \frac{\binom{8}{2} \times 1}{\binom{9}{3}} = \frac{28}{84} = \frac{1}{3}$$

b) We need to select 3 books from the 9 books on the shelf. Our preferred choice would be to select two novels (from five) and one science book (from three).

$$\mathrm{P}(B) = \frac{\binom{5}{2}\binom{3}{1}}{\binom{9}{3}} = \frac{10 \times 3}{84} = \frac{5}{14}$$

16 a) We have to select 5 cards (from 52) and we need 3 kings (from 4). That means that the remaining 2 drawn cards can be any from the 48 non-king cards.

$$\mathrm{P}(A) = \frac{\binom{4}{3}\binom{48}{2}}{\binom{52}{5}} = \frac{4 \times 1128}{2\,598\,960} = \frac{94}{54\,145} \approx 0.001\,74$$

b) Again, we need to select 5 cards altogether. We have to select 4 hearts (from 13) and 1 diamond (from 13).

$$P(B) = \frac{\binom{13}{4}\binom{13}{1}}{\binom{52}{5}} = \frac{715 \times 13}{2\,598\,960} = \frac{143}{39\,984} \approx 0.003\,58$$

17 a) We have to select 6 students from a class of 22. We would like to have 1 out of the 12 boys and 5 out of the 10 girls.

$$P(A) = \frac{\binom{12}{1}\binom{10}{5}}{\binom{22}{6}} = \frac{12 \times 252}{74\,613} = \frac{144}{3553} \approx 0.0405$$

b) It could be 4, 5 or 6 boys and 2, 1 or no girls respectively in the team of 6 students.

$$P(B) = \frac{\binom{12}{4}\binom{10}{2}}{\binom{22}{6}} + \frac{\binom{12}{5}\binom{10}{1}}{\binom{22}{6}} + \frac{\binom{12}{6}\binom{10}{0}}{\binom{22}{6}} = \frac{495 \times 45 + 792 \times 10 + 924}{74\,613} = \frac{943}{2261} \approx 0.417$$

18 In questions where conditions are given, we need to fulfil the conditions first and then we have to see what possibilities are available with the remaining elements. We select 6 people from a group of 15.

a) We need to select both married couples, that is, all 4 from the group of 4, and then we look at the remaining 11 people from which we need to select a further 2 people.

$$P(A) = \frac{\binom{4}{4}\binom{11}{2}}{\binom{15}{6}} = \frac{1 \times 55}{5005} = \frac{1}{91} \approx 0.0110$$

b) If we select the three youngest members in the group, we can select any of the remaining members of the group for the final three places.

$$P(B) = \frac{\binom{3}{3}\binom{12}{3}}{\binom{15}{6}} = \frac{1 \times 220}{5005} = \frac{4}{91} \approx 0.0440$$

19 a) $$P(A) = \frac{\binom{10}{3}\binom{15}{3}}{\binom{25}{6}} = \frac{120 \times 455}{177\,100} = \frac{78}{253} \approx 0.308$$

b) 'At least 3' means 3, 4, 5 or 6. We will use the complementary event since it has fewer calculations. The complementary event is at most 2, which is 0, 1 or 2 colour laser printers.

$$\mathrm{P}(B) = 1 - \mathrm{P}(B') = 1 - \left(\frac{\binom{10}{0}\binom{15}{6}}{\binom{25}{6}} + \frac{\binom{10}{1}\binom{15}{5}}{\binom{25}{6}} + \frac{\binom{10}{2}\binom{15}{4}}{\binom{25}{6}} \right)$$

$$= 1 - \frac{1 \times 5005 + 10 \times 3003 + 45 \times 1365}{177\,100}$$

$$= 1 - \frac{689}{1265} = \frac{576}{1265} \approx 0.445$$

20 a) Since there are 30 buses and we need to select 6 for inspection, there are $\binom{30}{6} = 593\,775$ ways to select six buses.

b) Half means that 3 buses have cracks on the instrument panel.

$$\mathrm{P}(B) = \frac{\binom{10}{3}\binom{20}{3}}{\binom{30}{6}} = \frac{608}{2639} \approx 0.230$$

c) 'At least half' means 3, 4, 5 or 6. Again, we will calculate the probability by using the complementary event, which is 0, 1 or 2 buses have cracks.

$$\mathrm{P}(C) = 1 - \mathrm{P}(C') = 1 - \left(\frac{\binom{10}{0}\binom{20}{6}}{\binom{30}{6}} + \frac{\binom{10}{1}\binom{20}{5}}{\binom{30}{6}} + \frac{\binom{10}{2}\binom{20}{4}}{\binom{30}{6}} \right)$$

$$= 1 - \frac{38\,760 + 155\,040 + 218\,025}{593\,775}$$

$$= 1 - \frac{5491}{7917} = \frac{2426}{7917} \approx 0.306$$

d) 'At most half' means 0, 1, 2 or 3. Again, we will calculate the probability by using the complementary event, which is 4, 5 or 6 buses have cracks.

$$\mathrm{P}(D) = 1 - \mathrm{P}(D') = 1 - \left(\frac{\binom{10}{4}\binom{20}{2}}{\binom{30}{6}} + \frac{\binom{10}{5}\binom{20}{1}}{\binom{30}{6}} + \frac{\binom{10}{6}\binom{20}{0}}{\binom{30}{6}} \right)$$

$$= 1 - \frac{39\,900 + 5040 + 210}{593\,775} = \frac{548\,625}{593\,775}$$

$$= \frac{1045}{1131} \approx 0.924$$

21 There are 67 workers in the factory altogether and we have to select 9.

a) $$P(A)=\frac{\binom{30}{9}}{\binom{67}{9}}=\frac{14\,307\,150}{42\,757\,703\,560}=\frac{10\,005}{29\,900\,492}\approx 0.000\,335$$

b) The same shift means either from the day, evening or morning shift.

$$P(B)=\frac{\binom{30}{9}}{\binom{67}{9}}+\frac{\binom{22}{9}}{\binom{67}{9}}+\frac{\binom{15}{9}}{\binom{67}{9}}=\frac{14\,307\,150+497\,420+5005}{42\,757\,703\,560}=\frac{269\,265}{777\,412\,792}\approx 0.000\,346$$

c) We will calculate the probability that at least two of the shifts are represented by considering the complementary event, which is that only one of the shifts is represented (part **b** of this question).

$$P(C)=1-P(B)=1-\frac{269\,265}{777\,412\,792}=\frac{777\,143\,527}{777\,412\,792}\approx 0.9997$$

d) The probability that at least one of the shifts is unrepresented means that one shift is not selected, either from the day, evening or morning shift.

$$P(D)=\frac{\overbrace{\binom{52}{9}}^{\text{morning}}}{\binom{67}{9}}+\frac{\overbrace{\binom{45}{9}}^{\text{evening}}}{\binom{67}{9}}+\frac{\overbrace{\binom{37}{9}}^{\text{day}}}{\binom{67}{9}}=\frac{3\,679\,075\,400+886\,163\,135+124\,403\,620}{42\,757\,703\,560}$$

$$=\frac{85\,266\,221}{777\,412\,792}\approx 0.1097$$

22 a) Since we have to select 2 out of these 8 chips, there are $\binom{8}{2}=56$ different outcomes.

A sum of 7 is obtained with the following pairs: (1, 6), (6, 1), (2, 5), (5, 2), (3, 4) and (4, 3); therefore, the probability is $P(A)=\frac{6}{56}=\frac{3}{28}\approx 0.107$.

b) Since we have to select 2 out of these 20 chips, there are $\binom{20}{2}=190$ different outcomes.

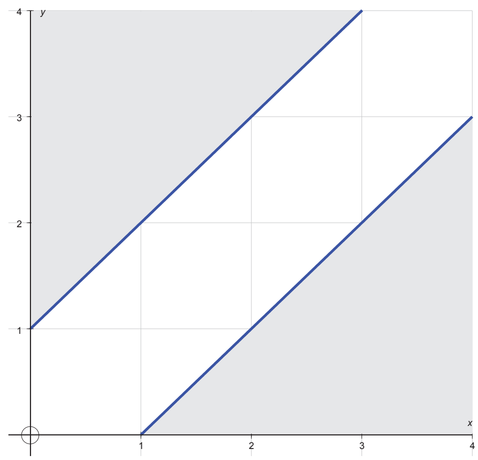
Since we do not care about the order of the numbers, we will take the smaller number first. We notice that there are 17 favourble outcomes, because 18, 19 and 20 don't have numbers in the box which differ by 3, so the probability is $P(B)=\frac{17}{190}\approx 0.0895$.

c) We have 190 different outcomes and we are going to use the complementary event to find the required probability. Again, we take the smaller number first and need to find all of those pairs in which the numbers differ by 1 or 2. As in the previous part, we will count the number of pairs that differ by 1, which is 19 because 20 doesn't have such a number in the box. The number of pairs in which the numbers differ by 2 is 18, since 19 and 20 don't have such numbers in the box.

$$P(B)=1-\frac{19+18}{190}=\frac{153}{190}\approx 0.805$$

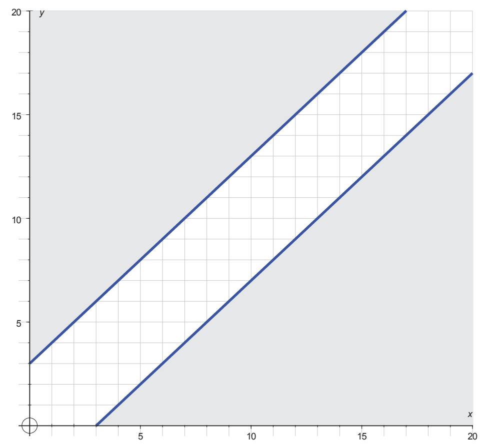
Note: It is impossible to draw two equal numbers.

23 To solve this problem we are going to use geometric probability. Let x be Tim's time of arrival and y Val's time of arrival. For ease of calculation, we are going to measure the time from 20:00, taking 30 minutes as a unit on both axes. Now, we can conclude that the range of both variables will be $0 \leqslant x, y \leqslant 4$. In order to have dinner together, they have to arrive at the restaurant within 1 unit (30 minutes) of each other; therefore, they have to satisfy the inequality $|x - y| \leqslant 1$. The unshaded area represents the favourable outcomes.



We notice that all the possible outcomes are represented by a square with an area of 16, whilst the unshaded part consists of four unit squares and six half-squares; therefore, the probability is: $\mathrm{P}(D) = \frac{7}{16}$.

24 Let x represent the time of the tram and y the time of the bus at the station. Since both stay for 3 minutes in the station, we are going to use minutes as units on both axes. Now, we can conclude that the range of both variables will be $0 \leqslant x, y \leqslant 20$. In order to be at the station at the same time, they have to arrive at the station within 3 minutes of each other; therefore, they have to satisfy the inequality $|x - y| \leqslant 3$. The unshaded area represents the favourable outcomes.



To avoid counting squares, we can divide the unshaded area into three parts: two isosceles right-angled triangles at each end and one rectangle. The two isosceles right triangles form a square with an area of 9. The rectangle in the middle has side lengths of $17\sqrt{2}$ and $3\sqrt{2}$; therefore, its area is 102. So, the total area of the unshaded figure is 111. Thus, the probability is: $\mathrm{P}(B) = \frac{111}{400}$.

25 We have to select 6 from 30 bottles.

a) $\mathrm{P}(A) = \dfrac{\binom{8}{2}\binom{10}{2}\binom{12}{2}}{\binom{30}{6}} = \dfrac{264}{1885} \approx 0.140$

b) $\mathrm{P}(B) = \dfrac{\binom{8}{6}+\binom{10}{6}+\binom{12}{6}}{\binom{30}{6}} = \dfrac{166}{84\,825} \approx 0.001\,96$

c) Serving only Italian and French wines means that we have $8 + 12 = 20$ bottles for the favourable event. Hence, $\mathrm{P}(C) = \dfrac{\binom{20}{6}}{\binom{30}{6}} = \dfrac{2584}{39\,585} \approx 0.0653$

26 There are 1000 cubes altogether. Eight cubes are painted green at three faces (vertices). There are eight cubes painted green at two sides per edge. There are 12 edges, so there are 96 such cubes. There are 64 cubes on each of six faces that have only a green face; therefore, there are 384 such cubes.

a) $P(A) = \frac{96}{1000} = \frac{12}{125}$

b) $P(B) = \frac{8}{1000} = \frac{1}{125}$

c) From the above explanation, we notice that there are $8 + 96 + 384 = 488$ cubes with at least one face painted. So, to find the probability that a cube does not have a green face, we are going to use the complementary event.

$$P(C) = 1 - P(C') \Rightarrow P(C) = 1 - \frac{488}{100} = \frac{64}{125}$$

Exercise 12.4

1 Using the addition rule:

$$P(A \cup B) = P(A) + P(B) - P(A \cap B) \Rightarrow P(B) = P(A \cup B) + P(A \cap B) - P(A)$$

$$P(B) = \frac{4}{5} + \frac{3}{10} - \frac{3}{4} = \frac{16 + 6 - 15}{20} = \frac{7}{20}$$

2 **a)** $P(B) = P(A \cup B) + P(A \cap B) - P(A) \Rightarrow P(B) = \frac{9}{10} + \frac{3}{10} - \frac{7}{10} = \frac{5}{10} = \frac{1}{2}$

b) Firstly, we need to write the event in a different form: $B' \cap A = A \setminus (A \cap B)$ and since $A \cap B \subseteq A$ we can now calculate the probability:

$$P(B' \cap A) = P(A \setminus (A \cap B)) = P(A) - P(A \cap B) = \frac{7}{10} - \frac{3}{10} = \frac{4}{10} = \frac{2}{5}$$

c) Similarly, $P(B \cap A') = P(B \setminus (A \cap B)) = P(B) - P(A \cap B) = \frac{5}{10} - \frac{3}{10} = \frac{2}{10} = \frac{1}{5}$

d) By using a Venn diagram, we can spot that the intersection of the complements of two sets can be written as the complement of the union of the sets. Hence:

$$B' \cap A' = (A \cup B)' \Rightarrow P(B' \cap A') = P((A \cup B)') = 1 - P(A \cup B) = 1 - \frac{9}{10} = \frac{1}{10}$$

e) Using the conditional probability formula: $P(B | A') = \frac{P(B \cap A')}{P(A')} = \frac{\frac{1}{\cancel{5}}}{\frac{3}{\cancel{10}\,2}} = \frac{2}{3}$

3 Given the addition rule, we can calculate the intersection.

$$P(A \cap B) = P(A) + P(B) - P(A \cup B) \Rightarrow P(A \cap B) = \frac{1}{3} + \frac{2}{9} - \frac{4}{9} = \frac{1}{9}$$

It is obvious that the events are not mutually exclusive since $P(A \cap B) = \frac{1}{9} \neq 0$, and also they are not independent since $P(A) \times P(B) = \frac{1}{3} \times \frac{2}{9} = \frac{2}{27} \neq \frac{1}{9} = P(A \cap B)$.

4 Firstly, we will calculate the probability of event B.

$$P(A \cap B) = P(A) \times P(B) \Rightarrow P(B) = \frac{\frac{3}{10}}{\frac{3}{7}} = \frac{7}{10}$$

From the addition rule, we obtain:

$$P(A \cup B) = P(A) + P(B) - P(A \cap B) \Rightarrow P(A \cup B) = \frac{3}{7} + \frac{7}{10} - \frac{3}{10} = \frac{58}{70} = \frac{29}{35}$$

5 In order to pass the test without having to wait 6 months, the new driver has to pass either the first time or fail at the first attempt and pass the second time.

$$P(A) = \overbrace{0.6}^{\text{pass 1st attempt}} + \overbrace{0.4}^{\text{fail 1st attempt}} \cdot \overbrace{0.75}^{\text{pass 2nd attempt}} = 0.9$$

6 **a)** Using the complementary event, we get $P(O') = 1 - P(O) \Rightarrow P(O') = 1 - 0.08 = 0.92$ (92%).

b) **i)** $P(OO) = 0.08 \cdot 0.08 = 0.0064 = 0.64\%$

ii) The complementary event of at least one of them has O-negative is none of them has O-negative.
$P(O'O') = 0.92 \times 0.92 = 0.8464 = 84.64\% \Rightarrow P(B) = 1 - 0.8464 = 0.1536 = 15.36\%$

iii) If only one of them has O-negative, that means that the other is not O-negative, so:
$P(OO') + P(O'O) = 2 \cdot 0.08 \cdot 0.92 = 0.1472 = 14.72\%$

c) Using the complementary event that none has O-negative:
$P(C) = 1 - P(8O') = 1 - 0.92^8 = 0.486\,781\,1269 = 48.7\%$, correct to three significant figures.

7 **a)** 10 different digits can be used to make each four-digit number; as such, there are $10^4 = 10\,000$ different possible PIN numbers.

b) Since the first digit cannot be zero, there are $9 \cdot 10^3 = 9000$ such codes; therefore, the probability is $P(B) = \frac{9000}{10\,000} = \frac{9}{10}$.

c) The complementary event is that the code doesn't contain a zero; therefore,

$$P(C) = 1 - \frac{9^4}{10^4} = \frac{10\,000 - 6561}{10\,000} = \frac{3439}{10\,000}.$$

d) Using the conditional probability formula: $P(D|C) = \frac{P(D \cap C)}{P(C)} = \frac{\frac{1000}{\cancel{10\,000}}}{\frac{3439}{\cancel{10\,000}}} = \frac{1000}{3439}$

8 **a)** We need to use the complementary event that no red ball is drawn:

$$P(A) = 1 - P(BB) = 1 - \frac{2}{8} \times \frac{2}{8} = \frac{64 - 4}{64} = \frac{60}{64} = \frac{15}{16}$$

b) Using the conditional probability formula: $P(B|A) = \frac{P(A \cap B)}{P(B)} = \frac{\frac{2}{8} \times \frac{6}{8} + \frac{6}{8} \times \frac{6}{8}}{\frac{15}{16}} = \frac{\frac{\cancel{48}\,12}{\cancel{64}\,4}}{\frac{15}{\cancel{16}}} = \frac{4}{5}$

c) $P(C|A) = \frac{P(A \cap C)}{P(A)} = \frac{\frac{\cancel{2}}{\cancel{8}} \times \frac{\cancel{6}\,3}{\cancel{8}\,4}}{\frac{\cancel{15}\,5}{\cancel{16}}} = \frac{1}{5}$

9 **a)** $S = \{(1,1),(1,2),(1,3),(1,4),(1,5),(1,6),(2,1),(2,2),(2,3),(2,4),(2,5),(2,6),$
$(3,1),(3,2),(3,3),(3,4),(3,5),(3,6),(4,1),(4,2),(4,3),(4,4),(4,5),(4,6),$
$(5,1),(5,2),(5,3),(5,4),(5,5),(5,6),(6,1),(6,2),(6,3),(6,4),(6,5),(6,6)\}$

b)

x	2	3	4	5	6	7	8	9	10	11	12
P(x)	$\frac{1}{36}$	$\frac{1}{18}$	$\frac{1}{12}$	$\frac{1}{9}$	$\frac{5}{36}$	$\frac{1}{6}$	$\frac{5}{36}$	$\frac{1}{9}$	$\frac{1}{12}$	$\frac{1}{18}$	$\frac{1}{36}$

c) Looking at the sample space, we notice that there are 11 pairs with at least one die showing 6, so: $\mathrm{P}(C) = \frac{11}{36}$.

d) For the sum is at most 10, we will use a complementary event, that is, 11 or 12. Looking at the table, we obtain: $\mathrm{P}(D) = 1 - \left(\frac{1}{18} + \frac{1}{36}\right) = 1 - \frac{1}{12} = \frac{11}{12}$.

e) Using the addition formula:

$$\mathrm{P}(X \cup Y) = \mathrm{P}(X) + \mathrm{P}(Y) - \mathrm{P}(X \cap Y) \Rightarrow \mathrm{P}(X \cup Y) = \frac{11}{36} + \frac{3}{36} - \frac{2}{36} = \frac{12}{36} = \frac{1}{3}$$

f) Using the conditional probability formula: $\mathrm{P}(Y|X) = \frac{\mathrm{P}(X \cap Y)}{\mathrm{P}(X)} \Rightarrow \mathrm{P}(Y|X) = \frac{\frac{2}{\cancel{36}}}{\frac{3}{\cancel{36}}} = \frac{2}{3}$

10 **a)** There are 1500 students altogether, 700 of whom are female, so: $\mathrm{P}(A) = \frac{700}{1500} = \frac{7}{15}$.

b) $\mathrm{P}(B) = \frac{220}{1500} = \frac{11}{75}$

c) Now the sample space contains only female students, so: $\mathrm{P}(C) = \frac{180}{700} = \frac{9}{35}$.

d) In this problem we need to use the addition formula. Let T be the event of selecting a student from grade 12, and F selecting a female student.

$$\mathrm{P}(T \cup F) = \mathrm{P}(T) + \mathrm{P}(F) - \mathrm{P}(T \cap F) \Rightarrow \mathrm{P}(T \cup F) = \frac{400 + 700 - 180}{1500} = \frac{46}{75}$$

e) There are 400 grade 12 students altogether, 220 of whom are male, so: $\mathrm{P}(E) = \frac{220}{400} = \frac{11}{20}$.

f) In order to confirm that the two events are independent, we need to check whether the probability of the intersection of the two events is equal to the product of the probabilities of each individual event. Let's focus on grade 12 (G) and female (F).

$\mathrm{P}(G) = \frac{400}{1500} = \frac{4}{15}$, $\mathrm{P}(F) = \frac{7}{15} \Rightarrow \mathrm{P}(G) \times \mathrm{P}(F) = \frac{4}{15} \times \frac{7}{15} = \frac{28}{225}$; whilst, by looking at the table, we notice that $\mathrm{P}(G \cap F) = \frac{180}{1500} = \frac{9}{75} = \frac{27}{225} \neq \frac{28}{225}$. Therefore, the events are dependent.

11 **a)** Using the table, we can read off the results:

i) $0.41 + 0.15 = 0.56$

ii) 0.15

b) In this case, we use the conditional probability formula: $\mathrm{P}(B)=\frac{0.15}{0.56}=\frac{15}{56}$.

c) Another way of showing independence is by considering whether $\mathrm{P}(C|A)=\mathrm{P}(C)$. Now, if we use the events C using glasses and A needing glasses, we obtain:

$$\mathrm{P}(C)=0.41+0.04=0.45,\ \mathrm{P}(A)=0.56,\ \mathrm{P}(C|A)=\frac{\mathrm{P}(C\cap A)}{\mathrm{P}(A)}=\frac{0.41}{0.56}=\frac{41}{56}\neq\frac{45}{100}=\mathrm{P}(C)$$

Hence, the events are dependent.

12 If the events are mutually exclusive, their intersection is an empty set and its probability is zero: $\mathrm{P}(A\cap B)=\mathrm{P}(\varnothing)=0$.

If they are independent, then the probability of their intersection is the product of their probabilities: $\mathrm{P}(A\cap B)=\mathrm{P}(A)\times\mathrm{P}(B)$.

To find the probability of the union, we need to use the addition formula: $\mathrm{P}(A\cup B)=\mathrm{P}(A)+\mathrm{P}(B)-\mathrm{P}(A)\times\mathrm{P}(B)$.

The conditional probability doesn't change the probability of the original event, so: $\mathrm{P}(A|B)=\mathrm{P}(A)$.

P(A)	P(B)	Conditions for events A and B	P($A \cap B$)	P($A \cup B$)	P(A|B)
0.3	0.4	Mutually exclusive	0	0.7	0
0.3	0.4	Independent	0.12	0.58	0.3
0.1	0.5	Mutually exclusive	0	0.6	0
0.2	0.5	Independent	0.1	0.6	0.2

13 a) Since the condition is that the chosen student is doing Maths/SL, we have to find the number of favourable outcomes, which is the intersection itself: $\mathrm{P}(A)=\frac{12}{40}=\frac{3}{10}$.

b) $\mathrm{P}(M)\times\mathrm{P}(F)=\frac{40}{100}\times\frac{30}{100}=\frac{3}{25}=\frac{12}{100}=\mathrm{P}(M\cap F)$, so the events are independent.

14 a) We need to use the addition formula: $\mathrm{P}(M\cup V)=0.21+0.57-0.13=0.65=65\%$.

b) We need to use the complementary event: $\mathrm{P}((M\cup V)')=1-0.65=0.35=35\%$.

c) Exactly one acceptable card means that the person cannot have both cards. So: $\mathrm{P}((M\cup V)\setminus(M\cap V))=0.65-0.13=0.52=52\%$.

15 Let S be the set of patients taking a swimming class, and A the set of patients taking an aerobics class. The following probabilities are given: $\mathrm{P}(S\cup A)=\frac{132}{300}=\frac{11}{25}$, $\mathrm{P}(S)=\frac{78}{300}=\frac{13}{50}$, $\mathrm{P}(A)=\frac{84}{300}=\frac{7}{25}$

a) $\mathrm{P}((S\cup A)')=1-\mathrm{P}(S\cup A)=1-\frac{11}{25}=\frac{14}{25}$

b) $\mathrm{P}(S\cap A)=\mathrm{P}(S)+\mathrm{P}(A)-\mathrm{P}(S\cup A)\Rightarrow\mathrm{P}(S\cap A)=\frac{13}{50}+\frac{7}{25}-\frac{11}{25}=\frac{5}{50}=\frac{1}{10}$

16 a) In each attempt, the probability of rolling a two is 1 out of 6.

$$\mathrm{P}(A)=\frac{1}{6}\cdot\frac{1}{6}\cdot\frac{1}{6}=\frac{1}{216}$$

b) We will calculate the probability of the event 'at least one two is rolled' by using the complementary event, which is 'no two is rolled'.

$$P(B) = 1 - P(B') = 1 - \frac{5}{6} \times \frac{5}{6} \times \frac{5}{6} = 1 - \frac{125}{216} = \frac{91}{216}$$

c) Exactly one 2 can be rolled in three different ways. A two can be rolled at the first, second or third rolling; therefore, the probability is $P(C) = 3 \cdot \frac{1}{6} \cdot \frac{5}{6} \cdot \frac{5}{6} = \frac{25}{72}$.

17 a) She needs to miss the centre with her first shot and then hit the centre with her second shot; therefore, the probability is $P(A) = 0.7 \cdot 0.3 = 0.21 = 21\%$.

b) She can hit the centre with her first, second or third shot; therefore, the probability is:

$$P(B) = \overbrace{3}^{\text{number of ways}} \cdot \overbrace{0.3}^{\text{hit}} \cdot \overbrace{0.7^2}^{\text{twice no centre}} = 0.441 = 44.1\%$$

c) We will calculate the probability of 'at least once' by using the complementary event, which is 'no hit in the centre'.

$$P(C) = 1 - P(C') = 1 - \overbrace{0.7^3}^{\text{no centre in 3 attempts}} = 1 - 0.343 = 0.657 = 65.7\%$$

18 Since each die has 12 different outcomes, rolling two dice has $12 \cdot 12 = 144$ outcomes.

a) We will calculate the probability of 'at least one twelve shows' by using the complementary event, which is 'no twelve shows on either die'.

$$P(A) = 1 - P(A') = 1 - \frac{11}{12} \times \frac{11}{12} = 1 - \frac{121}{144} = \frac{23}{144}$$

b) A sum of 12 can be achieved as follows: (1, 11), (2, 10), (3, 9), (4, 8), (5, 7), (6, 6), (7, 5), (8, 4), (9, 3), (10, 2), (11, 1). Therefore, the probability is $P(B) = \frac{11}{144}$.

c) A total score of at least 20 can be achieved as follows: (8, 12), (9, 11), (9, 12), (10, 10), (10, 11), (10, 12), (11, 9), (11, 10), (11, 11), (11, 12), (12, 8), (12, 9), (12, 10), (12, 11), (12, 12). Therefore, the probability is $P(C) = \frac{15}{144} = \frac{5}{48}$.

d) If 12 shows on a die, there are 23 different outcomes. From these 23 outcomes, we look for those that have a sum of at least 20. Looking at the previous part, we notice nine such pairs, so $P(D) = \frac{9}{23}$.

19 a) The event 'at least one of the numbers is a 10 and the sum is at most 15' is satisfied as follows: (10, 1), (10, 2), (10, 3), (10, 4), (10, 5), together with these pairs with reversed coordinates.

$$P(A \cap B) = \frac{10}{144} = \frac{5}{72}$$

b) The event is 'at least one number is 10 or the sum is at most 15'. At least one 10 can be obtained in 23 different ways, whilst the sum of at most 15 can be obtained in 99 ways (144 – 45, where the 45 outcomes represent a sum of greater than 15). By using the addition rule, and the result from the previous part, we get:

$$P(A \cup B) = P(A) + P(B) - P(A \cap B) = \frac{23 + 99 - 10}{144} = \frac{112}{144} = \frac{7}{9}$$

c) This is the complementary event of the event in part **a**. We need to list all the remaining pairs that are not listed in part **a**. The probability is: $P((A \cap B)') = 1 - P(A \cap B) = 1 - \frac{5}{72} = \frac{67}{72}$.

d) This is the complementary event of the event in part **b**. We need to list all the remaining pairs that are not listed in part **b**. The probability is: $\mathrm{P}((A \cup B)') = 1 - \mathrm{P}(A \cup B) = 1 - \frac{7}{9} = \frac{2}{9}$.

e) By using de Morgan's laws, the result is the same as that in part **c**.

f) By using de Morgan's laws, the result is the same as that in part **d**.

g) By using the symmetrical difference property, we obtain $(A' \cap B) \cup (A \cap B') = (A \cup B) - (A \cap B)$, and, since $A \cap B \subseteq A \cup B$, then we calculate the probability in the following way.

$\mathrm{P}((A' \cap B) \cup (A \cap B')) = \mathrm{P}(A \cup B) - \mathrm{P}(A \cap B) = \frac{7}{9} - \frac{5}{72} = \frac{51}{72} = \frac{17}{24}$

20 a) We can use the addition rule:

$\mathrm{P}(A \cup B) = \mathrm{P}(A) + \mathrm{P}(B) - \mathrm{P}(A \cap B) \Rightarrow \mathrm{P}(A \cap B) = \mathrm{P}(A) + \mathrm{P}(B) - \mathrm{P}(A \cup B)$. Since we know that $\mathrm{P}(A \cup B) \leqslant 1$, then we can conclude that $\mathrm{P}(A \cap B) \geqslant \mathrm{P}(A) + \mathrm{P}(B) - 1$.

b) We can use the addition rule twice:

$$\begin{aligned}\mathrm{P}((A \cup B) \cup C) &= \mathrm{P}(A \cup B) + \mathrm{P}(C) - \mathrm{P}((A \cup B) \cap C)\\ &= \mathrm{P}(A) + \mathrm{P}(B) - \mathrm{P}(A \cap B) + \mathrm{P}(C) - \mathrm{P}((A \cup B) \cap C)\end{aligned}$$

The last probability can be handled separately by using the addition rule once again and the distribution properties of set operations.

$\mathrm{P}((A \cup B) \cap C) = \mathrm{P}((A \cap C) \cup (B \cap C)) = \mathrm{P}(A \cap C) + \mathrm{P}(B \cap C) - \mathrm{P}((A \cap C) \cap (B \cap C))$

Now we can finish the proof.

$$\begin{aligned}&= \mathrm{P}(A) + \mathrm{P}(B) - \mathrm{P}(A \cap B) + \mathrm{P}(C) - \mathrm{P}((A \cup B) \cap C)\\ &= \mathrm{P}(A) + \mathrm{P}(B) - \mathrm{P}(A \cap B) + \mathrm{P}(C) - (\mathrm{P}(A \cap C) + \mathrm{P}(B \cap C) - \mathrm{P}(A \cap B \cap C))\\ &= \mathrm{P}(A) + \mathrm{P}(B) + \mathrm{P}(C) - \mathrm{P}(A \cap B) - \mathrm{P}(A \cap C) - \mathrm{P}(B \cap C) + \mathrm{P}(A \cap B \cap C)\end{aligned}$$

21 When we roll three fair 6-sided dice, there are 216 possible outcomes.

a) Triples can be rolled in 6 different ways; therefore, $\mathrm{P}(A) = \frac{6}{216} = \frac{1}{36}$.

b) A sum of eight or less can be rolled in 56 different triplets and only two are triples. The possible triplets containing the same number are those with 1 and 2. Then there are 10 possible combinations of triplets with two equal numbers and each combination will appear three times, for example, (1, 1, 2), (1, 1, 3)...(2, 2, 3)...(2, 2, 4),(2, 3, 3). At the end there are four combinations of triplets, each with different numbers, for example, (1, 2, 3), (1, 2, 4), (1, 2, 5) and (1, 3, 4); and each of these combinations appear six times. So, we have a total of 56 different triplets.

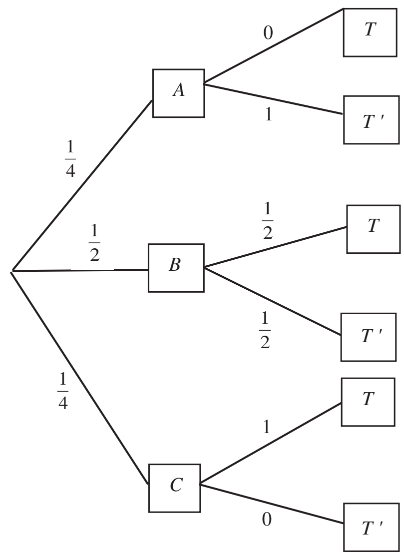
$\mathrm{P}(B) = \frac{2}{56} = \frac{1}{28}$

c) We will calculate the probability of 'at least one six' by using the complementary event, which is 'no six will appear': $\mathrm{P}(C) = 1 - \mathrm{P}(C') = 1 - \frac{5}{6} \times \frac{5}{6} \times \frac{5}{6} = 1 - \frac{125}{216} = \frac{91}{216}$.

Note: The result is the same as that in question 16 **b** since there is no difference in the appearance of a 2 or 6 on a die.

d) If all three dice have different numbers, there are $\binom{6}{3} = 20$ different combinations and each will appear $3! = 6$ times; therefore, we have 120 different triplets. Again, we will use the complementary event, 'no six will appear': $\mathrm{P}(D) = 1 - \mathrm{P}(D') = 1 - \frac{\binom{5}{3} \times 6}{120} = 1 - \frac{60}{120} = \frac{1}{2}$.

22 Let T be the event that the outcome is tails, and A, B and C the events that the selected coin has two heads, is fair, and has two tails respectively.



$$P(T|A) = \frac{P(T\cap A)}{P(A)} = \frac{\frac{2}{4}\times\frac{1}{2}}{\frac{2}{4}\times\frac{1}{2}+\frac{1}{4}\times 0+\frac{1}{4}\times 1} = \frac{\frac{1}{4}}{\frac{1}{2}} = \frac{1}{2}$$

We can also use Bayes' theorem (see the next chapter) for this problem. We have to state three hypotheses:

H_1: The coin is fair.

H_2: The coin has two heads.

H_3: The coin has two tails.

The event that is given is A, 'randomly selected coin is tossed and the outcome is a tail'.

We have to find the probability of the first hypothesis given that event A has occurred.

$$P(H_1|A) = \frac{P(H_1\cap A)}{P(A)} = \frac{\frac{2}{4}\times\frac{1}{2}}{\frac{2}{4}\times\frac{1}{2}+\frac{1}{4}\times 0+\frac{1}{4}\times 1} = \frac{\frac{1}{4}}{\frac{1}{2}} = \frac{1}{2}$$

23 There are five different ways of rolling a sum of 6: (1, 5), (2, 4), (3, 3), (4, 2) and (5, 1). So, the probability that either of the players will roll that sum is $\frac{5}{36}$.

a) $$P(A) = \overbrace{\frac{31}{36}}^{\text{K: no sum 6}} \times \overbrace{\frac{31}{36}}^{\text{G: no sum 6}} \times \overbrace{\frac{5}{36}}^{\text{K: sum 6}} = \frac{4805}{46\,656} \approx 0.103$$

b) $$P(B) = \overbrace{\frac{31}{36}}^{\text{K: no sum 6}} \times \overbrace{\frac{31}{36}}^{\text{G: no sum 6}} \times \overbrace{\frac{31}{36}}^{\text{K: no sum 6}} \times \overbrace{\frac{5}{36}}^{\text{G: sum 6}} = \frac{148\,955}{1\,679\,616} \approx 0.0887$$

c) To calculate the probability that Kassanthra wins, we need to find the sum of an infinite geometric sequence.

$$P(C) = \frac{5}{36} + \frac{31}{36}\cdot\frac{31}{36}\cdot\frac{5}{36} + \frac{31}{36}\cdot\frac{31}{36}\cdot\frac{31}{36}\cdot\frac{31}{36}\cdot\frac{5}{36} + \frac{31}{36}\cdot\frac{31}{36}\cdot\frac{31}{36}\cdot\frac{31}{36}\cdot\frac{31}{36}\cdot\frac{31}{36}\cdot\frac{5}{36} + \ldots$$

$$= \frac{5}{36}\left(1+\left(\frac{31}{36}\right)^2+\left(\frac{31}{36}\right)^4+\left(\frac{31}{36}\right)^6+\ldots\right) = \frac{5}{36}\,\frac{1}{1-\left(\frac{31}{36}\right)^2} = \frac{36}{67} \approx 0.537$$

24 **a)** The day has no effect on the observation, so we just need to find the probability that more than two requests will be made: $P(A) = 1-(0.1+0.3+0.5) = 0.1 = 10\%$.

b) Since the days are independent, we need to multiply the probabilities: $P(B) = 0.1^5 = 0.000\,01$.

25 The class has 11 students altogether and we need to select 4 students at random.

a) We will calculate the probability of 'at least one boy' by using the complementary event, which is 'no boy is selected': $\mathrm{P}(A) = 1 - \dfrac{\binom{6}{4}}{\binom{11}{4}} = 1 - \dfrac{15}{330} = 1 - \dfrac{1}{22} = \dfrac{21}{22}$.

b) There must be either 3 or 4 girls: $\mathrm{P}(B) = \dfrac{\binom{5}{1}\binom{6}{3} + \binom{6}{4}}{\binom{11}{4}} = \dfrac{100+15}{330} = \dfrac{23}{66}$.

c) Let C be the event that the boys are in the majority. The conditional probability formula gives us: $\mathrm{P}(C|A) = \dfrac{\mathrm{P}(C \cap A)}{\mathrm{P}(A)}$. Firstly, we will find the probability of the numerator.

$$\mathrm{P}(C \cap A) = \frac{\binom{5}{3}\binom{6}{1} + \binom{5}{4}}{\binom{11}{4}} = \frac{60+5}{330} = \frac{13}{66} \Rightarrow \mathrm{P}(C|A) = \frac{\frac{13}{\cancel{66}\,3}}{\frac{21}{\cancel{22}\,1}} = \frac{13}{63}$$

26 a) $\mathrm{P}(B_1 \cup B_2) = \mathrm{P}(B_1) + \mathrm{P}(B_2) - \mathrm{P}(B_1 \cap B_2) = 0.22 + 0.25 - 0.11 = 0.36$

b) $\mathrm{P}(B_1' \cap B_2') = \mathrm{P}((B_1 \cup B_2)') = 1 - \mathrm{P}(B_1 \cup B_2) = 1 - 0.36 = 0.64$

c) $\mathrm{P}((B_1' \cap B_2') \cup B_3) = \mathrm{P}(B_1' \cap B_2') + \mathrm{P}(B_3) - \mathrm{P}(B_1' \cap B_2' \cap B_3)$. To find this probability, we will firstly find: $\mathrm{P}(B_1' \cap B_2' \cap B_3) = \mathrm{P}(B_3) - \mathrm{P}(B_1 \cap B_3) - \mathrm{P}(B_2 \cap B_3) + \mathrm{P}(B_1 \cap B_2 \cap B_3)$

$= 0.28 - 0.05 - 0.07 + 0.01 = 0.17$.

Now, we need to use part **b** and the above result to find the required probability:
$\mathrm{P}(B_1' \cap B_2' \cup B_3) = 0.64 + 0.28 - 0.17 = 0.75$.

d) As calculated in the previous part, we can see that $\mathrm{P}(B_1' \cap B_2' \cap B_3) = 0.17$.

e) $\mathrm{P}(B_2 \cap B_3 | B_1) = \dfrac{\mathrm{P}(B_1 \cap B_2 \cap B_3)}{\mathrm{P}(B_1)} = \dfrac{0.01}{0.22} = \dfrac{1}{22}$

f) $\mathrm{P}(B_2 \cup B_3 | B_1) = \dfrac{\mathrm{P}(B_1 \cup B_2 \cap B_3)}{\mathrm{P}(B_1)}$. Firstly, we need to find the probability of the numerator:

$\mathrm{P}(B_1 \cup B_2 \cap B_3) = \mathrm{P}(B_1 \cap B_3) + \mathrm{P}(B_2 \cap B_3) - \mathrm{P}(B_1 \cap B_2 \cap B_3) = 0.11 + 0.05 - 0.01 = 0.15$. Hence:
$\mathrm{P}(B_2 \cup B_3 | B_1) = \dfrac{0.15}{0.22} = \dfrac{15}{22}$

27 Let N and D be the sets of all the joints found to be faulty by Nick and David respectively. Given the information in the question, we can find the probabilities $\mathrm{P}(N) = \dfrac{1448}{20\,000}$, $\mathrm{P}(D) = \dfrac{1502}{20\,000}$ and $\mathrm{P}(N \cup D) = \dfrac{2390}{20\,000}$.

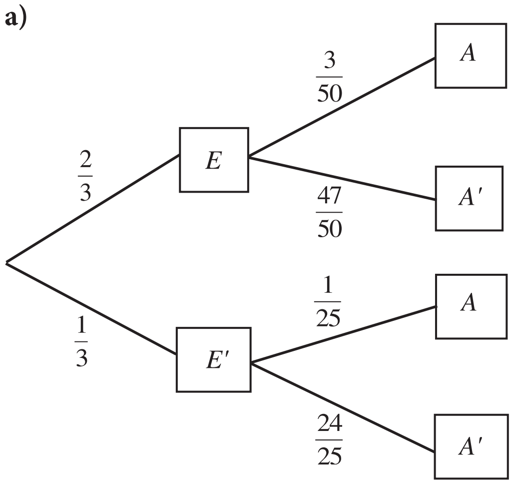
a) $\mathrm{P}((N \cup D)') = 1 - \mathrm{P}(N \cup D) = 1 - \dfrac{239}{2000} = \dfrac{1761}{2000} = 0.8805$

b) $\mathrm{P}(N \cap D) = \mathrm{P}(N) + \mathrm{P}(D) - \mathrm{P}(N \cup D) = \dfrac{1448 + 1502 - 2390}{20\,000} = \dfrac{560}{20\,000}$

$\mathrm{P}(D \cap N') = \mathrm{P}(D) - \mathrm{P}(D \cap N) = \dfrac{1502 - 560}{20\,000} = \dfrac{942}{20\,000} = \dfrac{471}{10\,000} = 0.0471$

Exercise 12.5

1 **a)**



b) $\mathrm{P}(A) = \dfrac{\cancel{2}\,1}{3} \cdot \dfrac{3}{\cancel{50}\,25} + \dfrac{1}{3} \cdot \dfrac{1}{25} = \dfrac{4}{75}$

c) $\mathrm{P}(E|A) = \dfrac{\mathrm{P}(E \cap A)}{\mathrm{P}(A)} = \dfrac{\dfrac{1}{\cancel{25}}}{\dfrac{4}{\cancel{75}\,3}} = \dfrac{3}{4}$

2 Let A be the event that a person is diagnosed with cancer, and E_1 and E_2 the events that the person does have and doesn't have the disease respectively.

a) $\mathrm{P}(A) = \dfrac{26}{100\,000} \times 0.78 + \dfrac{99\,974}{100\,000} \times 0.06 = 0.060\,1872 \approx 0.0602$

b) $\mathrm{P}(E_1|A) = \dfrac{0.000\,26 \times 0.78}{0.060\,1872} = 0.003\,369\,4872 \approx 0.003\,37$

3 Let S be the event that a driver is spotted speeding, and E_1 and E_2 the events that the driver used the east and west entrance respectively.

a) $\mathrm{P}(S) = 0.4 \cdot 0.4 + 0.6 \cdot 0.6 = 0.52$

b) $\mathrm{P}(E_2|S) = \dfrac{0.6 \cdot 0.6}{0.52} = 0.692$ (correct to 3 s.f.)

4 Let G be the event that a drawn ball is green, and E_1, E_2 and E_3 the events that the ball is drawn from box 1, 2 and 3 respectively.

a) $\mathrm{P}(G) = \dfrac{1}{3} \times \dfrac{4}{20} + \dfrac{1}{3} \times \dfrac{8}{16} + \dfrac{1}{3} \times \dfrac{6}{20} = \dfrac{1}{3} \times \left(\dfrac{1}{5} + \dfrac{1}{2} + \dfrac{3}{10}\right) = \dfrac{1}{3} \times \dfrac{2+5+3}{10} = \dfrac{1}{3}$

b) $\mathrm{P}(E_2|G) = \dfrac{\dfrac{1}{3} \cdot \dfrac{1}{2}}{\dfrac{1}{3}} = \dfrac{1}{2}$

5 Let H be the event that the selected coin lands on heads, and E_1 and E_2 the events that the selected coin is biased and unbiased respectively.

a) $\mathrm{P}(H) = 0.5 \cdot 0.6 + 0.5 \cdot 0.5 = 0.55$

b) $\mathrm{P}(E_2|H') = \dfrac{0.5 \times 0.4}{0.45} = \dfrac{4}{9} \approx 0.444$

6 Let C be the event that the question was correctly answered, and E_1 and E_2 the events that the student was well prepared and unprepared respectively.

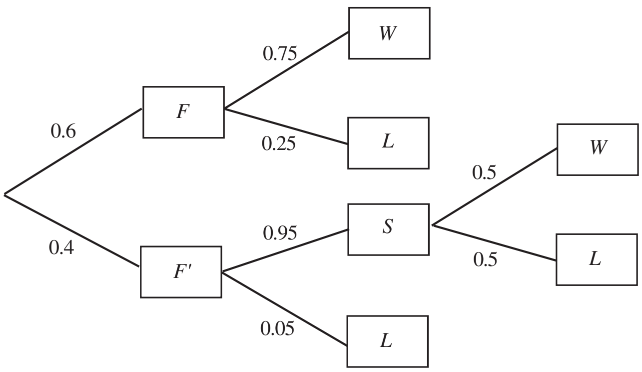
$$\mathrm{P}(E_1|C) = \frac{0.7 \cdot 0.6}{\underbrace{0.7}_{\text{well prepared}} \cdot \underbrace{0.6}_{\text{correct answer}} + \underbrace{0.3}_{\text{not prepared}} \cdot \underbrace{0.2}_{\text{guessing answer}}} = \frac{0.42}{0.48} = \frac{7}{8} = 0.875$$

7 Let T be the event that Nigel gets to his morning class on time, and E_1 and E_2 the events that the alarm clock was set and was not set respectively.

a) $\mathrm{P}(T) = 0.85 \cdot 0.9 + 0.15 \cdot 0.6 = 0.855 = 85.5\%$

b) $\mathrm{P}(E_2|T) = \dfrac{0.15 \times 0.6}{0.855} = \dfrac{2}{19} \approx 0.105 = 10.5\%$

8 We are going to use a probability tree. Let F represent a successful first serve, S a successful second serve, and W and L winning and losing the point respectively.



a) $\mathrm{P}(W) = 0.6 \cdot 0.75 + 0.4 \cdot 0.95 \cdot 0.5 = \dfrac{16}{25} = 0.64$

b) $\mathrm{P}(F|W) = \dfrac{\mathrm{P}(F \cap W)}{\mathrm{P}(W)} = \dfrac{0.6 \times 0.75}{0.64} = \dfrac{45}{64} = 0.703\,125 \approx 0.703$

9 Let F be the event that the first of February is a fine day, and S the event that the second of February is a fine day.

a) $\mathrm{P}(S) = 0.75 \cdot 0.8 + 0.25 \cdot 0.4 = 0.7$

b) $\mathrm{P}(F|S') = \dfrac{\mathrm{P}(F \cap S')}{\mathrm{P}(S')} = \dfrac{0.75 \times 0.2}{0.3} = \dfrac{1}{2} = 0.5$

10 Let A be the event that the person has arthritis, and P that the test is positive.

$$\mathrm{P}(A|P) = \frac{\mathrm{P}(A \cap P)}{\mathrm{P}(P)} = \frac{0.33 \times 0.87}{0.33 \times 0.87 + 0.67 \times 0.04} \approx 0.915$$

11 Let H represent students in the HL class, and L and A students studying locally and abroad respectively.

a) $\mathrm{P}(H \cap L) = 0.05 \times 0.72 = 0.036$

b) From the table, we know that: $\mathrm{P}(L) = 0.67 \Rightarrow \mathrm{P}(H \cap L) + \mathrm{P}(H' \cap L) = 0.67$

Therefore, $\mathrm{P}(H' \cap L) = 0.67 - \mathrm{P}(H \cap L) = 0.67 - 0.036 = 0.634$.

On the other hand: $\mathrm{P}(H' \cap L) = \mathrm{P}(H') \times \mathrm{P}(L|H') \Rightarrow \mathrm{P}(L|H') = \dfrac{0.634}{0.95} \approx 0.667$

c) $\mathrm{P}(H|L) = \dfrac{\mathrm{P}(H \cap L)}{\mathrm{P}(L)} = \dfrac{0.036}{0.67} \approx 0.0537$

d) $\mathrm{P}(H|A) = \dfrac{\mathrm{P}(H) \times \mathrm{P}(A|H)}{\mathrm{P}(A)} = \dfrac{0.05 \times 0.28}{0.33} = \dfrac{0.014}{0.33} \approx 0.0424$

12 Let U represent athletes who are users, and P and N a positive and negative test respectively.

$$\mathrm{P}(U'|P) = \frac{\mathrm{P}(U' \cap P)}{\mathrm{P}(P)} = \frac{0.1 \times 0.5}{0.1 \times 0.5 + 0.9 \times 0.09} \approx 0.382$$

13 Let A, R and M represent the estimates made by Antonio, Richard and Marco respectively, and E an error in the estimation. We would like to find the largest from $\mathrm{P}(A|E)$, $\mathrm{P}(R|E)$ and $\mathrm{P}(M|E)$. We notice that all three expressions have the same denominator, and therefore the largest one will be the one with the largest numerator. So, we calculate the following:

$\mathrm{P}(A \cap E) = 0.3 \times 0.03 = 0.009$, $\mathrm{P}(R \cap E) = 0.2 \times 0.02 = 0.004$, and

$\mathrm{P}(M \cap E) = 0.5 \times 0.01 = 0.005$

Thus, Antonio is probably responsible for most of the serious errors.

14 Let A represent the event that an aircraft is present, and S that there is a signal.

a) $\mathrm{P}(S) = \mathrm{P}(A) \cdot \mathrm{P}(S|A) + \mathrm{P}(A') \cdot \mathrm{P}(S|A') = 0.05 \cdot 0.99 + 0.95 \cdot 0.1 = 0.1445$

b) $\mathrm{P}(A|S') = \dfrac{\mathrm{P}(A) \times \mathrm{P}(S'|A)}{\mathrm{P}(S')} = \dfrac{0.05 \times 0.01}{1 - 0.1445} \approx 0.000\,584$

15 Let N, E and M represent a game against a novice, an experienced player and a master player respectively, and W the probability of winning the game.

a) $\mathrm{P}(W) = \mathrm{P}(N) \cdot \mathrm{P}(W|N) + \mathrm{P}(E) \cdot \mathrm{P}(W|E) + \mathrm{P}(M) \cdot \mathrm{P}(W|M)$

$= 0.5 \cdot 0.5 + 0.25 \cdot 0.4 + 0.25 \cdot 0.3 = 0.425$

b) $\mathrm{P}(M|W) = \dfrac{\mathrm{P}(M) \times \mathrm{P}(W|M)}{\mathrm{P}(W)} = \dfrac{0.25 \times 0.3}{0.425} \approx 0.176$

16 a) $\mathrm{P}(A) = \overbrace{0.8}^{\text{pass 1st}} + \overbrace{0.2}^{\text{fail 1st}} \cdot \overbrace{0.5}^{\text{pass 2nd}} + \overbrace{0.2}^{\text{fail 1st}} \cdot \overbrace{0.5}^{\text{fail 2nd}} \cdot \overbrace{0.3}^{\text{pass 3rd}} = 0.93$

b) $\mathrm{P}(B|A) = \dfrac{0.2 \times 0.5}{0.93} \approx 0.108$

17 a) $\mathrm{P}(F) = \dfrac{56 + 26 + 18}{250} = \dfrac{100}{250} = \dfrac{2}{5} = 0.4$

$\mathrm{P}(F \cap T) = \dfrac{56}{250} = \dfrac{28}{125} = 0.224$

$\mathrm{P}(F \cup A') = \dfrac{84 + 52 + 56 + 18 + 26}{250} = \dfrac{236}{250} = \dfrac{118}{125} = 0.944$

$\mathrm{P}(F'|A) = \dfrac{P(F' \cap A)}{P(A)} = \dfrac{\frac{14}{\cancel{250}}}{\frac{14 + 26}{\cancel{250}}} = \dfrac{14}{40} = \dfrac{7}{20} = 0.35$

b) T, A and S are independent of F because for all of them the probability of the intersection with F is equal to the product of the probabilities; for example,

$$\mathrm{P}(T) = \frac{140}{250} = \frac{14}{25} = 0.56, \mathrm{P}(F) \times \mathrm{P}(T) = \frac{2}{5} \times \frac{14}{25} = \frac{28}{125} = \mathrm{P}(F \cap T)$$

The event M is mutually exclusive since the probability of their intersection is zero.

c) i) $\mathrm{P}(C) = \mathrm{P}(T) \times \mathrm{P}(C|T) + \mathrm{P}(A) \times \mathrm{P}(C|A) + \mathrm{P}(S) \times \mathrm{P}(C|S) \Rightarrow$

$$\mathrm{P}(C) = \frac{14\not{0}}{25\not{0}} \cdot \frac{9}{10} + \frac{4\not{0}}{25\not{0}} \cdot \frac{8}{10} + \frac{7\not{0}}{25\not{0}} \cdot \frac{3}{10} = \frac{126 + 32 + 21}{250} = \frac{179}{250} = 0.716$$

ii) $$\mathrm{P}(T|C) = \frac{\mathrm{P}(T) \times \mathrm{P}(C|T)}{\mathrm{P}(C)} = \frac{\frac{126}{250}}{\frac{179}{250}} = \frac{126}{179} \approx 0.704$$

18 Let H, M and L represent high-risk, medium-risk and low-risk drivers respectively, and A those drivers who will have an accident.

a) $\mathrm{P}(H \cap A) = \mathrm{P}(H) \times \mathrm{P}(A|H) = 0.2 \times 0.06 = 0.012$

b) $\mathrm{P}(A) = \mathrm{P}(H) \cdot \mathrm{P}(A|H) + \mathrm{P}(M) \cdot \mathrm{P}(A|M) + \mathrm{P}(L) \cdot \mathrm{P}(A|L)$

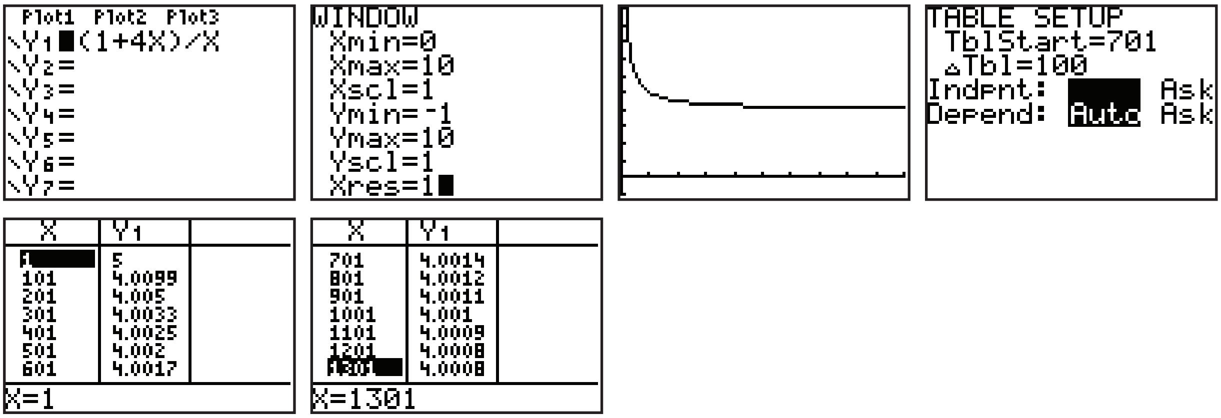
$= 0.2 \cdot 0.06 + 0.5 \cdot 0.03 + 0.3 \cdot 0.01 = 0.012 + 0.015 + 0.003 = 0.03$

c) $$\mathrm{P}(H|A) = \frac{\mathrm{P}(H) \cdot \mathrm{P}(A|H)}{\mathrm{P}(A)} = \frac{0.012}{0.03} = 0.4$$

Chapter 13

Exercise 13.1

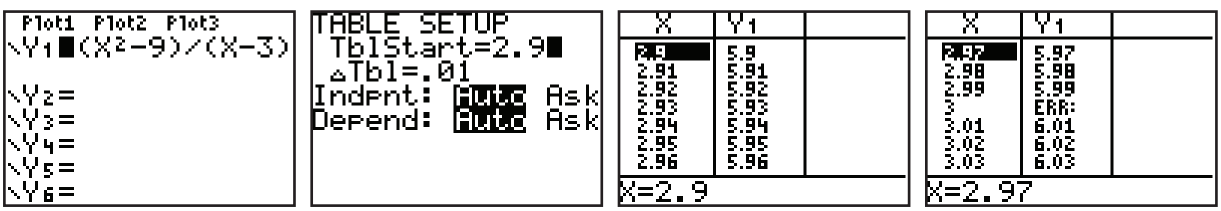
1 $\lim\limits_{n\to\infty} \dfrac{1+4n}{n} = \lim\limits_{n\to\infty}\left(\dfrac{1}{n} + \dfrac{4n}{n}\right) = \lim\limits_{n\to\infty} \dfrac{1}{n} + \lim\limits_{n\to\infty} 4 = 0 + 4 = 4$



2 $\lim\limits_{h\to 0}\left(3x^2 + 2hx + h^2\right) = \lim\limits_{h\to 0} 3x^2 + \lim\limits_{h\to 0} 2hx + \lim\limits_{h\to 0} h^2 = 3x^2 + 0 + 0 = 3x^2$

3 $\lim\limits_{d\to 0} \dfrac{(x+d)^2 - x^2}{d} = \lim\limits_{d\to 0} \dfrac{\cancel{x^2} + 2dx + d^2 - \cancel{x^2}}{d} = \lim\limits_{d\to 0} \dfrac{2dx}{d} + \lim\limits_{d\to 0} \dfrac{d^2}{d} = \lim\limits_{d\to 0} 2x + \lim\limits_{d\to 0} d = 2x + 0 = 2x$

4 $\lim\limits_{x\to 3} \dfrac{x^2 - 9}{x - 3} = \lim\limits_{x\to 3} \dfrac{\cancel{(x-3)}(x+3)}{\cancel{x-3}} = \lim\limits_{x\to 3} x + \lim\limits_{x\to 3} 3 = 3 + 3 = 6$



5 For $f(x) = \dfrac{3x+2}{x^2-3}$ we have:

$f(10) = 0.329\,8969$; $f(50) = 0.060\,873$; $f(100) = 0.030\,209$; $f(1000) = 0.003\,002$;

$f(10\,000) = 3.0002 \times 10^{-4}$; $f(1\,000\,000) = 3 \times 10^{-6}$

Hence, $\lim\limits_{x\to\infty} \dfrac{3x+2}{x^2-3} = 0$.

6 For $f(x) = \dfrac{5x-6}{2x+5}$ we have:

$f(10) = 1.76$; $f(50) = 2.323\,8095$; $f(100) = 2.409\,757\,61$; $f(1000) = 2.490\,7731$;

$f(10\,000) = 2.499\,0752$; $f(1\,000\,000) = 2.499\,9908$

Hence, $\lim\limits_{x\to\infty} \dfrac{5x-6}{2x+5} = 2.5 = \dfrac{5}{2}$.

7 For $f(x) = \dfrac{3x^2+2}{x-3}$ we have:

$f(10) = 43.1428$; $f(50) = 159.617$; $f(100) = 309.298\,97$; $f(1000) = 3009.0291$;

$f(10\,000) = 30\,009.003$; $f(1\,000\,000) = 3\,000\,009$

Hence, $\lim\limits_{x\to\infty} \dfrac{3x^2+2}{x-3} = \infty$, i.e. the function increases without bound.

8 $$\lim_{x\to4}\frac{x-4}{x^2-16} = \lim_{x\to4}\frac{x-4}{(x-4)(x+4)} = \frac{1}{4+4} = \frac{1}{8}$$

9 $$\lim_{x\to1}\frac{x^2+x-2}{x^2-1} = \lim_{x\to1}\frac{(x-1)(x+2)}{(x-1)(x+1)} = \lim_{x\to1}\frac{(x+2)}{(x+1)} = \frac{1+2}{1+1} = \frac{3}{2}$$

10 $$\lim_{x\to0}\frac{\sqrt{2+x}-\sqrt{2}}{x} = \lim_{x\to0}\frac{\sqrt{2+x}-\sqrt{2}}{x}\cdot\frac{\sqrt{2+x}+\sqrt{2}}{\sqrt{2+x}+\sqrt{2}} = \lim_{x\to0}\frac{\left(\sqrt{2+x}\right)^2-\left(\sqrt{2}\right)^2}{x\left(\sqrt{2+x}+\sqrt{2}\right)}$$

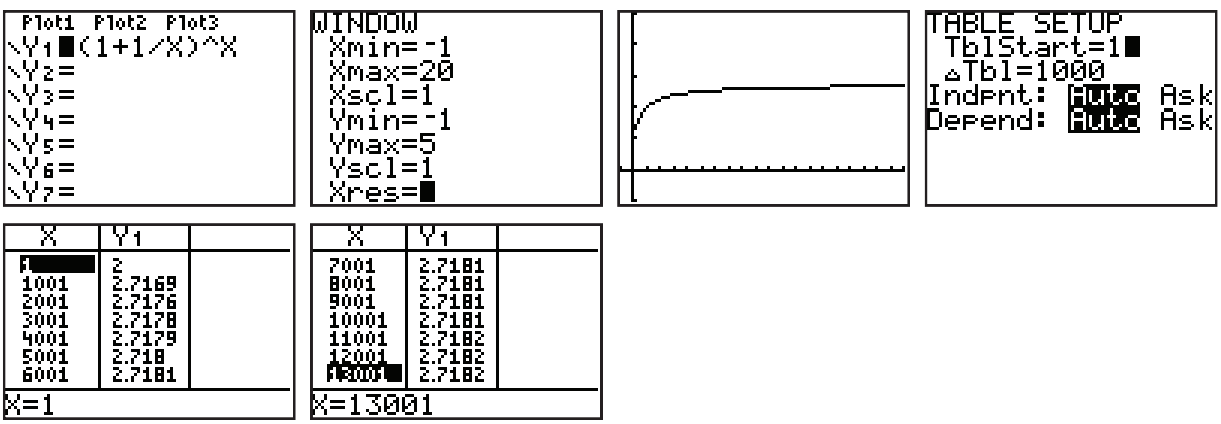
$$= \lim_{x\to0}\frac{2+x-2}{x\left(\sqrt{2+x}+\sqrt{2}\right)} = \lim_{x\to0}\frac{x}{x\left(\sqrt{2+x}+\sqrt{2}\right)} = \lim_{x\to0}\frac{1}{\left(\sqrt{2+x}+\sqrt{2}\right)} = \frac{1}{\sqrt{2}+\sqrt{2}} = \frac{1}{2\sqrt{2}} = \frac{\sqrt{2}}{4}$$

11 $$\lim_{x\to\infty}\frac{x^3-1}{4x^3-3x+1} = \lim_{x\to\infty}\frac{\frac{x^3}{x^3}-\frac{1}{x^3}}{\frac{4x^3}{x^3}-\frac{3x}{x^3}+\frac{1}{x^3}} = \lim_{x\to\infty}\frac{1-\frac{1}{x^3}}{4-\frac{3}{x^2}+\frac{1}{x^3}} = \frac{1-0}{4-0+0} = \frac{1}{4}$$

12 $$\lim_{x\to0}\frac{\tan x}{x} = \lim_{x\to0}\frac{\sin x}{\cos x}\cdot\frac{1}{x} = \lim_{x\to0}\frac{\sin x}{x}\cdot\lim_{x\to0}\frac{1}{\cos x} = 1\cdot\frac{1}{1} = 1$$

13 $$\lim_{\theta\to0}\frac{\sin 3\theta}{\theta} = \lim_{\theta\to0} 3\cdot\frac{\sin 3\theta}{3\theta} = 3\lim_{\theta\to0}\frac{\sin 3\theta}{3\theta} = 3\cdot1 = 3$$

14



Since $e \approx 2.718\,281\,828$, we can conclude that $\lim\limits_{c\to\infty}\left(1+\dfrac{1}{c}\right)^c = e$.

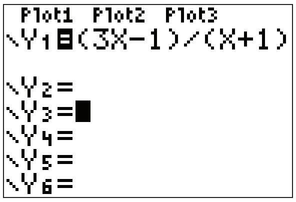
15 If the line $y = 3$ is a horizontal asymptote, the function $f(x)$ approaches the line as $x \to \pm\infty$, so $\lim\limits_{x\to-\infty} f(x) = \lim\limits_{x\to\infty} f(x) = 3$.

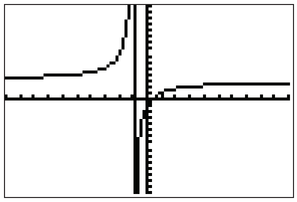
16 If the line $x = a$ is a vertical asymptote and $g(x) > 0$, the function $g(x)$ is increasing as $x \to a$, so $\lim\limits_{x\to a} g(x) = \infty$.

17 a) $\lim\limits_{x\to\pm\infty} \dfrac{3x-1}{x+1} = \lim\limits_{x\to\pm\infty} \dfrac{3-\dfrac{1}{x}}{1+\dfrac{1}{x}} = \dfrac{3-0}{1+0} = 3$

$\lim\limits_{x\to-1} \dfrac{3x-1}{x+1} = \dfrac{-4}{0} = \infty$

Horizontal asymptote is $y = 3$; vertical asymptote is $x = -1$.

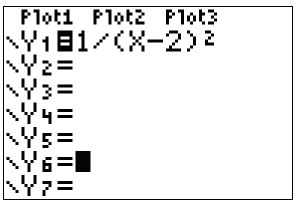


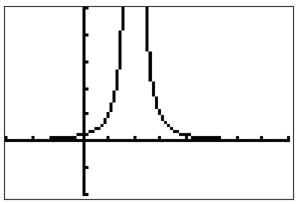


b) $\lim\limits_{x\to\pm\infty} \dfrac{1}{(x-2)^2} = \dfrac{1}{\infty} = 0$

$\lim\limits_{x\to2} \dfrac{1}{(x-2)^2} = \dfrac{1}{0} = \infty$

Horizontal asymptote is $y = 0$; vertical asymptote is $x = 2$.



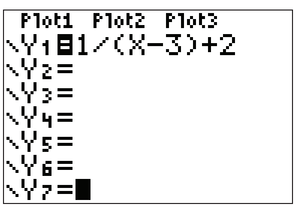


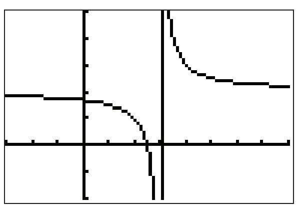
c) $\lim\limits_{x\to\pm\infty} \left(\dfrac{1}{x-a} + b\right) = \dfrac{1}{\pm\infty} + b = 0 + b = b$

$\lim\limits_{x\to a} \left(\dfrac{1}{x-a} + b\right) = \dfrac{1}{0} + b = \infty + b = \infty$

Horizontal asymptote is $y = b$; vertical asymptote is $x = a$.

Example for $a = 3, b = 2$:



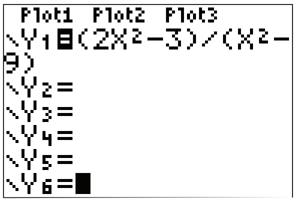


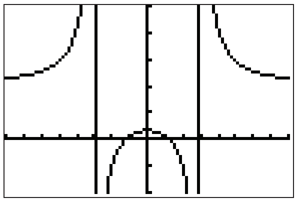
d) $\lim\limits_{x\to\pm\infty} \dfrac{2x^2-3}{x^2-9} = \lim\limits_{x\to\pm\infty} \dfrac{\dfrac{2x^2}{x^2}-\dfrac{3}{x^2}}{\dfrac{x^2}{x^2}-\dfrac{9}{x^2}} = \lim\limits_{x\to\pm\infty} \dfrac{2-\dfrac{3}{x^2}}{1-\dfrac{9}{x^2}} = \dfrac{2-0}{1-0} = 2$

$\lim\limits_{x\to-3} \dfrac{2x^2-3}{x^2-9} = \dfrac{2(-3)^2-3}{(-3)^2-9} = \dfrac{15}{0} = \infty$

$\lim\limits_{x\to3} \dfrac{2x^2-3}{x^2-9} = \dfrac{2\cdot3^2-3}{3^2-9} = \dfrac{15}{0} = \infty$

Horizontal asymptote is $y = 2$; vertical asymptote is $x = \pm 3$.



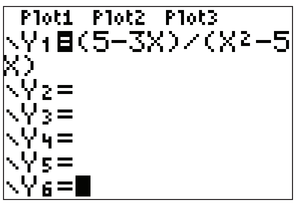


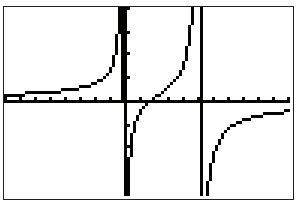
e) $\lim\limits_{x\to\pm\infty} \dfrac{5-3x}{x^2-5x} = \lim\limits_{x\to\pm\infty} \dfrac{\frac{5}{x}-\frac{3x}{x}}{\frac{x^2}{x}-\frac{5x}{x}} = \lim\limits_{x\to\pm\infty} \dfrac{\frac{5}{x}-3}{x-5} = \dfrac{0-3}{\pm\infty-5} = \dfrac{-3}{\pm\infty} = 0$

$\lim\limits_{x\to 0} \dfrac{5-3x}{x^2-5x} = \lim\limits_{x\to 0} \dfrac{5-3x}{x(x-5)} = \dfrac{5}{0\cdot(-5)} = \infty$

$\lim\limits_{x\to 5} \dfrac{5-3x}{x^2-5x} = \lim\limits_{x\to 5} \dfrac{5-3x}{x(x-5)} = \dfrac{5-15}{5\cdot 0} = \infty$

Horizontal asymptote is $y = 0$; vertical asymptotes are $x = 0$ and $x = 5$.

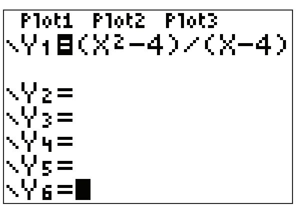


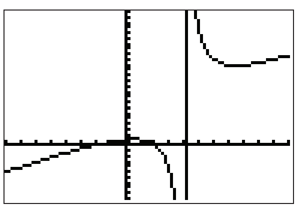


f) $\lim\limits_{x\to\pm\infty} \dfrac{x^2-4}{x-4} = \lim\limits_{x\to\pm\infty} \dfrac{\frac{x^2}{x}-\frac{4}{x}}{\frac{x}{x}-\frac{4}{x}} = \lim\limits_{x\to\pm\infty} \dfrac{x-\frac{4}{x}}{1-\frac{4}{x}} = \dfrac{\pm\infty-0}{1-0} = \pm\infty$

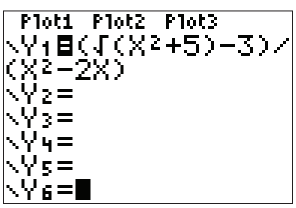
$\lim\limits_{x\to 4} \dfrac{x^2-4}{x-4} = \dfrac{16-4}{0} = \infty$

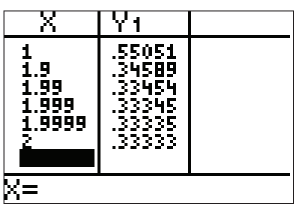
There is no horizontal asymptote; vertical asymptote is $x = 4$.





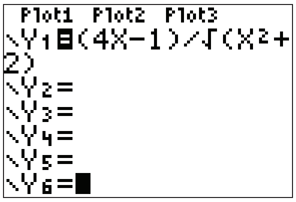
18 a)

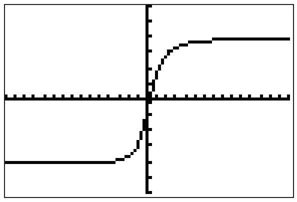




b) $\lim\limits_{x\to 2} \dfrac{\sqrt{x^2+5}-3}{x^2-2x} = \lim\limits_{x\to 2} \dfrac{\sqrt{x^2+5}-3}{x(x-2)} \cdot \dfrac{\sqrt{x^2+5}+3}{\sqrt{x^2+5}+3} = \lim\limits_{x\to 2} \dfrac{x^2+5-9}{x(x-2)} \cdot \dfrac{1}{\sqrt{x^2+5}+3}$

$= \lim\limits_{x\to 2} \dfrac{x^2-4}{x(x-2)} \cdot \dfrac{1}{\sqrt{x^2+5}+3} = \lim\limits_{x\to 2} \dfrac{\cancel{(x-2)}(x+2)}{x\,\cancel{(x-2)}} \cdot \dfrac{1}{\sqrt{x^2+5}+3} = \dfrac{2+2}{2(\sqrt{4+5}+3)} = \dfrac{4}{12} = \dfrac{1}{3}$

19 **a)** 



b) $$\lim_{x\to+\infty}\frac{4x-1}{\sqrt{x^2+2}}=\lim_{x\to+\infty}\frac{\frac{4x}{x}-\frac{1}{x}}{\frac{\sqrt{x^2+2}}{x}}=\lim_{x\to+\infty}\frac{4-\frac{1}{x}}{\sqrt{1+\frac{2}{x^2}}}=\frac{4-0}{\sqrt{1+0}}=4$$

20 $$\lim_{h\to0}\frac{\sqrt{x+h}-\sqrt{x}}{h}=\lim_{h\to0}\frac{\sqrt{x+h}-\sqrt{x}}{h}\cdot\frac{\sqrt{x+h}+\sqrt{x}}{\sqrt{x+h}+\sqrt{x}}=\lim_{h\to0}\frac{\left(\sqrt{x+h}\right)^2-\left(\sqrt{x}\right)^2}{h\left(\sqrt{x+h}+\sqrt{x}\right)}$$

$$=\lim_{h\to0}\frac{x+h-x}{h\left(\sqrt{x+h}+\sqrt{x}\right)}=\lim_{h\to0}\frac{\cancel{h}}{\cancel{h}\left(\sqrt{x+h}+\sqrt{x}\right)}=\lim_{h\to0}\frac{1}{\left(\sqrt{x+h}+\sqrt{x}\right)}=\frac{1}{\sqrt{x+0}+\sqrt{x}}=\frac{1}{2\sqrt{x}}$$

21 $$\lim_{h\to0}\frac{\frac{1}{x+h}-\frac{1}{x}}{h}=\lim_{h\to0}\frac{\frac{x-(x+h)}{x(x+h)}}{h}=\lim_{h\to0}\frac{-\cancel{h}}{\cancel{h}x(x+h)}=\frac{-1}{x(x+0)}=-\frac{1}{x^2}$$

Exercise 13.2

1 For $f(x)=1-x^2$ we have:

$$f'(x)=\lim_{h\to0}\frac{f(x+h)-f(x)}{h}=\lim_{h\to0}\frac{\left[1-(x+h)^2\right]-\left[1-x^2\right]}{h}$$

$$=\lim_{h\to0}\frac{\cancel{1}-\cancel{x^2}-2xh-h^2-\cancel{1}+\cancel{x^2}}{h}=\lim_{h\to0}(-2x-h)=-2x-0=-2x$$

2 For $g(x)=x^3+2$ we have:

$$g'(x)=\lim_{h\to0}\frac{g(x+h)-g(x)}{h}=\lim_{h\to0}\frac{\left[(x+h)^3+2\right]-\left[x^3+2\right]}{h}$$

$$=\lim_{h\to0}\frac{\cancel{x^3}+3x^2h+3xh^2+h^3+\cancel{2}-\cancel{x^3}-\cancel{2}}{h}=\lim_{h\to0}\left(3x^2+3xh+h^2\right)=3x^2+0+0=3x^2$$

3 For $h(x)=\sqrt{x}$ we have:

$$h'(x)=\lim_{h\to0}\frac{h(x+h)-h(x)}{h}=\lim_{h\to0}\frac{\sqrt{x+h}-\sqrt{x}}{h}\cdot\frac{\sqrt{x+h}+\sqrt{x}}{\sqrt{x+h}+\sqrt{x}}=\lim_{h\to0}\frac{\left(\sqrt{x+h}\right)^2-\left(\sqrt{x}\right)^2}{h\left(\sqrt{x+h}+\sqrt{x}\right)}$$

$$=\lim_{h\to0}\frac{\cancel{x}+h-\cancel{x}}{h\left(\sqrt{x+h}+\sqrt{x}\right)}=\lim_{h\to0}\frac{1}{\left(\sqrt{x+h}+\sqrt{x}\right)}=\frac{1}{\sqrt{x+0}+\sqrt{x}}=\frac{1}{2\sqrt{x}}$$

4 For $r(x) = \frac{1}{x^2}$ we have:

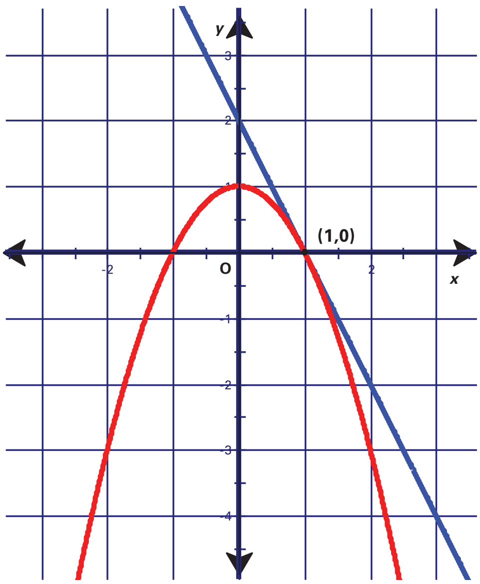
$$r'(x) = \lim_{h\to 0} \frac{r(x+h) - r(x)}{h} = \lim_{h\to 0} \frac{\frac{1}{(x+h)^2} - \frac{1}{x^2}}{h} = \lim_{h\to 0} \frac{\frac{x^2 - (x+h)^2}{x^2(x+h)^2}}{h}$$

$$= \lim_{h\to 0} \frac{\cancel{x^2} - \cancel{x^2} - 2xh - h^2}{hx^2(x+h)^2} = \lim_{h\to 0} \frac{\cancel{h}(-2x-h)}{\cancel{h}x^2(x+h)^2} = \frac{-2x-0}{x^2(x+0)^2} = \frac{-2x}{x^4} = -\frac{2}{x^3}$$

5 **i)** $f(1) = 1 - 1^2 = 0$

$f'(1) = -2 \cdot 1 = -2$

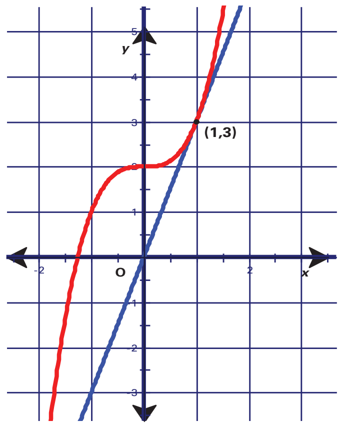
Slope at the point (1, 0) is −2.



ii) $g(1) = 1^3 + 2 = 3$

$g'(1) = 3 \cdot 1^2 = 3$

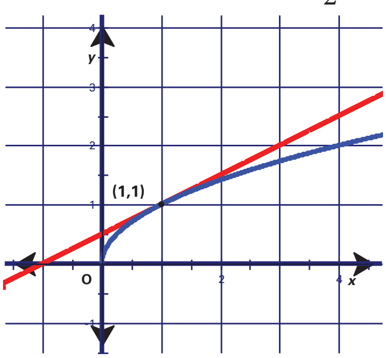
Slope at the point (1, 3) is 3.



iii) $h(1) = \sqrt{1} = 1$

$h'(1) = \frac{1}{2\sqrt{1}} = \frac{1}{2}$

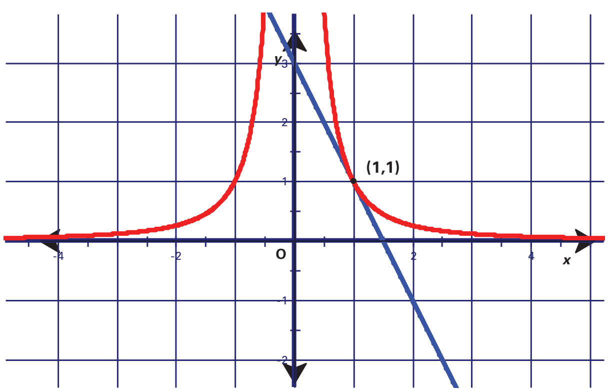
Slope at the point (1, 1) is $\frac{1}{2}$.



iv) $r(1) = \frac{1}{1^2} = 1$

$r'(1) = -\frac{2}{1^3} = -2$

Slope at the point (1, 1) is −2.



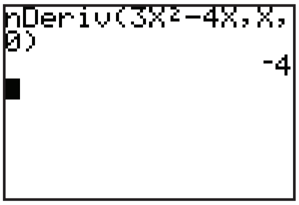
6 For $f(x) = 3x^2 - 4x$ we have:

a) $f'(x) = 3 \cdot 2x - 4 = 6x - 4$

b) $f'(0) = 6 \cdot 0 - 4 = -4$

Slope at the point $(0, 0)$ is -4.

GDC confirmation:

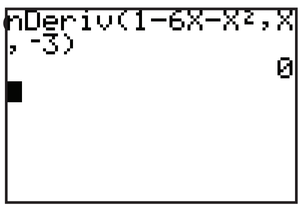


7 For $f(x) = 1 - 6x - x^2$ we have:

a) $f'(x) = 0 - 6 - 2x = -6 - 2x$

b) $f'(-3) = -6 - 2 \cdot (-3) = 0$

Slope at the point $(-3, 10)$ is 0.



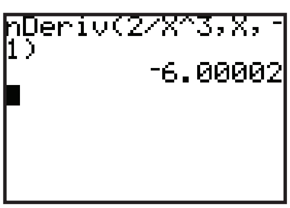
8 For $f(x) = \frac{2}{x^3} = 2x^{-3}$ we have:

a) $f'(x) = 2 \cdot (-3)x^{-3-1} = -\frac{6}{x^4}$

b) $f'(-1) = -\frac{6}{(-1)^4} = -6$

Slope at the point $(-1, 2)$ is -6.

GDC confirmation:



Note: The GDC confirms the slope, but its computations have incorporated a small amount of error.

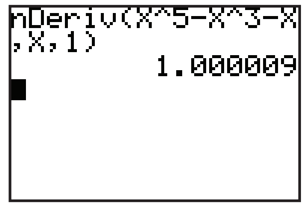
9 For $f(x) = x^5 - x^3 - x$ we have:

a) $f'(x) = 5x^4 - 3x^2 - 1$

b) $f'(1) = 5 - 3 - 1 = 1$

Slope at the point $(1, -1)$ is 1.

GDC confirmation:



Note: The GDC confirms the slope, but its computations have incorporated a small amount of error.

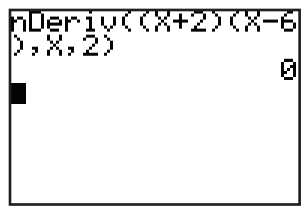
10 For $f(x) = (x + 2)(x - 6) = x^2 - 4x - 12$ we have:

a) $f'(x) = 2x - 4$

b) $f'(2) = 2 \cdot 2 - 4 = 0$

Slope at the point $(2, -16)$ is 0.

GDC confirmation:



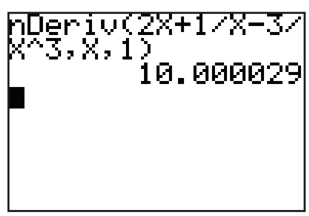
11 For $f(x) = 2x + \frac{1}{x} - \frac{3}{x^3} = 2x + x^{-1} - 3x^{-3}$ we have:

a) $f'(x) = 2 - x^{-1-1} - 3 \cdot (-3)x^{-3-1} = 2 - \frac{1}{x^2} + \frac{9}{x^4}$

b) $f'(1) = 2 - 1 + 9 = 10$

Slope at the point $(1, 0)$ is 10.

GDC confirmation:



Note: The GDC confirms the slope, but its computations have incorporated a small amount of error.

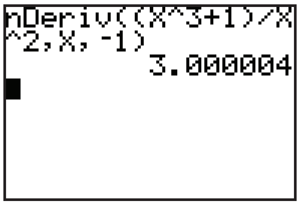
12 For $f(x) = \frac{x^3 + 1}{x^2} = x + x^{-2}$ we have:

a) $f'(x) = 1 - 2x^{-2-1} = 1 - \frac{2}{x^3} = \frac{x^3 - 2}{x^3}$

b) $f'(-1) = \frac{(-1)^3 - 2}{(-1)^3} = 3$

Slope at the point $(-1, 0)$ is 3.

GDC confirmation:



Note: The GDC confirms the slope, but its computations have incorporated a small amount of error.

13 The point $(2, -4)$ is on the curve $f(x) = y = x^2 + ax + b$ so it must be valid.

$f(2) = -4 \Rightarrow 2^2 + 2a + b = -4$

$\Rightarrow 2a + b = -8 \quad (1)$

The slope function is $f'(x) = 2x + a$. Since the slope at the point $(2, -4)$ is -1, we have:

$f'(2) = -1 \Rightarrow 2 \cdot 2 + a = -1 \Rightarrow a = -5$

By substituting $a = -5$ into equation (1) we get:

$2 \cdot (-5) + b = -8 \Rightarrow b = 2$

14 For $y = f(x) = x^2 + 3x$ we have:

$f'(x) = 2x + 3$

$f'(x) = 3 \Rightarrow 2x + 3 = 3 \Rightarrow x = 0$

$f(0) = 0$

So, the point on the graph where the slope is 3 is $(0, 0)$.

15 For $y = f(x) = x^3$ we have:

$f'(x) = 3x^2$

$f'(x) = 12 \Rightarrow 3x^2 = 12 \Rightarrow x_1 = -2, x_2 = 2$

$f(-2) = (-2)^3 = -8, f(2) = 2^3 = 8$

So, the points on the graph where the slope is 12 are $(-2, -8)$ and $(2, 8)$.

16 For $y = f(x) = x^2 - 5x + 1$ we have:

$f'(x) = 2x - 5$

$f'(x) = 0 \Rightarrow 2x - 5 = 0 \Rightarrow x = \frac{5}{2}$

$f\left(\frac{5}{2}\right) = \left(\frac{5}{2}\right)^2 - 5\left(\frac{5}{2}\right) + 1 = -\frac{21}{4}$

So, the point on the graph where the slope is 0 is $\left(\frac{5}{2}, -\frac{21}{4}\right)$.

17 For $y = f(x) = x^2 - 3x$ we have:

$f'(x) = 2x - 3$

$f'(x) = -1 \Rightarrow 2x - 3 = -1 \Rightarrow x = 1$

$f(1) = 1^2 - 3 \cdot 1 = -2$

So, the point on the graph where the slope is -1 is $(1, -2)$.

18 **a)** The average rate of change is greatest between points *A* and *B*, because the line through these points is the steepest (biggest slope).

b) The instantaneous rate of change is positive at points *A*, *B* and *F*, because the tangents at those points are increasing (positive slope).

The instantaneous rate of change is negative at points *D* and *E*, because the tangents at those points are decreasing (negative slope).

The instantaneous rate of change is zero at point *C*, because the tangent at this point is horizontal (zero slope).

c) The average rate of change is approximately equal for pairs *B* and *D*, and *E* and *F* (approximately parallel lines through these pairs).

19 The slope function of the curve $f(x) = x^2 - 4x + 6$ is $f'(x) = 2x - 4$. The slope at the point $(3, 3)$ is $f'(3) = 2 \cdot 3 - 4 = 2$.

The function $g(x) = 8x - 3x^2$ passes through the point (a, b), so we have: $g(a) = 8a - 3a^2 = b$.

The slope function is $g'(x) = 8 - 6x$ and is equal to 2 at (a, b). This gives us: $g'(a) = 8 - 6a = 2$.

To determine a and b we have to solve the system of equations:

$$\begin{cases} b = 8a - 3a^2 \\ 8 - 6a = 2 \end{cases} \Rightarrow a = 1, b = 5$$

20 For $y = f(x) = ax^3 - 2x^2 - x + 7$ we have:

$f'(x) = 3ax^2 - 4x - 1$

$f'(2) = 3 \Rightarrow 12a - 8 - 1 = 3 \Rightarrow a = 1$

21 For $y = f(x) = x^2 - x$, $f'(x) = 2x - 1$.

The tangent at the point $(x, f(x))$ should have a slope equal to 5, since it is parallel to the line $y = 5x$. So, we have:

$f'(x) = 5 \Rightarrow 2x - 1 = 5 \Rightarrow x = 3$

$f(3) = 3^2 - 3 = 6$

The point is $(3, 6)$.

22 **a)** For $f(x) = x^3 + 1$ and $h = 0.1$, we have:

$$\frac{f(2+h) - f(2)}{h} = \frac{f(2.1) - f(2)}{0.1} = \frac{(2.1^3 + 1) - (2^3 + 1)}{0.1} = 12.61$$

b) $f'(x) = 3x^2$

$$\lim_{h \to 0} \frac{f(2+h) - f(2)}{h} = f'(2) = 3 \cdot 2^2 = 12$$

23 For $f(x) = ax^2 + bx + c$ we have:

$$\lim_{h \to 0} \frac{f(x+h) - f(x)}{h} = \lim_{h \to 0} \frac{\left[a(x+h)^2 + b(x+h) + c\right] - \left[ax^2 + bx + c\right]}{h}$$

$$= \lim_{h \to 0} \frac{a(x^2 + 2xh + h^2) + b(x+h) + c - ax^2 - bx - c}{h}$$

$$= \lim_{h \to 0} \frac{2ahx + bh + ah^2}{h} = \lim_{h \to 0} (2ax + b + ah) = 2ax + b = f'(x)$$

i) For $a = 1, b = 0, c = 0$: $f'(x) = 2x$.

ii) For $a = 3, b = -4, c = 2$: $f'(x) = 2 \cdot 3x - (-4) = 16$.

24 For $C(t) = 2\sqrt{t^3} + 17 = 2t^{\frac{3}{2}} + 17, 0 < t < 5$:

a) Average rate of change $= \dfrac{C(4) - C(1)}{4 - 1} = \dfrac{2\sqrt{4^3} + 17 - (2\sqrt{1^3} + 17)}{3} = \dfrac{14}{3} = 4.\bar{6}$ degrees Celsius per hour.

b) $C'(t) = 2 \cdot \dfrac{3}{2} t^{\frac{3}{2} - 1} = 3\sqrt{t}$

c) $C'(t) = \dfrac{14}{3} \Rightarrow 3\sqrt{t} = \dfrac{14}{3} \Rightarrow t = \left(\dfrac{14}{9}\right)^2 \approx 2.42$ hours

25 **a)** If $h(-x) = h(x)$, then: $h'(-x) = \dfrac{d}{dx} h(-x) \cdot \dfrac{d}{dx}(-x) = \dfrac{d}{dx} h(x) \cdot (-1) = -h'(x)$

b) If $p(-x) = -p(x)$, then:

$$p'(-x) = \frac{d}{dx} p(-x) \cdot \frac{d}{dx}(-x) = \frac{d}{dx}\left[-p(-x)\right] \cdot (-1) = -\frac{d}{dx} p(x) \cdot (-1) = -p'(x) \cdot (-1) = p'(x)$$

26 $$\lim_{x \to 0} \frac{\cos x - 1}{x} = \lim_{x \to 0} \frac{1 - 2\sin^2\left(\frac{x}{2}\right) - 1}{x} = \lim_{x \to 0} \frac{-\cancel{2}\sin^2\left(\frac{x}{2}\right)}{\frac{x}{2} \cdot \cancel{2}} = -\lim_{x \to 0} \frac{\sin\left(\frac{x}{2}\right)}{\frac{x}{2}} \cdot \lim_{x \to 0} \sin\left(\frac{x}{2}\right) = -1 \cdot 0 = 0$$

27 $\dfrac{d}{dx}\left(\sqrt{x}\right) = \lim_{h\to 0}\dfrac{\sqrt{x+h}-\sqrt{x}}{h} = \lim_{h\to 0}\dfrac{\sqrt{x+h}-\sqrt{x}}{h}\cdot\dfrac{\sqrt{x+h}+\sqrt{x}}{\sqrt{x+h}+\sqrt{x}}$

$= \lim_{h\to 0}\dfrac{\left(\sqrt{x+h}\right)^2-\left(\sqrt{x}\right)^2}{h\left(\sqrt{x+h}+\sqrt{x}\right)} = \lim_{h\to 0}\dfrac{\cancel{h}}{\cancel{h}\left(\sqrt{x+h}+\sqrt{x}\right)} = \dfrac{1}{\sqrt{x+0}+\sqrt{x}} = \dfrac{1}{2\sqrt{x}}$

28 $\dfrac{d}{dx}\left(\dfrac{1}{x}\right) = \lim_{h\to 0}\dfrac{\dfrac{1}{x+h}-\dfrac{1}{x}}{h} = \lim_{h\to 0}\dfrac{\dfrac{x-(x+h)}{x(x+h)}}{h} = \lim_{h\to 0}\dfrac{-\cancel{h}}{\cancel{h}x(x+h)} = \dfrac{-1}{x(x+0)} = -\dfrac{1}{x^2}$

29 $\dfrac{d}{dx}\left(\dfrac{2+x}{3-x}\right) = \lim_{h\to 0}\dfrac{\dfrac{2+(x+h)}{3-(x+h)}-\dfrac{2+x}{3-x}}{h} = \lim_{h\to 0}\dfrac{(2+x+h)(3-x)-(2+x)(3-x-h)}{h(3-x)(3-x-h)}$

$= \lim_{h\to 0}\dfrac{(6+x+3h-xh-x^2)-(6+x-2h-xh-x^2)}{h(3-x)(3-x-h)} = \lim_{h\to 0}\dfrac{5\cancel{h}}{\cancel{h}(3-x)(3-x-h)}$

$= \dfrac{5}{(3-x)(3-x-0)} = \dfrac{5}{(3-x)^2}$

30 $\dfrac{d}{dx}\left(\dfrac{1}{\sqrt{x+2}}\right) = \lim_{h\to 0}\dfrac{\dfrac{1}{\sqrt{x+h+2}}-\dfrac{1}{\sqrt{x+2}}}{h} = \lim_{h\to 0}\dfrac{\sqrt{x+2}-\sqrt{x+h+2}}{h\sqrt{x+2}\sqrt{x+h+2}}\cdot\dfrac{\sqrt{x+2}+\sqrt{x+h+2}}{\sqrt{x+2}+\sqrt{x+h+2}}$

$= \lim_{h\to 0}\dfrac{\left(\sqrt{x+2}\right)^2-\left(\sqrt{x+h+2}\right)^2}{h\sqrt{x+2}\sqrt{x+h+2}\left(\sqrt{x+2}+\sqrt{x+h+2}\right)} = \lim_{h\to 0}\dfrac{-\cancel{h}}{\cancel{h}\sqrt{x+2}\sqrt{x+h+2}\left(\sqrt{x+2}+\sqrt{x+h+2}\right)}$

$= \dfrac{-1}{\sqrt{x+2}\sqrt{x+0+2}\left(\sqrt{x+2}+\sqrt{x+0+2}\right)} = -\dfrac{1}{2(x+2)\sqrt{x+2}}$

31 $\dfrac{d}{dx}(c) = \lim_{h\to 0}\dfrac{c-c}{h} = \lim_{h\to 0} 0 = 0$

Exercise 13.3

1 For $y = x^2 - 2x - 6$ we have: $\dfrac{dy}{dx} = 2x - 2 \Rightarrow 2x - 2 = 0 \Rightarrow x = 1$

$y = 1^2 - 2\cdot 1 - 6 = -7$

The vertex of the parabola is the point $(1, -7)$.

2 For $y = 4x^2 + 12x + 17$ we have: $\dfrac{dy}{dx} = 8x + 12 \Rightarrow 8x + 12 = 0 \Rightarrow x = -\dfrac{3}{2}$

$y = 4\left(-\dfrac{3}{2}\right)^2 + 12\left(-\dfrac{3}{2}\right) + 17 = 8$

The vertex of the parabola is the point $\left(-\dfrac{3}{2}, 8\right)$.

3 For $y = -x^2 + 6x - 7$ we have:

$\dfrac{dy}{dx} = -2x + 6 \Rightarrow -2x + 6 = 0 \Rightarrow x = 3$

$y = -3^2 + 6\cdot 3 - 7 = 2$

The vertex of the parabola is the point (3, 2).

4 For $y = f(x) = x^2 - 5x + 6$ we have:

a) $f'(x) = 2x - 5 = 0 \Rightarrow x = \frac{5}{2}$

Since the function is continuous with a domain of all real numbers, the point with $x = \frac{5}{2}$ is the only stationary point. One stationary point means that there are two intervals that need to be tested.

b) $f'(3) = 2 \cdot 3 - 5 = 1 > 0 \Rightarrow f(x)$ is increasing for $x > \frac{5}{2}$

c) $f'(1) = 2 - 5 = -3 < 0 \Rightarrow f(x)$ is decreasing for $x < \frac{5}{2}$

5 For $y = f(x) = 7 - 4x - 3x^2$ we have:

a) $f'(x) = -4 - 6x = 0 \Rightarrow x = -\frac{2}{3}$

Since the function is continuous with a domain of all real numbers, the point with $x = -\frac{2}{3}$ is the only stationary point. One stationary point means that there are two intervals that need to be tested.

b) $f'(-1) = -4 - 6 \cdot (-1) = 2 > 0 \Rightarrow f(x)$ is increasing for $x < -\frac{2}{3}$

c) $f'(1) = -4 - 6 = -10 < 0 \Rightarrow f(x)$ is decreasing for $x > -\frac{2}{3}$

6 For $y = f(x) = \frac{1}{3}x^3 - x$ we have:

a) $f'(x) = x^2 - 1 = 0 \Rightarrow x = -1$ or $x = 1$

Since the function is continuous with a domain of all real numbers, there are two stationary points with $x = -1$ and $x = 1$. Two stationary points means that there are three intervals that need to be tested.

b) $f'(-2) = (-2)^2 - 1 = 3 > 0,\ f'(2) = 4 - 1 = 3 > 0 \Rightarrow f(x)$ is increasing for $x < -1$ and $x > 1$

c) $f'(0) = -1 < 0 \Rightarrow f(x)$ is decreasing for $-1 < x < 1$

7 For $y = f(x) = x^4 - 4x^3$ we have:

a) $f'(x) = 4x^3 - 12x^2 = 0 \Rightarrow 4x^2(x-3) = 0 \Rightarrow x = 0$ or $x = 3$

Since the function is continuous with a domain of all real numbers, there are two stationary points with $x = 0$ and $x = 3$. So, three intervals need to be tested.

b) $f'(4) = 4 \cdot 4^3 - 12 \cdot 4^2 = 64 > 0 \Rightarrow f(x)$ is increasing for $x > 3$

c) $f'(-1) = 4(-1)^3 - 12 \cdot (-1)^2 = -16 < 0,\ f'(1) = 4 - 12 = -8 < 0 \Rightarrow f(x)$ is decreasing for $x < 0$ and $x < 3$

8 For $y = f(x) = 2x^3 + 3x^2 - 72x + 5$ we have:

a) $f'(x) = 6x^2 + 6x - 72 = 0 \Rightarrow x^2 + x - 12 = 0 \Rightarrow x = \frac{-1 \pm \sqrt{1^2 - 4 \cdot 1 \cdot (-12)}}{2 \cdot 1} = \frac{-1 \pm 7}{2} \Rightarrow$

$x = -4$ and $x = 3$

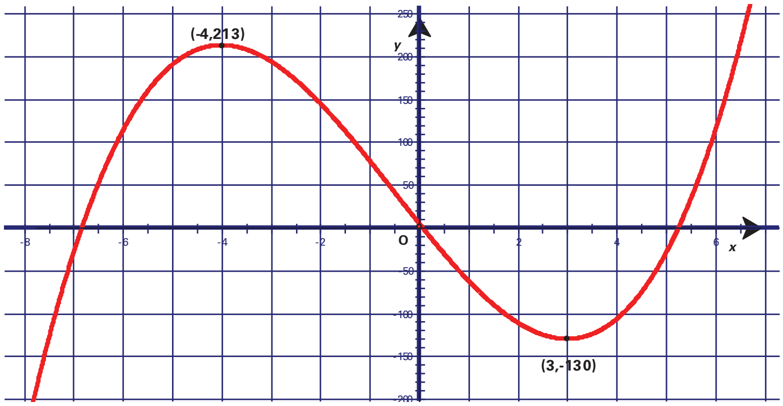
$f(-4) = 2 \cdot (-4)^3 + 3 \cdot (-4)^2 - 72 \cdot (-4) + 5 = 213,\ f(3) = 2 \cdot 3^3 + 3 \cdot 3^2 - 72 \cdot 3 + 5 = -130$

Therefore, the stationary points are $(-4, 213)$ and $(3, -130)$.

b) $f''(x) = 12x + 6$

$f''(-4) = -48 + 6 = -42 < 0 \Rightarrow$ maximum at $(-4, 213)$

$f''(3) = 36 + 6 = 42 > 0 \Rightarrow$ minimum at $(3, -130)$

c) 

9 For $y = f(x) = \frac{1}{6}x^3 - 5$ we have:

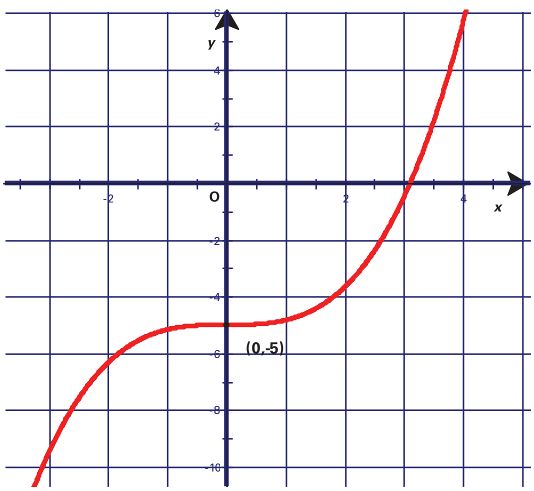
a) $f'(x) = \frac{1}{2}x^2 = 0 \Rightarrow x = 0$

$f(0) = -5$

Therefore, the stationary point is $(0, -5)$.

b) $f''(x) = x,\ f''(0) = 0$

$f'(-1) = \frac{1}{2}\cdot(-1)^2 = \frac{1}{2} > 0,\ f'(1) = \frac{1}{2}\cdot 1^2 = \frac{1}{2} > 0 \Rightarrow (0, -5)$ is neither a minimum nor a maximum because the first derivative is always positive, i.e. the function is constantly increasing.

c) 

10 For $y = f(x) = x(x-3)^2 = x^3 - 6x^2 + 9x$ we have:

a) $f'(x) = 3x^2 - 12x + 9 = 0 \Rightarrow x^2 - 4x + 3 = 0 \Rightarrow x = \frac{4 \pm \sqrt{(-4)^2 - 4\cdot 1\cdot 3}}{2\cdot 1} = \frac{4 \pm 2}{2} \Rightarrow$

$x = 1$ and $x = 3$

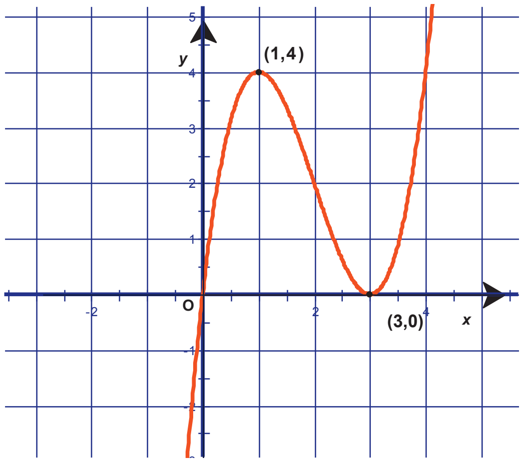
$f(1) = 1 - 6 + 9 = 4,\ f(3) = 3^3 - 6\cdot 3^2 + 9\cdot 3 = 0$

Therefore, the stationary points are (1, 4) and (3, 0).

b) $f''(x) = 6x - 12$

$f''(1) = 6 - 12 = -6 < 0 \Rightarrow$ maximum at (1, 4)

$f''(3) = 18 - 12 = 6 > 0 \Rightarrow$ minimum at (3, 0)

c) 

11 For $y = f(x) = x^4 - 2x^3 - 5x^2 + 6$ we have:

a) $f'(x) = 4x^3 - 6x^2 - 10x = 0 \Rightarrow 2x(2x^2 - 3x - 5) = 0 \Rightarrow x = 0$ or

$(2x^2 - 3x - 5) = 0 \Rightarrow x = \dfrac{3 \pm \sqrt{(-3)^2 - 4 \cdot 2 \cdot (-5)}}{2 \cdot 2} = \dfrac{3 \pm 7}{4} \Rightarrow x = -1$ and $x = \dfrac{5}{2}$

$f(0) = 6,\ f(-1) = (-1)^4 - 2\cdot(-1)^3 - 5\cdot(-1)^2 + 6 = 4,\ f\left(\dfrac{5}{2}\right) = \left(\dfrac{5}{2}\right)^4 - 2\cdot\left(\dfrac{5}{2}\right)^3 - 5\cdot\left(\dfrac{5}{2}\right)^2 + 6 = -\dfrac{279}{16}$

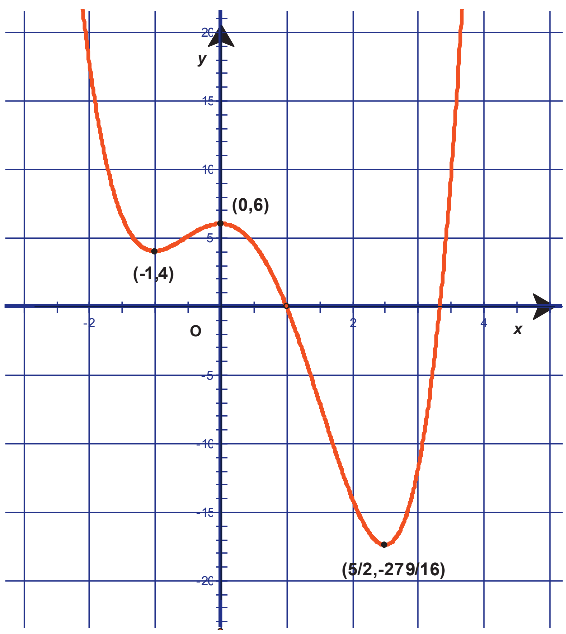
Therefore, the stationary points are $(0, 6), (-1, 4)$ and $\left(\dfrac{5}{2}, -\dfrac{279}{16}\right)$.

b) $f''(x) = 12x^2 - 12x - 10$

$f''(0) = -10 < 0 \Rightarrow$ maximum at $(0, 6)$

$f''(-1) = 12\cdot(-1)^2 - 12\cdot(-1) - 10 = 14 > 0 \Rightarrow$ minimum at $(-1, 4)$

$f''\left(\dfrac{5}{2}\right) = 12\cdot\left(\dfrac{5}{2}\right)^2 - 12\cdot\dfrac{5}{2} - 10 = 35 > 0 \Rightarrow$ minimum at $\left(\dfrac{5}{2}, -\dfrac{279}{16}\right)$

c) 

12 For $y = f(x) = x^3 - 2x^2 - 7x + 10$ we have:

a) $f'(x) = 3x^2 - 4x - 7 = 0 \Rightarrow x = \dfrac{4 \pm \sqrt{(-4)^2 - 4 \cdot 3 \cdot (-7)}}{2 \cdot 3} = \dfrac{4 \pm 10}{6} \Rightarrow x = -1$ and $x = \dfrac{7}{3}$

$f(-1) = -1 - 2 + 7 + 10 = 14,\ f\left(\dfrac{7}{3}\right) = \left(\dfrac{7}{3}\right)^3 - 2 \cdot \left(\dfrac{7}{3}\right)^2 - 7 \cdot \dfrac{7}{3} + 10 = -\dfrac{122}{27}$

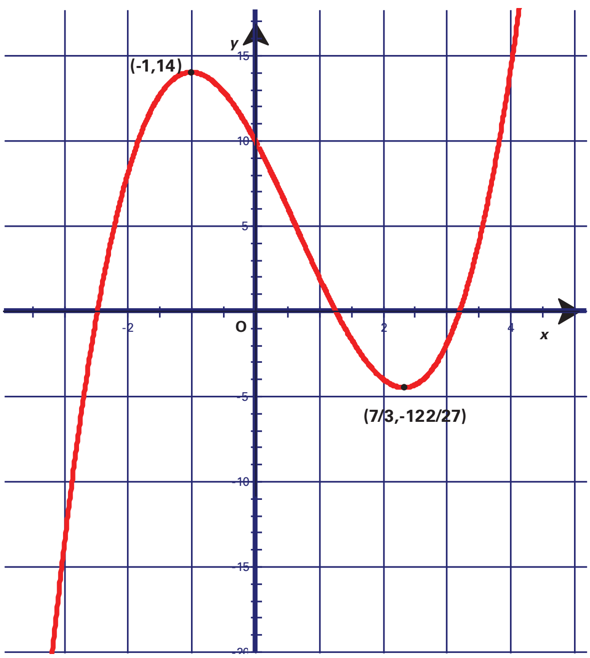
Therefore, the stationary points are $(-1, 14)$ and $\left(\dfrac{7}{3}, -\dfrac{122}{27}\right)$.

b) $f''(x) = 6x - 4$

$f''(-1) = -6 - 4 = -10 < 0 \Rightarrow$ maximum at $(-1, 14)$

$f''\left(\dfrac{7}{3}\right) = 6 \cdot \dfrac{7}{3} - 4 = 10 > 0 \Rightarrow$ minimum at $\left(\dfrac{7}{3}, -\dfrac{122}{27}\right)$

c)



13 For $y = f(x) = x - \sqrt{x} = x^{3} - x^{\frac{1}{2}}$ we have:

a) $f'(x) = 1 - \dfrac{1}{2}x^{-\frac{1}{2}} = 1 - \dfrac{1}{2\sqrt{x}} = 0 \Rightarrow$

$2\sqrt{x} - 1 = 0 \Rightarrow x = \dfrac{1}{4}$

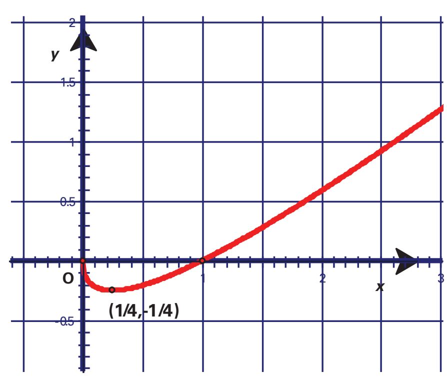
$f\left(\dfrac{1}{4}\right) = \dfrac{1}{4} - \sqrt{\dfrac{1}{4}} = -\dfrac{1}{4}$

Therefore, the stationary point is $\left(\dfrac{1}{4}, -\dfrac{1}{4}\right)$.

b) $f''(x) = \dfrac{1}{4}x^{-\frac{3}{2}}$

$f''\left(\dfrac{1}{4}\right) = \dfrac{1}{4\sqrt{\left(\dfrac{1}{4}\right)^3}} = 2 > 0 \Rightarrow$ minimum at $\left(\dfrac{1}{4}, -\dfrac{1}{4}\right)$

c)

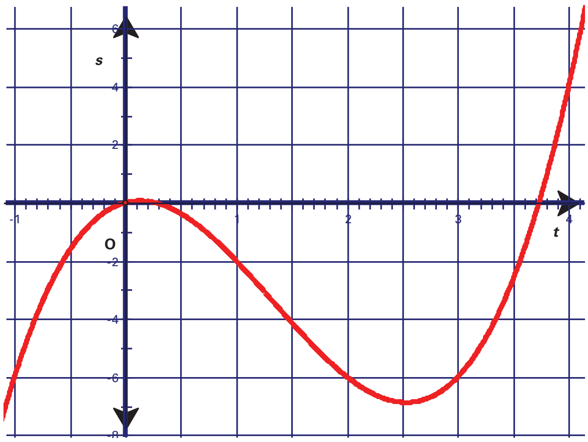


14 For $s(t) = t^3 - 4t^2 + t$ we have:

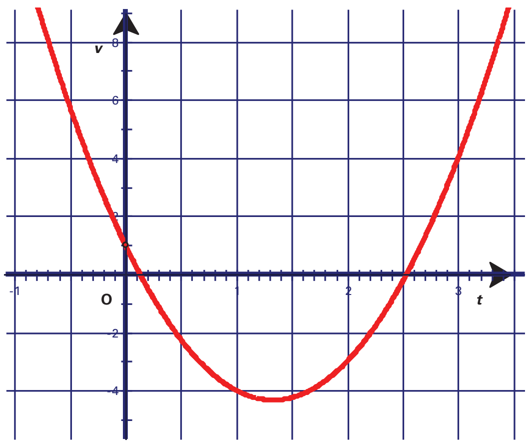
a) $v(t) = \frac{ds}{dt} = 3t^2 - 8t + 1$

$a(t) = \frac{dv}{dt} = \frac{d^2s}{dt^2} = 6t - 8$

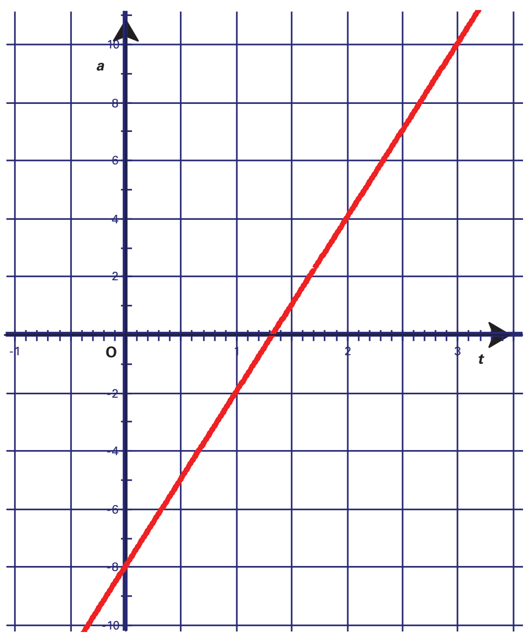
b)



Displacement function
$s(t) = t^3 - 4t^2 + t$



Velocity function
$v(t) = 3t^2 - 8t + 1$



Acceleration function
$a(t) = 6t - 8$

c) $s'(t) = 0 \Rightarrow 3t^2 - 8t + 1 = 0 \Rightarrow t = \dfrac{8 \pm \sqrt{(-8)^2 - 4 \cdot 3 \cdot 1}}{2 \cdot 3} = \dfrac{4 \pm \sqrt{13}}{3} \Rightarrow$

$t \approx 0.1315$ and $t = 2.5352$

$s(0.1315) = 0.1315^3 - 4 \cdot 0.1315^2 + 0.1315 \approx 0.0646$

Therefore, the object has a maximum displacement of approximately 0.0646 m at $t \approx 0.1315$.

d) $v'(t) = 0 \Rightarrow 6t - 8 = 0 \Rightarrow t = \dfrac{4}{3} = 1.\overline{3}$

$v\left(\dfrac{4}{3}\right) = 3\left(\dfrac{4}{3}\right)^2 - 8\left(\dfrac{4}{3}\right) + 1 = -\dfrac{13}{3} = -4.\overline{3}$

Therefore, the object has a minimum velocity of $-4.\overline{3}$ metres per second at $t = 1.\overline{3}$.

e) The object moves right at a decreasing velocity, then turns left with increasing velocity, then slows down and turns right with increasing velocity.

15 For $f(x) = x^3 - 12x$ we have:

$f'(x) = 3x^2 - 12 = 0 \Rightarrow x^2 - 4 = 0 \Rightarrow (x+2)(x-2) = 0 \Rightarrow x = -2$ and $x = 2$

Since the function is continuous with a domain of all real numbers, there are two stationary points with $x = -2$ and $x = 2$. So, there are three intervals that need to be tested:

$f'(-3) = 3 \cdot (-3)^2 - 12 = 15 > 0 \Rightarrow f(x) \nearrow$ on $(-\infty, -2)$

$f'(0) = -12 < 0 \Rightarrow f(x) \searrow$ on $(-2, 2)$

$f'(3) = 3 \cdot 3^2 - 12 = 15 > 0 \Rightarrow f(x) \nearrow$ on $(2, \infty)$

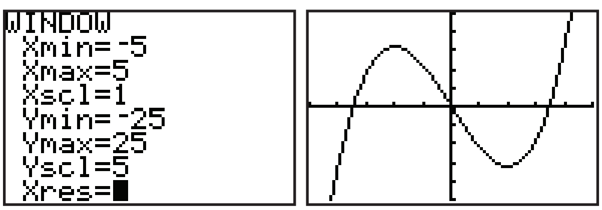
$f(-2) = (-2)^3 - 12 \cdot (-2) = 16,\ f(2) = 2^3 - 12 \cdot 2 = -16$

Therefore, f has a relative maximum at $(-2, 16)$ and a relative minimum at $(2, -16)$.

$f''(x) = 6x = 0 \Rightarrow x = 0$

$f(0) = 0$

Therefore, f has a point of inflexion at $(0, 0)$.



16 For $f(x) = \dfrac{1}{4}x^4 - 2x^2$ we have:

$f'(x) = x^3 - 4x = 0 \Rightarrow x(x^2 - 4) = 0 \Rightarrow x(x+2)(x-2) = 0 \Rightarrow x = 0, x = -2$ and $x = 2$

Since the function is continuous with a domain of all real numbers, there are three stationary points with $x = 0, x = -2$ and $x = 2$. So, there are four intervals that need to be tested:

$f'(-3) = (-3)^3 - 4 \cdot (-3) = -15 < 0 \Rightarrow f(x) \searrow$ on $(-\infty, -2)$

$f'(-1) = (-1)^3 - 12(-1) = 11 > 0 \Rightarrow f(x) \nearrow$ on $(-2, 0)$

$f'(1) = 1 - 12 = -11 < 0 \Rightarrow f(x) \searrow$ on $(0, 2)$

$f'(3) = 3^3 - 4 \cdot 3 = 15 > 0 \Rightarrow f(x) \nearrow$ on $(2, \infty)$

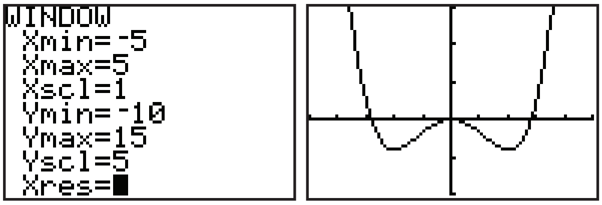
$f(-2) = \dfrac{1}{4}(-2)^4 - 2 \cdot (-2)^2 = -4,\ f(0) = 0,\ f(2) = \dfrac{1}{4}2^4 - 2 \cdot 2^2 = -4$

Therefore, f has an absolute minimum at $(-2,-4)$ and $(2,-4)$, and a relative maximum at $(0,0)$.

$$f''(x)=3x^2-4=0 \Rightarrow x=\pm\frac{2}{\sqrt{3}} \Rightarrow x=-\frac{2\sqrt{3}}{3} \text{ and } x=\frac{2\sqrt{3}}{3}$$

$$f\left(-\frac{2\sqrt{3}}{3}\right)=\frac{1}{4}\cdot\left(\frac{2\sqrt{3}}{3}\right)^4-2\cdot\left(\frac{2\sqrt{3}}{3}\right)^2=-\frac{20}{9}=f\left(\frac{2\sqrt{3}}{3}\right)$$

Therefore, f has a point of inflexion at $\left(-\frac{2\sqrt{3}}{3},-\frac{20}{9}\right)$ and $\left(\frac{2\sqrt{3}}{3},-\frac{20}{9}\right)$.



17 For $f(x)=x+\frac{4}{x}=x+4x^{-1}$ we have:

$$f'(x)=1-4x^{-2}=1-\frac{4}{x^2}=\frac{x^2-4}{x^2}=0 \Rightarrow (x-2)(x+2)=0 \Rightarrow x=-2 \text{ and } x=2$$

Since the function is not defined for $x=0$ (vertical asymptote), there are two stationary points with $x=-2$ and $x=2$ but four intervals that need to be tested:

$f'(-3)=\frac{5}{9}>0 \Rightarrow f(x)\nearrow$ on $(-\infty,-2)$

$f'(-1)=-3<0 \Rightarrow f(x)\searrow$ on $(-2,0)$

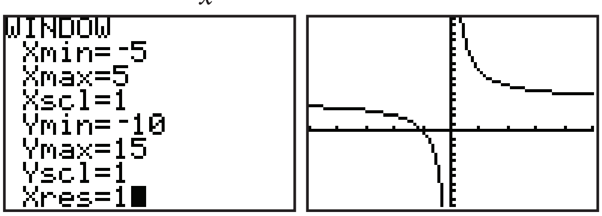
$f'(1)=-3<0 \Rightarrow f(x)\searrow$ on $(0,2)$

$f'(3)=-3<0 \Rightarrow f(x)\searrow$ on $(2,\infty)$

$f(-2)=-4,\ f(2)=4$

Therefore, f has a relative maximum at $(-2,-4)$ and a relative minimum at $(2,4)$.

$f''(x)=8x^{-3}=\frac{8}{x^3}\neq 0$ for all x from the domain, so there are no points of inflexion.



18 For $f(x)=x^2-\frac{1}{x}=x^2-x^{-1}$ we have:

$$f'(x)=2x+x^{-2}=\frac{2x^3+1}{x^2}=0 \Rightarrow 2x^3+1=0 \Rightarrow x=\sqrt[3]{-\frac{1}{2}}=-\frac{\sqrt[3]{4}}{2}\approx -0.79$$

Since the function is not defined for $x=0$ (vertical asymptote), there is one stationary point with $x=-\frac{\sqrt[3]{4}}{2}$ but three intervals that need to be tested:

$$f'(-2)=\frac{2\cdot(-2)^3+1}{(-2)^2}=-\frac{15}{4}<0\Rightarrow f(x)\searrow \text{ on }\left(-\infty,-\frac{\sqrt[3]{4}}{2}\right)$$

$$f'\left(-\frac{1}{2}\right)=\frac{2\cdot\left(-\frac{1}{2}\right)^3+1}{\left(-\frac{1}{2}\right)^2}=3>0\Rightarrow f(x)\nearrow \text{ on }\left(-\frac{\sqrt[3]{4}}{2},0\right)$$

$$f'(1)=\frac{2+1}{1}=3>0\Rightarrow f(x)\nearrow \text{ on }(0,\infty)$$

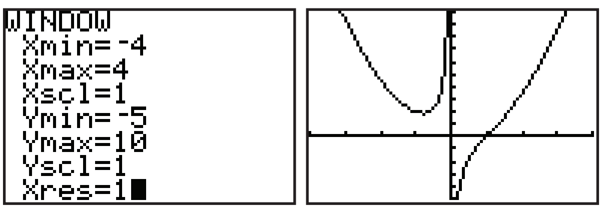
$$f\left(-\frac{\sqrt[3]{4}}{2}\right)=\left(-\frac{\sqrt[3]{4}}{2}\right)^2-\left(-\frac{\sqrt[3]{4}}{2}\right)^{-1}=\frac{3\sqrt[3]{2}}{2}$$

Therefore, f has a relative minimum at $\left(-\frac{\sqrt[3]{4}}{2},\frac{3\sqrt[3]{2}}{2}\right)$.

$$f''(x)=2-2x^{-3}=\frac{2(x^3-1)}{x^3}=0\Rightarrow x=1$$

$$f(1)=1-1=0$$

Therefore, f has a point of inflexion at $(1, 0)$.



19 For $f(x)=-3x^5+5x^3$ we have:

$$f'(x)=-15x^4+15x^2=0\Rightarrow -15x^2(x^2-1)=0\Rightarrow -15x^2(x+1)(x-1)=0\Rightarrow x=0,\ x=-1 \text{ and } x=1$$

Since the function is continuous with a domain of all real numbers, there are three stationary points with $x=0$, $x=-1$ and $x=1$. So, there are four intervals that need to be tested:

$$f'(-2)=-15(-2)^4+15\cdot(-2)^2=-180<0\Rightarrow f(x)\searrow \text{ on }(-\infty,-1)$$

$$f'\left(-\frac{1}{2}\right)=-15\left(-\frac{1}{2}\right)^4+15\left(-\frac{1}{2}\right)^2=\frac{45}{16}>0\Rightarrow f(x)\nearrow \text{ on }(-1,0)$$

$$f'\left(\frac{1}{2}\right)=-15\left(\frac{1}{2}\right)^4+15\left(\frac{1}{2}\right)^2=\frac{45}{16}>0\Rightarrow f(x)\nearrow \text{ on }(0,1)$$

$$f'(2)=-15\cdot 2^4+15\cdot 2^2=-180<0\Rightarrow f(x)\searrow \text{ on }(1,\infty)$$

$$f(-1)=-3\cdot(-1)^5+5\cdot(-1)^3=-2,\ f(0)=0,\ f(1)=-3+5=2$$

Therefore, f has a relative minimum at $(-1,-2)$ and a relative maximum at $(1, 2)$. Since $f'(x)$ does not change sign at $x=0$, the point $(0, 0)$ is not a relative extremum.

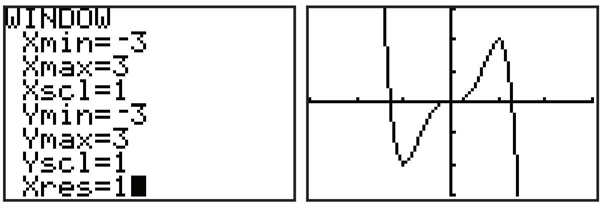
$$f''(x) = -60x^3 + 30x = 0 \Rightarrow -30x(2x^2 - 1) = 0 \Rightarrow x = 0,\ x = -\frac{\sqrt{2}}{2} \text{ and } x = \frac{\sqrt{2}}{2}$$

$$f\left(-\frac{\sqrt{2}}{2}\right) = -3\cdot\left(-\frac{\sqrt{2}}{2}\right)^5 + 5\cdot\left(-\frac{\sqrt{2}}{2}\right)^3 = -\frac{7\sqrt{2}}{8}$$

$$f(0) = 0$$

$$f\left(\frac{\sqrt{2}}{2}\right) = -3\cdot\left(\frac{\sqrt{2}}{2}\right)^5 + 5\cdot\left(\frac{\sqrt{2}}{2}\right)^3 = \frac{7\sqrt{2}}{8}$$

Therefore, f has points of inflexion at $\left(-\frac{\sqrt{2}}{2}, -\frac{7\sqrt{2}}{8}\right)$, $(0, 0)$ and $\left(\frac{\sqrt{2}}{2}, \frac{7\sqrt{2}}{8}\right)$.



20 For $f(x) = 3x^4 - 4x^3 - 12x^2 + 5$ we have:

$$f'(x) = 12x^3 - 12x^2 - 24x = 12x(x^2 - x - 2) = 0 \Rightarrow 12x(x+1)(x-2) = 0 \Rightarrow x = 0,\ x = -1 \text{ and } x = 2$$

Since the function is continuous with a domain of all real numbers, there are three stationary points with $x = 0$, $x = -1$ and $x = 2$. So, there are four intervals that need to be tested:

$f'(-2) = 12(-2)^3 - 12\cdot(-2)^2 - 24\cdot(-2) = -96 < 0 \Rightarrow f(x) \searrow$ on $(-\infty, -1)$

$f'\left(-\frac{1}{2}\right) = 12\left(-\frac{1}{2}\right)^3 - 12\left(-\frac{1}{2}\right)^2 - 24\left(-\frac{1}{2}\right) = \frac{15}{2} > 0 \Rightarrow f(x) \nearrow$ on $(-1, 0)$

$f'(1) = 12 - 12 - 24 = -24 < 0 \Rightarrow f(x) \searrow$ on $(0, 2)$

$f'(3) = 12\cdot 3^3 - 12\cdot 3^2 - 24\cdot 3 = 144 > 0 \Rightarrow f(x) \nearrow$ on $(2, \infty)$

$f(-1) = 3 + 4 - 12 + 5 = 0,\ f(0) = 5,\ f(2) = 3\cdot 2^4 - 4\cdot 2^3 - 12\cdot 2^2 + 5 = -27$

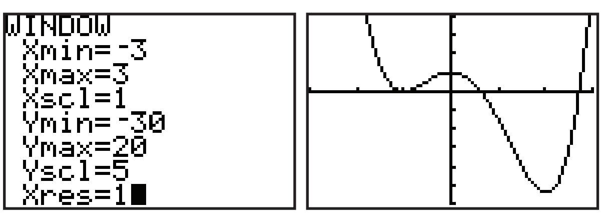
Therefore, f has a relative minimum at $(-1, 0)$, a relative maximum at $(0, 5)$, and an absolute minimum at $(2, -27)$.

$$f''(x) = 36x^2 - 24x - 24 = 0 \Rightarrow 3x^2 - 2x - 2 = 0 \Rightarrow x = \frac{2 \pm \sqrt{(-2)^2 - 4\cdot 3\cdot(-2)}}{2\cdot 3} = \frac{1 \pm \sqrt{7}}{3} \Rightarrow$$

$x \approx -0.549$ and $x \approx 1.22$

$f(-0.549) \approx 2.32,\ f(1.22) \approx -13.4$

Therefore, f has points of inflexion at $(-0.549, 2.32)$ and $(1.22, -13.4)$.



21 For the displacement function $s(t) = t(t-3)(8t-9) = 8t^3 - 33t^2 + 27t$, we have:

a) Velocity function: $v(t) = \dfrac{ds}{dt} = 24t^2 - 66t + 27$

Initial velocity: $v(0) = 27$ m/s

Acceleration function: $a(t) = \dfrac{dv}{dt} = \dfrac{d^2s}{dt^2} = 48t - 66$

Initial acceleration: $a(0) = -66$ m/s^2

b) At $t = 3$ seconds:

$v(3) = 24 \cdot 3^2 - 66 \cdot 3 + 27 = 45$ m/s

$a(3) = 48 \cdot 3 - 66 = 78$ m/s^2

c) The object changes direction when the displacement function has a relative maximum or minimum.

$s'(t) = v(t) = 24t^2 - 66t + 27 = 0 \Rightarrow 8t^2 - 22t + 9 = 0 \Rightarrow t = \dfrac{22 \pm \sqrt{(-22)^2 - 4 \cdot 8 \cdot 9}}{2 \cdot 8} \Rightarrow$

$t = \dfrac{1}{2}$ and $t = \dfrac{9}{4}$

At $t = \frac{1}{2}$ and $t = \frac{9}{4}$, the displacement has a relative maximum or minimum.

d) Velocity is a minimum when the acceleration is zero:

$v'(t) = a(t) = 48t - 66 = 0 \Rightarrow t = \dfrac{11}{8}$

At $t = \frac{11}{8}$, the acceleration is zero.

22 For the delivery cost function $D(x) = 3x + \dfrac{100}{x}$, $x > 0$, we have:

$D'(x) = 3 - \dfrac{100}{x^2} = 0 \Rightarrow 3x^2 - 100 = 0 \Rightarrow x = \sqrt{\dfrac{100}{3}} \approx 5.77$

$D(5.77) = 3 \cdot 5.77 + \dfrac{100}{5.77} \approx 34.6$

$D'(4) = 3 - \dfrac{100}{4^2} = -3.25 < 0 \Rightarrow D(x) \searrow$

$D'(6) = 3 - \dfrac{100}{6^2} = 0.22 > 0 \Rightarrow D(x) \nearrow$

Therefore, the delivery cost per tonne of bananas is a minimum for 5.77 tonnes, and the minimum delivery cost is \$34 600. This cost is a minimum because $D(x)$ decreases to this value then increases.

23 The curve $y = f(x) = x^4 + ax^2 + bx + c$ passes through the point $(-1, -8)$, so $1 + a - b + c = -8$.

Then:

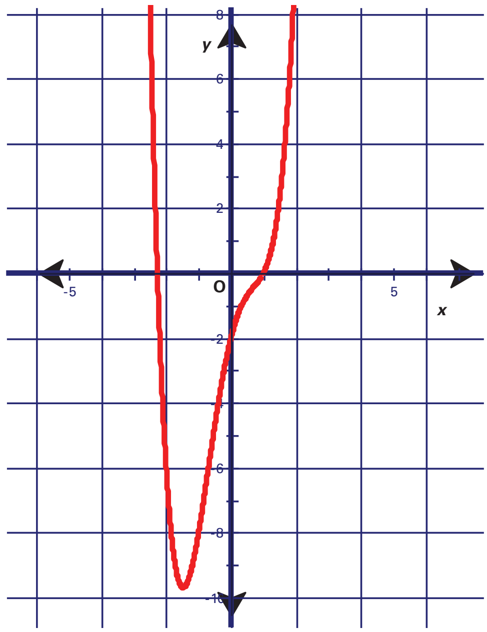
$f'(x) = \dfrac{dy}{dx} = 4x^3 + 2ax + b$ and $\dfrac{dy}{dx} = 6$; at $x = -1 \Rightarrow -4 - 2a + b = 6$

$f''(x) = \dfrac{d^2y}{dx^2} = 12x^2 + 2a$ and $\dfrac{d^2y}{dx^2} = 6$; at $x = -1 \Rightarrow 12 + 2a = 6$

These conditions give us a system of linear equations:

$$\begin{cases} a - b + c = -9 \\ -2a + b = 10 \\ 2a = -6 \end{cases} \Rightarrow a = -3, b = 4, c = -2$$

Therefore, the curve is $y = x^4 - 3x^2 + 4x - 2$.



24 For $f(x) = \dfrac{x^3 + 3x - 1}{x^2} = x + 3x^{-1} - x^{-2}$ we have:

$$f'(x) = 1 - 3x^{-2} + 2x^{-3} = \frac{x^3 - 3x + 2}{x^3} = 0 \Rightarrow x^3 - 3x + 2 = 0 \Rightarrow (x-1)(x^2 + x - 2) = 0 \Rightarrow$$

$(x-1)^2(x+2) = 0 \Rightarrow x = -2$ and $x = 1$

$$f(-2) = \frac{(-2)^3 + 3(-2) - 1}{(-2)^2} = -\frac{15}{4},\ f(1) = \frac{1 + 3 - 1}{1} = 3$$

$$f''(x) = 6x^{-3} - 6x^{-4} = \frac{6(x-1)}{x^4} = 0 \Rightarrow x = 1$$

Since the function is not defined for $x = 0$ (vertical asymptote), there are two stationary points with $x = -2$ and $x = 1$ but four intervals that need to be tested:

$$f'(-3) = \frac{(-3)^3 - 3(-3) + 2}{(-3)^3} = \frac{16}{27} > 0 \Rightarrow f(x) \nearrow \text{ on } (-\infty, -2)$$

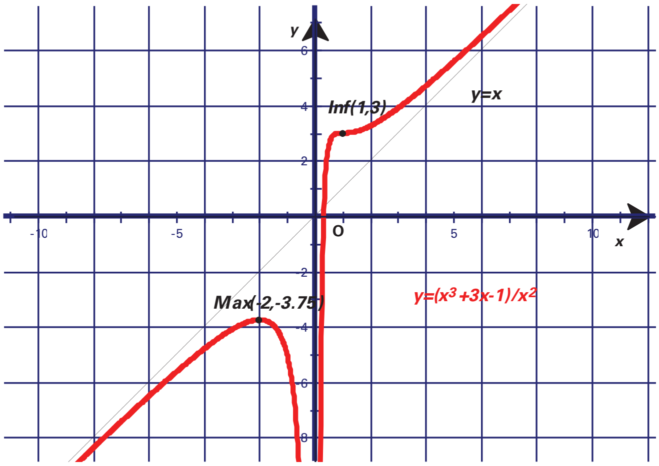
$$f'(-1) = \frac{(-1)^3 - 3(-1) + 2}{(-1)^3} = -4 < 0 \Rightarrow f(x) \searrow \text{ on } (-2, 0)$$

$$f'\left(\frac{1}{2}\right) = \frac{\left(\frac{1}{2}\right)^3 - 3\left(\frac{1}{2}\right) + 2}{\left(\frac{1}{2}\right)^3} = 5 > 0 \Rightarrow f(x) \nearrow \text{ on } (0, 1)$$

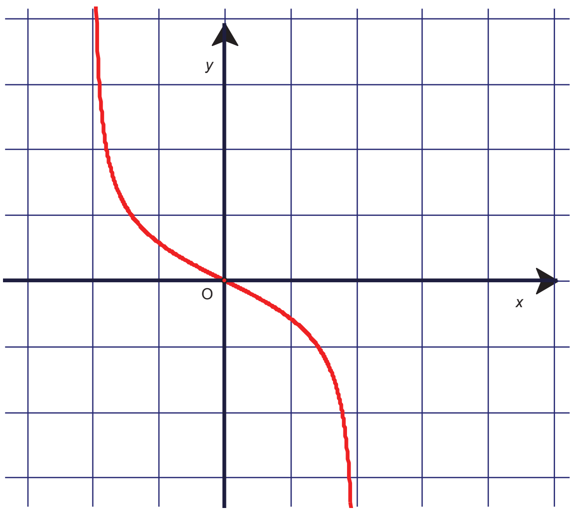
$$f'(2) = \frac{2^3 - 3 \cdot 2 + 2}{2^3} = \frac{1}{2} > 0 \Rightarrow f(x) \nearrow \text{ on } (2, \infty)$$

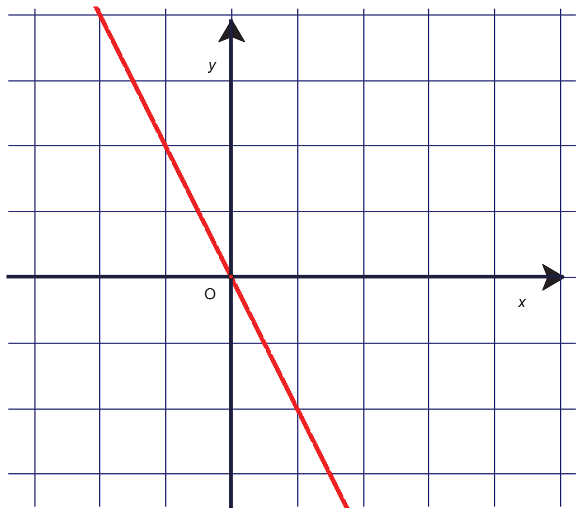
Therefore, f has a relative maximum at $\left(-2, -\dfrac{15}{4}\right)$ and a stationary point of inflexion at (1, 3).

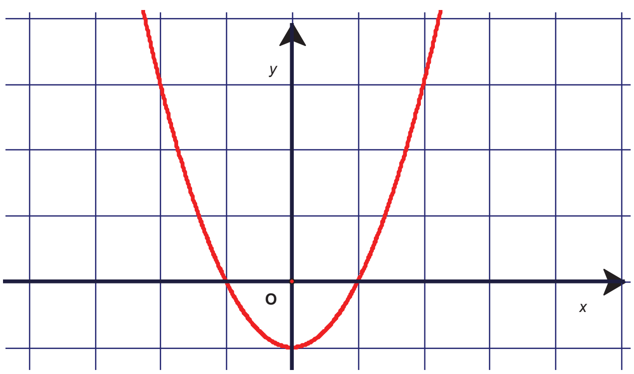
As the function can be written as $f(x) = x + \dfrac{3}{x} - \dfrac{1}{x^2}$, we can see that $f(x) \to x$ as $x \to \pm\infty$.

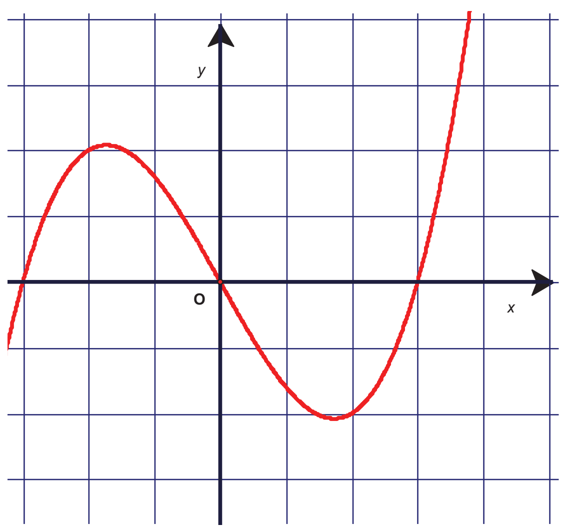


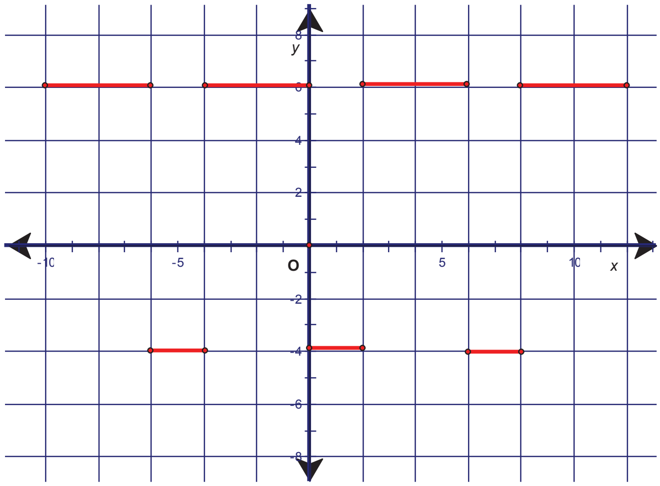
25











26 **a)** $f'(x) < 0 \Rightarrow f(x)$ decreasing for $x < 1$ and $x > 5$

$f'(x) > 0 \Rightarrow f(x)$ increasing on $x \in (1, 5)$

b) $f'(1) = 0 \Rightarrow$ minimum at $x = 1$

$f'(5) = 0 \Rightarrow$ maximum at $x = 5$

27 **a)** $f'(x) < 0 \Rightarrow f(x)$ decreasing on $x \in (1, 3) \cup (5, \infty)$

$f'(x) > 0 \Rightarrow f(x)$ increasing on $x \in (0, 1) \cup (3, 5)$

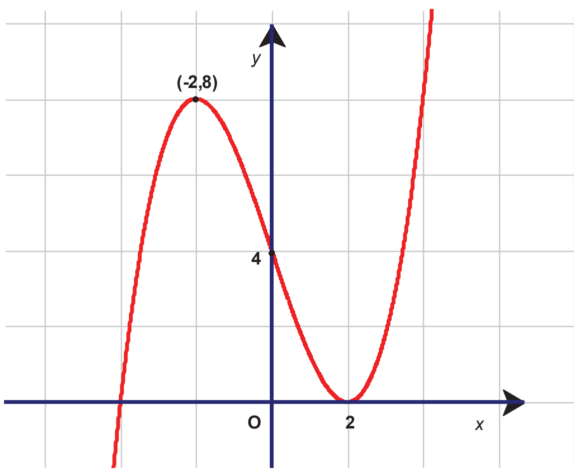
b) $f'(1) = 0 \Rightarrow$ maximum at $x = 1$

$f'(3) = 0 \Rightarrow$ minimum at $x = 3$

$f'(5) = 0 \Rightarrow$ maximum at $x = 5$

28 $f''(0.5) = 0$, $f''(7.5) = 0$ and changing the sign at these points $\Rightarrow$ inflexion points at $x \approx 0.5$, $x \approx 7.5$

29



30 For the displacement function $s(t) = -2t^3 + 15t^2 - 24t, t \geqslant 0$, we have:

a) $v(t) = s'(t) = -6t^2 + 30t - 24 = -6(t-1)(t-4)$

$s'(t) = 0 \Rightarrow t = 1, t = 4$

$s'(t) < 0$ for $0 \leqslant t < 1$ and $t > 4 \Rightarrow$ the object is moving to the left

$s'(t) > 0$ for $1 < t < 4$ and $t > 4 \Rightarrow$ the object is moving to the right

b) **i)** $v(0) = -24$ m/s

ii) $a(t) = v'(t) = -12t + 30 \Rightarrow a(0) = 30$ m/s^2

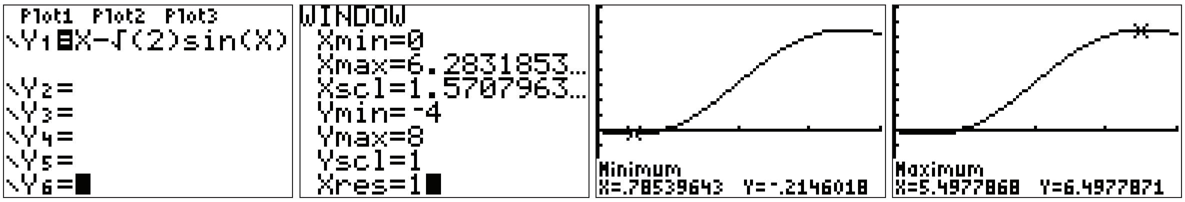
c) **i)** The maximum displacement is: $s(4) = -2 \cdot 4^3 + 15 \cdot 4^2 - 24 \cdot 4 = 16$ m

ii) $v'(t) = 0 \Rightarrow -12t + 30 = 0 \Rightarrow t = 2.5$

The maximum velocity is: $v(2.5) = -6 \cdot 2.5^2 + 30 \cdot 2.5 - 24 = 13.5$ m/s

d) The acceleration is equal to zero at $t = 2.5$, when the object is moving with a velocity of 13.5 m/s.

31 **a)**



Minimum value: $y \approx -0.215$; maximum value: $y \approx 6.5$.

b) For $f(x) = x - \sqrt{2}\sin x,\ 0 \leqslant x \leqslant 2\pi$, we have:

$$f'(x) = 1 - \sqrt{2}\cos x = 0 \Rightarrow \cos x = \frac{1}{\sqrt{2}} = \frac{\sqrt{2}}{2} \Rightarrow x = \frac{\pi}{4},\ x = \frac{7\pi}{4}$$

Minimum: $f\left(\frac{\pi}{4}\right) = \frac{\pi}{4} - \sqrt{2}\sin\left(\frac{\pi}{4}\right) = \frac{\pi}{4} - \sqrt{2} \cdot \frac{\sqrt{2}}{2} = \frac{\pi}{4} - 1$

Maximum: $f\left(\frac{7\pi}{4}\right) = \frac{7\pi}{4} - \sqrt{2}\sin\left(\frac{7\pi}{4}\right) = \frac{7\pi}{4} - \sqrt{2} \cdot \left(-\frac{\sqrt{2}}{2}\right) = \frac{7\pi}{4} + 1$

Exercise 13.4

1 To find an equation of the tangent line to the graph of a function $f(x)$ at a given value of x, we first have to find the y-coordinate of the point of tangency by substituting the given value of x into the equation. Then the value of the first derivative at this value of x will give us the slope of the tangent. We then substitute in the point-slope form of the equation of the line to find the equation of the tangent.

a) $a(x) = y = x^2 + 2x + 1$

$a(-3) = 9 - 6 + 1 = 4 \Rightarrow$ point of tangency is $(-3, 4)$

$\frac{dy}{dx} = a'(x) = 2x + 2 \Rightarrow$ slope of the tangent is $a'(-3) = -6 + 2 = -4$

Therefore, the equation of the tangent is: $y - 4 = -4(x + 3) \Rightarrow y = -4x - 8$

b) $b(x) = y = x^3 + x^2$

$b\left(-\frac{2}{3}\right) = -\frac{8}{27} + \frac{4}{9} = \frac{4}{27} \Rightarrow$ point of tangency is $\left(-\frac{2}{3}, \frac{4}{27}\right)$

$\frac{dy}{dx} = b'(x) = 3x^2 + 2x \Rightarrow$ slope of the tangent is $b'\left(-\frac{2}{3}\right) = 3 \cdot \frac{4}{9} + 2 \cdot \left(-\frac{2}{3}\right) = 0$

Therefore, the equation of the tangent is: $y - \frac{4}{27} = 0 \cdot \left(x + \frac{2}{3}\right) \Rightarrow y = \frac{4}{27}$

c) $c(x) = y = 3x^2 - x + 1$

$c(0) = 1 \Rightarrow$ point of tangency is $(0, 1)$

$\frac{dy}{dx} = c'(x) = 6x - 1 \Rightarrow$ slope of the tangent is $c'(0) = -1$

Therefore, the equation of the tangent is: $y - 1 = -1(x - 0) \Rightarrow y = -x + 1$

d) $d(x) = y = 2x + \frac{1}{x} = 2x + x^{-1}$

$d\left(\frac{1}{2}\right) = 1 + 2 = 3 \Rightarrow$ point of tangency is $\left(\frac{1}{2}, 3\right)$

$\frac{dy}{dx} = d'(x) = 2 - x^{-2} = 2 - \frac{1}{x^2} \Rightarrow$ slope of the tangent is $d'\left(\frac{1}{2}\right) = 2 - 4 = -2$

Therefore, the equation of the tangent is: $y - 3 = -2 \cdot \left(x - \frac{1}{2}\right) \Rightarrow y = -2x + 4$

2 The slope of the normal is the opposite sign of the reciprocal of the slope of the tangent.

a) $a(x) = y = x^2 + 2x + 1$

Point: $(-3, 4)$

Slope of the normal is: $-\frac{1}{-4} = \frac{1}{4}$

Therefore, the equation of the normal is: $y - 4 = \frac{1}{4}(x + 3) \Rightarrow y = \frac{1}{4}x + \frac{19}{4}$

b) $b(x) = y = x^3 + x^2$

Point: $\left(-\frac{2}{3}, \frac{4}{27}\right)$

Slope of the normal is not defined: $\frac{1}{0}$

Therefore, the equation of the normal is: $x = -\frac{2}{3}$

c) $c(x) = y = 3x^2 - x + 1$

Point: $(0, 1)$

Slope of the normal is: $-\frac{1}{-1} = 1$

Therefore, the equation of the normal is: $y - 1 = 1(x - 0) \Rightarrow y = x + 1$

d) $d(x) = y = 2x + \frac{1}{x}$

Point: $\left(\frac{1}{2}, 3\right)$

Slope of the normal is: $-\frac{1}{-2} = \frac{1}{2}$

Therefore, the equation of the normal is: $y - 3 = \frac{1}{2} \cdot \left(x - \frac{1}{2}\right) \Rightarrow y = \frac{1}{2}x + \frac{11}{4}$

3 Intersections of the curve $y = x^3 - 3x^2 + 2x$ and the x-axis:

$x^3 - 3x^2 + 2x = 0 \Rightarrow x(x^2 - 3x + 2) = 0 \Rightarrow x(x-2)(x-1) = 0 \Rightarrow x = 0, x = 1, x = 2$

The points of tangency are (0, 0), (1, 0) and (2, 0).

$\frac{dy}{dx} = 3x^2 - 6x + 2$

The slope of a tangent is equal to the value of the first derivative at the given value of x, so:

At $x = 0$, slope is 2; at $x = 1$, slope is $3 - 6 + 2 = -1$; at $x = 2$, slope is $3 \cdot 4 - 6 \cdot 2 + 2 = 2$.

Therefore, the equations of the tangents are:

$y - 0 = 2(x - 0) \Rightarrow y = 2x$

$y - 0 = -1(x - 1) \Rightarrow y = -x + 1$

$y - 0 = 2(x - 2) \Rightarrow y = 2x - 4$

4 The tangent to the curve $y = x^2 - 2x$ is perpendicular to the line $x - 2y = 1$, or $y = \frac{1}{2}x - \frac{1}{2}$, so its slope is $-\frac{1}{\frac{1}{2}} = -2$.

We must determine the point of tangency at which the first derivative is equal to -2.

$\frac{dy}{dx} = 2x - 2 \Rightarrow 2x - 2 = -2 \Rightarrow x = 0, y = 0$

Therefore, the equation of the tangent is: $y - 0 = -2(x - 0) \Rightarrow y = -2x$

5 For the curves $y = x^2 - 6x + 20$ and $y = x^3 - 3x^2 - x$, we must find the x-coordinate, $0 \leqslant x \leqslant \frac{3}{2}$, of the point where the slopes of the tangents, i.e. the first derivatives, are the same.

a) $\frac{d}{dx}\left(x^2 - 6x + 20\right) = \frac{d}{dx}\left(x^3 - 3x^2 - x\right) \Rightarrow 2x - 6 = 3x^2 - 6x - 1 \Rightarrow$

$3x^2 - 8x + 5 = 0 \Rightarrow x = \frac{8 \pm \sqrt{(-8)^2 - 4 \cdot 3 \cdot 5}}{2 \cdot 3} = \frac{8 \pm 2}{6} \Rightarrow x = 1,$ ~~$x = \frac{5}{3}$~~

b) For the curve $y = x^2 - 6x + 20$, at $x = 1$, $y = 1 - 6 + 20 = 15$.

$\frac{dy}{dx} = 2x - 6$; slope at $x = 1$ is $2 - 6 = -4$

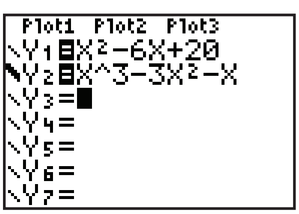
Equation of the tangent at (1, 15) is: $y - 15 = -4(x - 1) \Rightarrow y = -4x + 19$

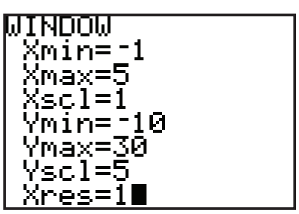
For the curve $y = x^3 - 3x^2 - x$, at $x = 1$, $y = 1 - 3 - 1 = -3$.

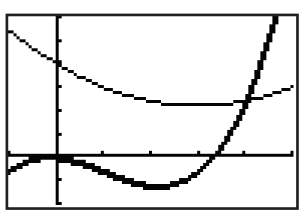
$\frac{dy}{dx} = 3x^2 - 6x - 1$; slope at $x = 1$ is $3 - 6 - 1 = -4$

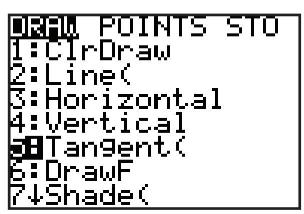
Equation of the tangent at (1, –3) is: $y + 3 = -4(x - 1) \Rightarrow y = -4x + 1$

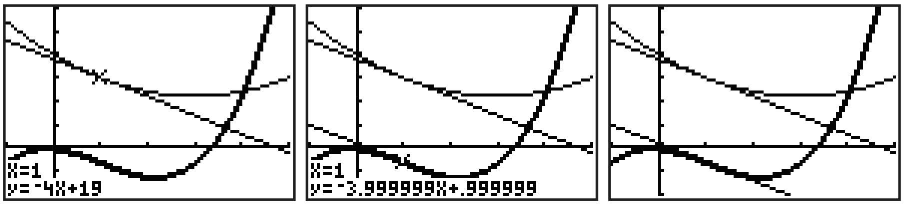
Using our GDC to sketch the curves:











6 For the curve $y = x^2 + 4x - 2$, at $x = -3$, $y = (-3)^2 + 4\cdot(-3) - 2 = -5$.

$\dfrac{dy}{dx} = 2x + 4 \Rightarrow$ slope of the tangent at $x = -3$ is: $-6 + 4 = -2$

Slope of the normal is: $-\dfrac{1}{-2} = \dfrac{1}{2}$

Equation of the normal at the point $(-3, -5)$ is: $y + 5 = \dfrac{1}{2}(x + 3) \Rightarrow y = \dfrac{1}{2}x - \dfrac{7}{2}$

To find the other point where this normal intersects the curve, we will solve a system of equations (using the substitution method):

$$\begin{cases} y = x^2 + 4x - 2 \\ y = \dfrac{1}{2}x - \dfrac{7}{2} \end{cases} \Rightarrow x^2 + 4x - 2 = \frac{1}{2}x - \frac{7}{2} \Rightarrow 2x^2 + 7x + 3 = 0 \Rightarrow$$

$$x = \frac{-7 \pm \sqrt{7^2 - 4\cdot 2\cdot 3}}{2\cdot 2} = \frac{-7 \pm 5}{4} \Rightarrow x = -\frac{1}{2}, \ \cancel{x = -3}$$

For $x = -\dfrac{1}{2}$: $y = \dfrac{1}{2}\left(-\dfrac{1}{2}\right) - \dfrac{7}{2} = -\dfrac{15}{4}$.

Therefore, the other point of intersection is $\left(-\dfrac{1}{2}, -\dfrac{15}{4}\right)$.

7 For the function $g(x) = \dfrac{1 - x^3}{x^4} = x^{-4} - x^{-1}$, we have:

$g'(x) = -4x^{-5} + x^{-2} = \dfrac{x^3 - 4}{x^5}$

Slope of the tangent at $x = 1$ is: $g'(1) = \dfrac{1 - 4}{1} = -3$

Equation of the tangent at $(1, 0)$ is: $y - 0 = -3(x - 1) \Rightarrow y = -3x + 3$

Slope of the normal at $x = 1$ is: $-\dfrac{1}{-3} = \dfrac{1}{3}$

Equation of the normal at $(1, 0)$ is: $y - 0 = \dfrac{1}{3}(x - 1) \Rightarrow y = \dfrac{1}{3}x - \dfrac{1}{3}$

8 The normal to the curve has a slope of 1 and it passes through $(0, -4)$, so its equation is: $y + 4 = 1(x - 0) \Rightarrow y = x - 4$.

The point with $x = 1$ is common for the normal $\Rightarrow y = 1 - 4 = -3$, and the curve $y = ax^{\frac{1}{2}} + bx \Rightarrow y = a + b$. So, we have $a + b = -3$.

The slope function for the curve is: $\dfrac{dy}{dx} = \dfrac{1}{2}ax^{-\frac{1}{2}} + b$.

If the slope of the normal is 1 at $x = 1$, the slope of the tangent at $x = 1$ is -1.

So, at $x = 1$, we have: $\frac{1}{2}a \cdot 1^{-\frac{1}{2}} + b = -1 \Rightarrow a + 2b = -2$.

The system of equations is:

$$\begin{cases} a + 2b = -2 \\ a + b = -3 \end{cases} \Rightarrow a = -4, b = 1$$

9 For $f(x) = x^3 + \frac{1}{2}x^2 + 1$ we have:

a) $f'(x) = 3x^2 + x$

Slope of the tangent at $\left(-1, \frac{1}{2}\right)$ is: $f'(-1) = 3(-1)^2 - 1 = 2$

Equation of the tangent is: $y - \frac{1}{2} = 2(x + 1) \Rightarrow y = 2x + \frac{5}{2}$

b) $f'(x) = 2 \Rightarrow 3x^2 + x = 2 \Rightarrow 3x^2 + x - 2 = 0 \Rightarrow$

$$x = \frac{-1 \pm \sqrt{(-1)^2 - 4 \cdot 3 \cdot (-2)}}{2 \cdot 3} = \frac{-1 \pm 5}{6} \Rightarrow x = \frac{2}{3}, \cancel{x = -1}$$

$$f\left(\frac{2}{3}\right) = \left(\frac{2}{3}\right)^3 + \frac{1}{2} \cdot \left(\frac{2}{3}\right)^2 + 1 = \frac{41}{27}$$

The other point where the tangent is parallel is $\left(\frac{2}{3}, \frac{41}{27}\right)$.

10 For the curve $y = \sqrt{x}\left(1 - \sqrt{x}\right) = x^{\frac{1}{2}} - x$, we have:

$x = 4$, $y = \sqrt{4}\left(1 - \sqrt{4}\right) = -2$; point is $(4, -2)$

$$\frac{dy}{dx} = \frac{1}{2}x^{-\frac{1}{2}} - 1$$

Slope of the tangent at $x = 4$ is: $\frac{1}{2}4^{-\frac{1}{2}} - 1 = -\frac{3}{4}$

Slope of the normal at $x = 4$ is: $-\frac{1}{-\frac{3}{4}} = \frac{4}{3}$

Equation of the tangent is: $y + 2 = -\frac{3}{4}(x - 4) \Rightarrow y = -\frac{3}{4}x + 1$

Equation of the normal is: $y + 2 = \frac{4}{3}(x - 4) \Rightarrow y = \frac{4}{3}x - \frac{22}{3}$

11 For the function $f(x) = (1 + x)^2(5 - x) = 5 + 9x + 3x^2 - x^3$, we have:

a) $f(1) = 5 + 9 + 3 - 1 = 16$

$f'(x) = 9 + 6x - 3x^2$

Slope of the tangent at $x = 1$ is $f'(1) = 9 + 6 - 3 = 12$.

The equation of the tangent at $(1, 16)$ is: $y - 16 = 12(x - 1) \Rightarrow y = 12x + 4$.

To check for a point of intersection, we will try to solve the following system of equations:

$$\begin{cases} y = 5 + 9x + 3x^2 - x^3 \\ y = 12x + 4 \end{cases} \Rightarrow x^3 - 3x^2 + 3x - 1 = 0 \Rightarrow (x - 1)^3 = 0$$

The only solution is $x = 1$, so the curve and the tangent meet only at the tangency point.

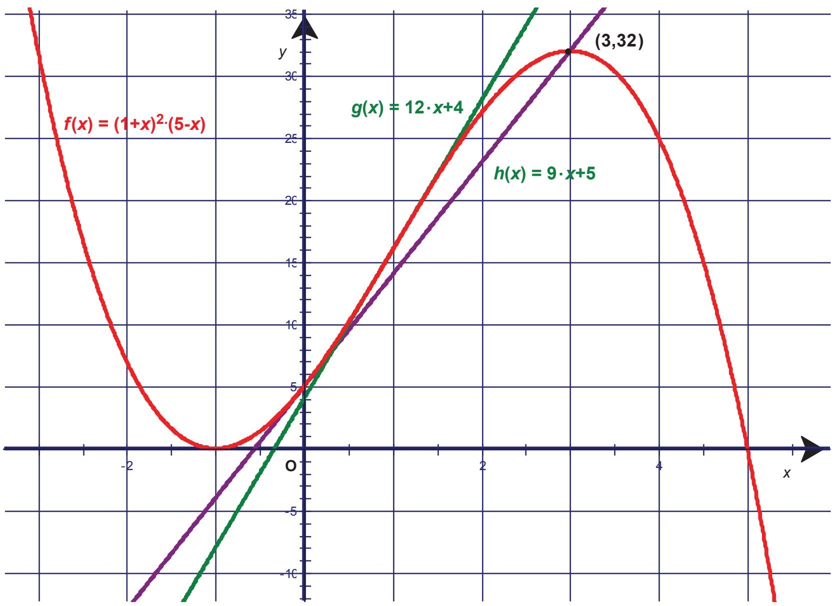
b) Slope of the tangent at $x = 0$ is $f'(0) = 9$.

The equation of the tangent at $(0, 5)$ is: $y - 5 = 9(x - 0) \Rightarrow y = 9x + 5$.

To find the intersection, we will solve the following system of equations:

$$\begin{cases} y = 5 + 9x + 3x^2 - x^3 \\ y = 9x + 5 \end{cases} \Rightarrow x^3 - 3x^2 = 0 \Rightarrow x^2(x-3) = 0 \Rightarrow x = 0, x = 3$$

The tangent meets the curve again at the point where $x = 3$ ($y = f(3) = 32$). We also have: $f'(3) = 9 + 6 \cdot 3 - 3 \cdot 3^2 = 0$, which means that $(3, 32)$ is the turning point.



12 For $y = x^2 + x$, the slope function is $\frac{dy}{dx} = 2x + 1$. Let the point of tangency be (a, b). Then the slope of the tangent is $2a + 1$, and the equation of the tangent that passes through $(2, -3)$ is: $y + 3 = (2a+1)(x-2) \Rightarrow y = (2a+1)x - 4a - 5$. Since the point (a, b) is on both the tangent and the curve, we have the following system of equations:

$$\begin{cases} b = a^2 + a \\ b = (2a+1)a - 4a - 5 \end{cases} \Rightarrow$$

$a^2 + a = (2a+1)a - 4a - 5 \Rightarrow a^2 - 4a - 5 = 0 \Rightarrow (a-5)(a+1) = 0 \Rightarrow a = 5, a = -1$

For $a = 5$, the tangent is $y = 11x - 25$; for $a = -1$, the tangent is $y = -x - 1$.

13 For $y = 1 + (x-1)^2 = x^2 - 2x + 2$, the slope function is $\frac{dy}{dx} = 2x - 2$. Let the point of tangency be (a, b). Then the slope of the tangent is $2a - 2$, and the equation of the tangent that passes through $(0, 0)$ is $y = (2a-2)x$. Since the point (a, b) is on both the tangent and the curve, we have the following system of equations:

$$\begin{cases} b = a^2 - 2a + 2 \\ b = (2a-2)a \end{cases} \Rightarrow a^2 - 2a + 2 = (2a-2)a \Rightarrow a^2 - 2 = 0 \Rightarrow a = \sqrt{2}, a = -\sqrt{2}$$

For $a = \sqrt{2}$, the tangent is $y = (2\sqrt{2} - 2)x$; for $a = -\sqrt{2}$, the tangent is $y = -(2\sqrt{2} + 2)x$.

14 **a)** For $y = \sqrt[3]{x} = x^{\frac{1}{3}}$, we have $\frac{dy}{dx} = \frac{1}{3}x^{-\frac{2}{3}}$, so the slope at $x = 8$ is $\frac{1}{3\sqrt[3]{8^2}} = \frac{1}{12}$.

The equation of the tangent at $x = 8$, $y = \sqrt[3]{8} = 2$ is: $y - 2 = \frac{1}{12}(x - 8) \Rightarrow y = \frac{1}{12}x + \frac{4}{3}$.

b) $\sqrt[3]{9} \approx \frac{1}{12} \cdot 9 + \frac{4}{3} = \frac{25}{12} \approx 2.08$

15 For $f(x) = \frac{1}{\sqrt{x}} = x^{-\frac{1}{2}}$ we have:

$f'(x) = -\frac{1}{2}x^{-\frac{3}{2}}$, $f'(a) = -\frac{1}{2}a^{-\frac{3}{2}} = -\frac{1}{2a\sqrt{a}}$, $f(a) = \frac{1}{\sqrt{a}}$

The equation of the tangent is: $y - \frac{1}{\sqrt{a}} = -\frac{1}{2a\sqrt{a}}(x - a) \Rightarrow y = -\frac{1}{2a\sqrt{a}}x + \frac{3}{2\sqrt{a}}$

16 For $y = x^3$, we have $\frac{dy}{dx} = 3x^2$. Let a be the x-coordinate of point P. Then the slope of the tangent at $P(a, a^3)$ is $3a^2$, and the equation of the tangent at P is: $y - a^3 = 3a^2(x - a) \Rightarrow y = 3a^2x - 2a^3$.

To find point Q, the point where the tangent intersects the curve again, we must solve the following system of equations:

$$\begin{cases} y = x^3 \\ y = 3a^2x - 2a^3 \end{cases} \Rightarrow x^3 = 3a^2x - 2a^3 \Rightarrow x^3 - 3a^2x + 2a^3 = 0$$

This equation can be solved by factorizing:

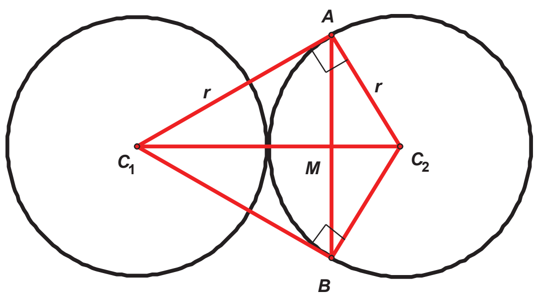
$x^3 - 3a^2x + 2a^3 = 0 \Rightarrow x^3 - a^2x - 2a^2x + 2a^3 = 0 \Rightarrow x(x^2 - a^2) - 2a^2(x - a) = 0 \Rightarrow$

$x(x - a)(x + a) - 2a^2(x - a) = 0 \Rightarrow (x - a)\left[x(x + a) - 2a^2\right] = 0 \Rightarrow$

$(x - a)(x^2 + ax - 2a^2) = 0 \Rightarrow (x - a)(x - a)(x + 2a) = 0 \Rightarrow x = a, x = -2a$

For $x = -2a$, $y = (-2a)^3 = -8a^3$, so $Q(-2a, -8a^3)$.

17



Since the tangents are perpendicular to the radius at the point of tangency, we have:

$|C_1C_2| = 2r, |C_1A| = |C_1B| = \sqrt{(2r)^2 - r^2} = r\sqrt{3}$

$\text{Area}_{C_1C_2A} = \text{Area}_{C_1C_2B} = \frac{|AC_1| \cdot |AC_2|}{2} = \frac{r\sqrt{3} \cdot r}{2} = \frac{r^2\sqrt{3}}{2} \Rightarrow \text{Area}_{C_1AC_2B} = r^2\sqrt{3}$

On the other hand, AB intersects C_1C_2 at M, so we also have:

$\text{Area}_{C_1AC_2B} = \text{Area}_{BC_1A} + \text{Area}_{AC_2B} = \frac{|AB| \cdot |MC_1|}{2} + \frac{|AB| \cdot |MC_2|}{2} = \frac{|AB| \cdot |MC_1|}{2} + \frac{|AB| \cdot (2r - |MC_1|)}{2}$

These two areas are equal; hence:

$$\frac{|AB| \cdot |MC_1|}{2} + \frac{|AB| \cdot (2r - |MC_1|)}{2} = r^2\sqrt{3}$$

$$|AB| \cdot |MC_1| + 2r\,|AB| - |AB| \cdot |MC_1| = r^2\sqrt{3}$$

$$2r\,|AB| = r^2\sqrt{3} \Rightarrow |AB| = \frac{r^2\sqrt{3}}{2r} = \frac{r\sqrt{3}}{2}$$

18 For the curve $y = 4 - x^2$, we have $\frac{dy}{dx} = -2x$. The slope of the tangent at any point $(a, 4 - a^2)$ on the curve is $-2a$, so the equation of the tangent is: $y - (4 - a^2) = -2a(x - a) \Rightarrow y = -2ax + a^2 + 4$.

The tangent should pass through the point (1, 2), so:

$2 = -2a + a^2 + 4 \Rightarrow a^2 - 2a + 2 = 0$

This equation has no real solutions for a because $\Delta = (-2)^2 - 4 \cdot 2 = -4 < 0$, so there is no line through (1, 2) that is tangent to the curve $y = 4 - x^2$.

Chapter 14

Exercise 14.1

1 **a)** $\overrightarrow{AB} = \overrightarrow{OB} - \overrightarrow{OA} = (x_B - x_A, y_B - y_A, z_B - z_A) = \left(1 - \left(-\frac{3}{2}\right), -\frac{5}{2} - \left(-\frac{1}{2}\right), 1 - 1\right) = \left(\frac{5}{2}, -2, 0\right)$

b) $\overrightarrow{AB} = \overrightarrow{OB} - \overrightarrow{OA} = (x_B - x_A, y_B - y_A, z_B - z_A) = \left(1 - (-2), \sqrt{3} - (-\sqrt{3}), -\frac{1}{2} - \left(-\frac{1}{2}\right)\right) = \left(3, 2\sqrt{3}, 0\right)$

c) $\overrightarrow{AB} = \overrightarrow{OB} - \overrightarrow{OA} = (x_B - x_A, y_B - y_A, z_B - z_A) = (1 - 2, -1 - (-3), 3 - 5) = (-1, 2, -2)$

d) $\overrightarrow{AB} = \overrightarrow{OB} - \overrightarrow{OA} = (x_B - x_A, y_B - y_A, z_B - z_A) = (-a - a, -2a - (-a), a - 2a) = (-2a, -a, -a)$

2 **a)** Given $\overrightarrow{PQ} = \overrightarrow{OQ} - \overrightarrow{OP} = (x_Q - x_P, y_Q - y_P, z_Q - z_P) = \left(1, -\frac{5}{2}, 1\right)$

$\Rightarrow \left(x_Q - \left(-\frac{3}{2}\right), y_Q - \left(-\frac{1}{2}\right), z_Q - 1\right) = \left(1, -\frac{5}{2}, 1\right)$. Therefore:

$x_Q - \left(-\frac{3}{2}\right) = 1 \Rightarrow x_Q = 1 - \frac{3}{2} = -\frac{1}{2}$

$y_Q - \left(-\frac{1}{2}\right) = -\frac{5}{2} \Rightarrow y_Q = -\frac{5}{2} - \frac{1}{2} = -3$

$z_Q - 1 = 1 \Rightarrow z_Q = 1 + 1 = 2$. So, $Q\left(-\frac{1}{2}, -3, 2\right)$.

b) Given $\overrightarrow{PQ} = \overrightarrow{OQ} - \overrightarrow{OP} = (x_Q - x_P, y_Q - y_P, z_Q - z_P) = \left(-\frac{3}{2}, -\frac{1}{2}, 1\right)$

$\Rightarrow \left(1 - x_P, -\frac{5}{2} - y_P, 1 - z_P\right) = \left(-\frac{3}{2}, -\frac{1}{2}, 1\right)$. Therefore:

$1 - x_P = -\frac{3}{2} \Rightarrow x_P = 1 + \frac{3}{2} = \frac{5}{2}$

$-\frac{5}{2} - y_P = -\frac{1}{2} \Rightarrow y_P = -\frac{5}{2} + \frac{1}{2} = -2$

$1 - z_P = 1 \Rightarrow z_P = 1 - 1 = 0$. So, $P\left(\frac{5}{2}, -2, 0\right)$.

c) Given $\overrightarrow{PQ} = \overrightarrow{OQ} - \overrightarrow{OP} = (x_Q - x_P, y_Q - y_P, z_Q - z_P) = (-a, -2a, a)$

$\Rightarrow (x_Q - a, y_Q - (-2a), z_Q - 2a) = (-1, -2a, a)$. Therefore:

$x_Q - a = -a \Rightarrow x_Q = 0$

$y_Q - (-2a) = -2a \Rightarrow y_Q = -2a - 2a = -4a$

$z_Q - 2a = a \Rightarrow z_Q = a + 2a = 3a$. So, $Q\,(0, -4a, 3a)$.

3 **a)** For points M, A and B to be collinear, it is sufficient to make $\overrightarrow{AM}$ parallel to $\overrightarrow{AB}$. If the two vectors are parallel, then one of them is a scalar multiple of the other, for example, $\overrightarrow{AM} = t\,\overrightarrow{AB}$.

$\overrightarrow{AM} = (x_M - x_A, y_M - y_A, z_M - z_A) = (x - 0, y - 0, z - 5)$

$\overrightarrow{AB} = (x_B - x_A, y_B - y_A, z_B - z_A) = (1 - 0, 1 - 0, 0 - 5)$

Therefore: $(x, y, z - 5) = t\,(1, 1, -5) \Rightarrow (x, y, z - 5) = (t, t, -5t) \Rightarrow x = t, y = t, z - 5 = -5t$

So, $M\,(t, t, 5-5t)$, where $t \in \mathbb{R}$.

Note: We can find M by letting $\overrightarrow{BM} = t\,\overrightarrow{AB}$. Then we have:

$\overrightarrow{BM} = \left(x_M - x_B, y_M - y_B, z_M - z_b\right) = \left(x-1, y-1, z-0\right)$, $\overrightarrow{AB} = (1, 1, -5)$

Therefore: $\left(x-1, y-1, z\right) = (t, t, -5t) \Rightarrow x = 1+t, y = 1+t, z = -5t$

So, $M\,(1+t, 1+t, -5t)$, where $t \in \mathbb{R}$.

Both conditions describe the same set of points; for example, we can obtain point $M(0, 0, 5)$ by putting $t = 0$ in $M\,(t, t, 5-5t)$, or $t = -1$ in $M\,(1+t, 1+t, -5t)$; or $M(2, 2, -5)$ by putting $t = 2$ in $M\,(t, t, 5-5t)$, or $t = 1$ in $M\,(1+t, 1+t, -5t)$.

b) For points M, A and B to be collinear, it is sufficient to make $\overrightarrow{AM}$ parallel to $\overrightarrow{AB}$. So, let's say, $\overrightarrow{AM} = t\,\overrightarrow{AB}$.

$\overrightarrow{AM} = \left(x_M - x_A, y_M - y_A, z_M - z_A\right) = \left(x-(-1), y-0, z-1\right)$

$\overrightarrow{AB} = \left(x_B - x_A, y_B - y_A, z_B - z_A\right) = (3-(-1), 5-0, -2-1)$

Therefore:

$\left(x+1, y, z-1\right) = t\,(4, 5, -3) \Rightarrow \left(x+1, y, z-1\right) = (4t, 5t, -3t) \Rightarrow x = -1+4t, y = 5t, z = 1-3t$

So, $M\,(-1+4t, 5t, 1-3t)$, where $t \in \mathbb{R}$.

Note: If we start with the condition $\overrightarrow{BM} = t\,\overrightarrow{AB}$, we will have $\overrightarrow{BM} = \left(x-3, y-5, z+2\right)$; therefore, from $\overrightarrow{BM} = t\,\overrightarrow{AB}$, we will find $x = 3+4t, y = 5+5t, z = -2-3t$.

c) For points M, A and B to be collinear, it is sufficient to make $\overrightarrow{AM}$ parallel to $\overrightarrow{AB}$. So, let's say, $\overrightarrow{AM} = t\,\overrightarrow{AB}$.

$\overrightarrow{AM} = \left(x_M - x_A, y_M - y_A, z_M - z_A\right) = \left(x-2, y-3, z-4\right)$

$\overrightarrow{AB} = \left(x_B - x_A, y_B - y_A, z_B - z_A\right) = (-2-2, -3-3, 5-4)$

Therefore:

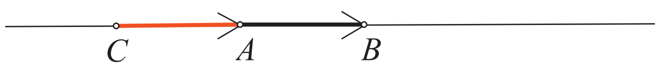
$\left(x-2, y-3, z-4\right) = t\,(-4, -6, 1) \Rightarrow \left(x-2, y-3, z-4\right) = (-4t, -6t, t) \Rightarrow$

$x = 2-4t, y = 3-6t, z = 4+t$

So, $M\,(2-4t, 3-6t, 4+t)$, where $t \in \mathbb{R}$.

Note: If we start with the condition $\overrightarrow{BM} = t\,\overrightarrow{AB}$, we will have $\overrightarrow{BM} = \left(x+2, y+3, z-5\right)$; therefore, from $\overrightarrow{BM} = t\,\overrightarrow{AB}$, we will find $x = -2-4t, y = -3-6t, z = 5+t$.

4 If C is the symmetric image of B with respect to A, points A, B and C are on the line and their positions are as shown below. Their relationship can be expressed using different vector relationships; for example, $\overrightarrow{AB} = \overrightarrow{CA}$, $\overrightarrow{AB} = -\overrightarrow{AC}$, $\overrightarrow{CB} = 2\,\overrightarrow{AB}$, ... Here, we will use $\overrightarrow{AB} = \overrightarrow{CA}$.



a) For $C(x, y, z)$:

$\overrightarrow{AB} = (-1-3, 0-(-4), 1-0) = (-4, 4, 1)$, $\overrightarrow{CA} = \left(3-x, -4-y, 0-z\right)$. Therefore:

$(3-x, -4-y, 0-z) = (-4, 4, 1) \Rightarrow 3+4 = x, -4-4 = y, z = -1$; so, $C(7, -8, -1)$.

b) For $C(x, y, z)$:

$\overrightarrow{AB} = \left(-1-(-1), \frac{1}{2}-3, \frac{1}{3}-5\right) = \left(0, -\frac{5}{2}, -\frac{14}{3}\right)$, $\overrightarrow{CA} = (-1-x, 3-y, 5-z)$. Therefore:

$\left(0, -\frac{5}{2}, -\frac{14}{3}\right) = (-1-x, 3-y, 5-z) \Rightarrow x = -1, y = 3+\frac{5}{2} = \frac{11}{2}, z = 5+\frac{14}{3} = \frac{29}{3}$; so,

$C\left(-1, \frac{11}{2}, \frac{29}{3}\right)$.

c) For $C(x, y, z)$:

$\overrightarrow{AB} = (a-1, 2a-2, b-(-1))$, $\overrightarrow{CA} = (1-x, 2-y, -1-z)$. Therefore:

$(a-1, 2a-2, b+1) = (1-x, 2-y, -1-z)$

$\Rightarrow x = 1-a+1 = 2-a, y = 2-2a+2 = 4-2a, z = -1-b-1 = -2-b$; so,

$C(2-a, 4-2a, -2-b)$.

5 **a)** For $G(x, y, z)$:

$\vec{0} = \overrightarrow{GA} + \overrightarrow{GB} + \overrightarrow{GC} = (-1-x, -1-y, -1-z) + (-1-x, 2-y, -1-z) + (1-x, 2-y, 3-z)$

$\Rightarrow (0, 0, 0) = (-1-3x, 3-3y, 1-3z)$

$\Rightarrow x = -\frac{1}{3}, y = 1, z = \frac{1}{3}$. So, $G\left(-\frac{1}{3}, 1, \frac{1}{3}\right)$.

b) For $G(x, y, z)$:

$\vec{0} = \overrightarrow{GA} + \overrightarrow{GB} + \overrightarrow{GC} = (2-x, -3-y, 1-z) + (1-x, -2-y, -5-z) + (0-x, 0-y, 1-z)$

$\Rightarrow (0, 0, 0) = (3-3x, -5-3y, -3-3z)$

$\Rightarrow x = 1, y = -\frac{5}{3}, z = -1$. So, $G\left(1, -\frac{5}{3}, -1\right)$.

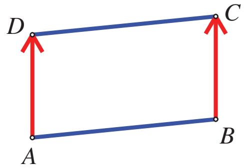
c) For $G(x, y, z)$:

$\vec{0} = \overrightarrow{GA} + \overrightarrow{GB} + \overrightarrow{GC} = (a-x, 2a-y, 3a-z) + (b-x, 2b-y, 3b-z) + (c-x, 2c-y, 3c-z)$

$\Rightarrow (0, 0, 0) = (a+b+c-3x, 2a+2b+2c-3y, 3a-3b-3c-3z)$

$\Rightarrow x = \frac{a+b+c}{3}, y = \frac{2a+2b+2c}{3}, z = a+b+c$. So, $G\left(\frac{a+b+c}{3}, \frac{2a+2b+2c}{3}, a+b+c\right)$.

6 The relationship between points A, B, C and D of parallelogram $ABCD$ can be expressed using different vector relationships; for example, $\overrightarrow{AB} = \overrightarrow{DC}$, $\overrightarrow{AD} = \overrightarrow{BC}$, $\overrightarrow{BA} = \overrightarrow{CD}$, ... Here, we will use $\overrightarrow{AD} = \overrightarrow{BC}$.



a) For $D(x, y, z)$:

$\overrightarrow{BC} = (-\sqrt{3}-1, 2-3, -5-0) = (-\sqrt{3}-1, -1, -5)$

$\overrightarrow{AD} = (x-\sqrt{3}, y-2, z-(-1)) = (x-\sqrt{3}, y-2, z+1)$

Therefore: $-\sqrt{3}-1 = x-\sqrt{3} \Rightarrow x = -1, y-2 = -1 \Rightarrow y = 1, z+1 = -5 \Rightarrow z = -6$

So, $D(-1, 1, -6)$.

b) For $D(x, y, z)$:

$\overrightarrow{BC} = \left(-2\sqrt{2} - 3\sqrt{2}, \sqrt{3} - \left(-\sqrt{3}\right), -3\sqrt{5} - 5\sqrt{5}\right) = \left(-5\sqrt{2}, 2\sqrt{3}, -8\sqrt{5}\right)$

$\overrightarrow{AD} = \left(x - \sqrt{2}, y - \sqrt{3}, z - \sqrt{5}\right)$

Therefore: $-5\sqrt{2} = x - \sqrt{2} \Rightarrow x = -4\sqrt{2},\ y - \sqrt{3} = 2\sqrt{3} \Rightarrow y = 3\sqrt{3},\ z - \sqrt{5} = -8\sqrt{5} \Rightarrow z = -7\sqrt{5}$

So, $D\left(-4\sqrt{2}, 3\sqrt{3}, -7\sqrt{5}\right)$.

c) For $D(x, y, z)$:

$\overrightarrow{BC} = \left(\frac{7}{2} - \frac{1}{2}, -\frac{1}{3} - \frac{2}{3}, 1 - 5\right) = (3, -1, -4)$

$\overrightarrow{AD} = \left(x + \frac{1}{2}, y - \frac{1}{3}, z - 0\right) = \left(x + \frac{1}{2}, y - \frac{1}{3}, z\right)$

Therefore: $3 = x + \frac{1}{2} \Rightarrow x = \frac{5}{2},\ y - \frac{1}{3} = -1 \Rightarrow y = -\frac{2}{3},\ z = -4$

So, $D\left(\frac{5}{2}, -\frac{2}{3}, -4\right)$.

7 Two vectors **v** and **w** have the same direction if, for $t > 0$, $\mathbf{v} = t\mathbf{w}$. Therefore:

$(m - 2, m + n, -2m + n) = t\,(2, 4, -6)$.

$$\begin{cases} m - 2 = 2t \\ m + n = 4t \\ -2m + n = -6t \end{cases}$$

$$m = 2t + 2$$

$$\Rightarrow \begin{cases} m + n = 4t \\ -2m + n = -6t \end{cases} \Rightarrow \begin{matrix} 2t + 2 + n = 4t \Rightarrow n = 2t - 2 \\ -2m + n = -6t \end{matrix} \Rightarrow -2(2t + 2) + (2t - 2) = -6t \Rightarrow t = \frac{3}{2}$$

Therefore: $m = 2t + 2 = 5, n = 2t - 2 = 1$.

8 a) The length of the vector $\mathbf{v} = 2\mathbf{i} + 2\mathbf{j} - \mathbf{k}$ is $\sqrt{2^2 + 2^2 + (-1)^2} = \sqrt{9} = 3$, so the unit vector is $\frac{1}{3}(2\mathbf{i} + 2\mathbf{j} - \mathbf{k}) = \frac{2}{3}\mathbf{i} + \frac{2}{3}\mathbf{j} - \frac{1}{3}\mathbf{k}$.

b) The length of the vector $\mathbf{v} = 6\mathbf{i} - 4\mathbf{j} + 2\mathbf{k}$ is $\sqrt{6^2 + (-4)^2 + 2^2} = \sqrt{56} = 2\sqrt{14}$, so the unit vector is $\frac{1}{2\sqrt{14}}(6\mathbf{i} - 4\mathbf{j} + 2\mathbf{k}) = \frac{3}{\sqrt{14}}\mathbf{i} - \frac{2}{\sqrt{14}}\mathbf{j} + \frac{1}{\sqrt{14}}\mathbf{k}$.

c) The length of the vector $\mathbf{v} = 2\mathbf{i} - \mathbf{j} - 2\mathbf{k}$ is $\sqrt{2^2 + (-1)^2 + (-2)^2} = \sqrt{9} = 3$, so the unit vector is $\frac{1}{3}(2\mathbf{i} - \mathbf{j} - 2\mathbf{k}) = \frac{2}{3}\mathbf{i} - \frac{1}{3}\mathbf{j} - \frac{2}{3}\mathbf{k}$.

9 a) The length of the vector $\mathbf{v} = 2\mathbf{i} + 2\mathbf{j} - \mathbf{k}$ is $\sqrt{2^2 + 2^2 + (-1)^2} = \sqrt{9} = 3$, so the unit vector in its direction is $\frac{1}{3}(2\mathbf{i} + 2\mathbf{j} - \mathbf{k})$, and the vector of magnitude 2 is: $\frac{2}{3}(2\mathbf{i} + 2\mathbf{j} - \mathbf{k})$.

b) The length of the vector $\mathbf{v} = 6\mathbf{i} - 4\mathbf{j} + 2\mathbf{k}$ is $\sqrt{6^2 + (-4)^2 + 2^2} = \sqrt{56} = 2\sqrt{14}$, so the unit vector in its direction is $\frac{1}{2\sqrt{14}}(6\mathbf{i} - 4\mathbf{j} + 2\mathbf{k})$, and the vector of magnitude 4 is:

$\frac{4}{2\sqrt{14}}(6\mathbf{i} - 4\mathbf{j} + 2\mathbf{k}) = \frac{2}{\sqrt{14}}(6\mathbf{i} - 4\mathbf{j} + 2\mathbf{k})$.

c) The length of the vector $\mathbf{v} = 2\mathbf{i} - \mathbf{j} - 2\mathbf{k}$ is $\sqrt{2^2 + (-1)^2 + (-2)^2} = \sqrt{9} = 3$, so the unit vector in its direction is $\frac{1}{3}(2\mathbf{i} - \mathbf{j} - 2\mathbf{k})$, and the vector of magnitude 2 is: $\frac{5}{3}(2\mathbf{i} - \mathbf{j} - 2\mathbf{k})$.

10 a) $\mathbf{u} + \mathbf{v} = (\mathbf{i} + 3\mathbf{j} - 2\mathbf{k}) + (2\mathbf{i} + \mathbf{j}) = 3\mathbf{i} + 4\mathbf{j} - 2\mathbf{k}$

$|\mathbf{u} + \mathbf{v}| = |3\mathbf{i} + 4\mathbf{j} - 2\mathbf{k}| = \sqrt{3^2 + 4^2 + (-2)^2} = \sqrt{29}$

b) $|\mathbf{u}| + |\mathbf{v}| = |\mathbf{i} + 3\mathbf{j} - 2\mathbf{k}| + |2\mathbf{i} + \mathbf{j}| = \sqrt{1^2 + 3^2 + (-2)^2} + \sqrt{2^2 + 1^2 + 0^2} = \sqrt{14} + \sqrt{5}$

c) $-3\mathbf{u} = -3(\mathbf{i} + 3\mathbf{j} - 2\mathbf{k}) = -3\mathbf{i} - 9\mathbf{j} + 6\mathbf{k}; 3\mathbf{v} = 3(2\mathbf{i} + \mathbf{j}) = 6\mathbf{i} + 3\mathbf{j}$

$|-3\mathbf{u}| + |3\mathbf{v}| = \sqrt{(-3)^2 + (-9)^2 + 6^2} + \sqrt{6^2 + 3^2 + 0^2} = \sqrt{126} + \sqrt{45} = 3\sqrt{14} + 3\sqrt{5}$

d) $\frac{1}{|\mathbf{u}|}\mathbf{u} = \frac{1}{\sqrt{14}}\mathbf{u} = \frac{1}{\sqrt{14}}(\mathbf{i} + 3\mathbf{j} - 2\mathbf{k}) = \frac{1}{\sqrt{14}}\mathbf{i} + \frac{3}{\sqrt{14}}\mathbf{j} - \frac{2}{\sqrt{14}}\mathbf{k}$

e) $\left|\frac{1}{|\mathbf{u}|}\mathbf{u}\right| = \left|\frac{1}{\sqrt{14}}\mathbf{i} + \frac{3}{\sqrt{14}}\mathbf{j} - \frac{2}{\sqrt{14}}\mathbf{k}\right| = \sqrt{\left(\frac{1}{\sqrt{14}}\right)^2 + \left(\frac{3}{\sqrt{14}}\right)^2 + \left(\frac{-2}{\sqrt{14}}\right)^2} = \sqrt{\frac{1+9+4}{14}} = 1$

11 a) Using $B(x, y, z)$ for the terminal point and $A(-1, 2, -3)$ for the initial point:

$\overrightarrow{AB} = (x - (-1), y - 2, z - (-3)) = (x + 1, y - 2, z + 3), \overrightarrow{AB} = \mathbf{w}$

Therefore: $\begin{pmatrix} x+1 \\ y-2 \\ z+3 \end{pmatrix} = \begin{pmatrix} 4 \\ 2 \\ -2 \end{pmatrix} \Rightarrow x = 3, y = 4, z = -5$. So, the terminal point is $(3, 4, -5)$.

b) Using $B(x, y, z)$ for the terminal point and $A(-2, 1, 4)$ for the initial point:

$\overrightarrow{AB} = (x - (-2), y - 1, z - 4) = (x + 2, y - 1, z - 4), \overrightarrow{AB} = \mathbf{v}$

Therefore: $\begin{pmatrix} x+2 \\ y-1 \\ z-4 \end{pmatrix} = \begin{pmatrix} 2 \\ -3 \\ 1 \end{pmatrix} \Rightarrow x = 0, y = -2, z = 5$. So, the terminal point is $(0, -2, 5)$.

12 a) A vector opposite in direction and a third the magnitude of $\mathbf{u}$ is $-\frac{1}{3}\mathbf{u}$. Therefore:

$$-\frac{1}{3}\mathbf{u} = -\frac{1}{3}\begin{pmatrix} -3 \\ 4 \end{pmatrix} = \begin{pmatrix} 1 \\ -\frac{4}{3} \end{pmatrix}.$$

b) A vector in the same direction as $\mathbf{w}$ and whose magnitude equals 12 is 12 times a unit vector in the direction of $\mathbf{w}$. Therefore, the vector is of the form:

$$12\frac{1}{|\mathbf{w}|}\mathbf{w} = 12\frac{1}{\sqrt{4^2 + 2^2 + (-2)^2}}(4\mathbf{i} + 2\mathbf{j} - 2\mathbf{k}) = \frac{12}{\sqrt{24}}(4\mathbf{i} + 2\mathbf{j} - 2\mathbf{k}) = \sqrt{6}(4\mathbf{i} + 2\mathbf{j} - 2\mathbf{k}).$$

c) If vectors are parallel, then one can be represented as t times the other. Therefore:

$x\mathbf{i} + y\mathbf{j} - 2\mathbf{k} = t(\mathbf{i} - 4\mathbf{j} + 3\mathbf{k})$. From the $\mathbf{k}$-coordinate, we can find the value of t:

$-2 = 3t \Rightarrow t = -\frac{2}{3}$. So, $x = -\frac{2}{3} \cdot 1 = -\frac{2}{3}$ and $y = -\frac{2}{3} \cdot (-4) = \frac{8}{3}$, and the vector is:

$-\frac{2}{3}\mathbf{i} + \frac{8}{3}\mathbf{j} - 2\mathbf{k}$.

13 Let **u** be the vector from the vertex A to the midpoint of side BC; so, $\mathbf{u} = \overrightarrow{AB} + \frac{1}{2}\overrightarrow{BC}$. Let **v** be the vector from the vertex B to the midpoint of side AC; so, $\mathbf{v} = \overrightarrow{BA} + \frac{1}{2}\overrightarrow{AC}$. Let **w** be the vector from the vertex C to the midpoint of side AB; so, $\mathbf{w} = \overrightarrow{CA} + \frac{1}{2}\overrightarrow{AB}$. Adding the vectors:

$$\mathbf{u}+\mathbf{v}+\mathbf{w} = \cancel{\overrightarrow{AB}} + \frac{1}{2}\overrightarrow{BC} + \cancel{\overrightarrow{BA}} + \frac{1}{2}\overrightarrow{AC} + \overrightarrow{CA} + \frac{1}{2}\overrightarrow{AB} = \frac{1}{2}\overrightarrow{BC} + \left(\frac{1}{2}\overrightarrow{AC} + \overrightarrow{CA}\right) + \frac{1}{2}\overrightarrow{AB}$$

$$= \frac{1}{2}\overrightarrow{BC} + \frac{1}{2}\overrightarrow{CA} + \frac{1}{2}\overrightarrow{AB} = \frac{1}{2}\left(\overrightarrow{BC} + \overrightarrow{CA} + \overrightarrow{AB}\right) = \frac{1}{2}\cdot\vec{0} = \vec{0}$$

14 The length of the vector $\mathbf{v} = t\mathbf{i} - 2t\mathbf{j} + 3t\mathbf{k}$ is $\sqrt{t^2 + (-2t)^2 + (3t)^2} = \sqrt{14t^2} = |t|\sqrt{14}$. Hence,

$|t|\sqrt{14} = 1 \Rightarrow |t| = \frac{1}{\sqrt{14}} \Rightarrow t = \pm\frac{\sqrt{14}}{14}$.

15 The length of the vector $\mathbf{v} = 2\mathbf{i} - 2t\mathbf{j} + 3t\mathbf{k}$ is $\sqrt{2^2 + (-2t)^2 + (3t)^2} = \sqrt{4 + 13t^2}$. Hence,

$\sqrt{4+13t^2} = 1 \Rightarrow 4 + 13t^2 = 1 \Rightarrow 13t^2 = -3$, so there is no solution.

16 The length of the vector $\mathbf{v} = 0.5\mathbf{i} - t\mathbf{j} + 1.5t\mathbf{k}$ is $\sqrt{0.5^2 + (-t)^2 + (1.5t)^2} = \sqrt{0.25 + 3.25t^2}$. Hence,

$\sqrt{0.25 + 3.25t^2} = 1 \Rightarrow 0.25 + 3.25t^2 = 1 \Rightarrow t^2 = \frac{3}{13} \Rightarrow t = \pm\sqrt{\frac{3}{13}}$.

17 a) $\mathbf{a} = \begin{pmatrix}8\\0\\0\end{pmatrix}, \mathbf{b} = \begin{pmatrix}8\\8\\0\end{pmatrix}, \mathbf{c} = \begin{pmatrix}0\\8\\0\end{pmatrix}, \mathbf{d} = \begin{pmatrix}0\\0\\8\end{pmatrix}, \mathbf{e} = \begin{pmatrix}8\\0\\8\end{pmatrix}, \mathbf{f} = \begin{pmatrix}8\\8\\8\end{pmatrix}, \mathbf{g} = \begin{pmatrix}0\\8\\8\end{pmatrix}$

b) $\mathbf{l} = \begin{pmatrix}8\\4\\8\end{pmatrix}, \mathbf{m} = \begin{pmatrix}4\\8\\8\end{pmatrix}, \mathbf{n} = \begin{pmatrix}8\\8\\4\end{pmatrix}$

c) $\overrightarrow{LM} + \overrightarrow{MN} + \overrightarrow{NL} = (\mathbf{m} - \mathbf{l}) + (\mathbf{n} - \mathbf{m}) + (\mathbf{l} - \mathbf{n}) = \vec{0}$

Note: We can verify the statement using coordinates:

$$\overrightarrow{LM} + \overrightarrow{MN} + \overrightarrow{NL} = \begin{pmatrix}4-8\\8-4\\8-8\end{pmatrix} + \begin{pmatrix}8-4\\8-8\\4-8\end{pmatrix} + \begin{pmatrix}8-8\\4-8\\8-4\end{pmatrix} = \begin{pmatrix}-4\\4\\0\end{pmatrix} + \begin{pmatrix}4\\0\\-4\end{pmatrix} + \begin{pmatrix}0\\-4\\4\end{pmatrix} = \begin{pmatrix}0\\0\\0\end{pmatrix} = \vec{0}$$

18 a) $\mathbf{c} = \begin{pmatrix}8\\0\\12\end{pmatrix}, \mathbf{d} = \begin{pmatrix}0\\10\\12\end{pmatrix}$ **b)** $\mathbf{f} = \begin{pmatrix}4\\5\\0\end{pmatrix}, \mathbf{g} = \begin{pmatrix}4\\5\\12\end{pmatrix}$

c) $\overrightarrow{AG} = \begin{pmatrix}4-8\\5-0\\12-0\end{pmatrix} = \begin{pmatrix}-4\\5\\12\end{pmatrix}$

$\overrightarrow{FD} = \begin{pmatrix}0-4\\10-5\\12-0\end{pmatrix} = \begin{pmatrix}-4\\5\\12\end{pmatrix}$

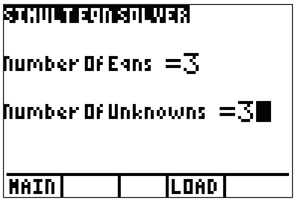
The vectors are the same because they connect a vertex and the midpoint of the parallel side in a parallelogram.

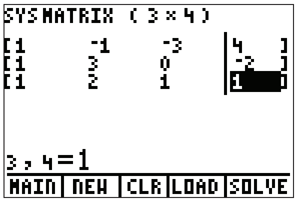
19 $|\alpha\mathbf{i} + (\alpha - 1)\,\mathbf{j} + (\alpha + 1)\,\mathbf{k}| = \sqrt{\alpha^2 + (\alpha - 1)^2 + (\alpha + 1)^2} = \sqrt{3\alpha^2 + 2}$. Hence, we have to solve the equation:

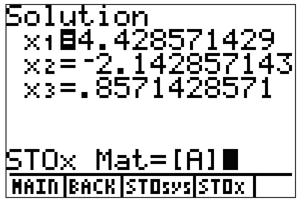
$$\sqrt{3\alpha^2 + 2} = 2 \Rightarrow 3\alpha^2 + 2 = 4 \Rightarrow a = \pm\sqrt{\frac{2}{3}} = \pm\frac{\sqrt{6}}{3}$$

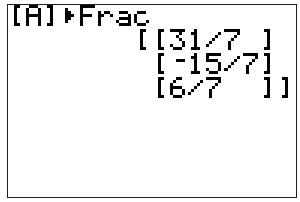
20 We have to solve a vector equation: $\begin{pmatrix} 4 \\ -2 \\ 1 \end{pmatrix} = \alpha\begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix} + \beta\begin{pmatrix} -1 \\ 3 \\ 2 \end{pmatrix} + \mu\begin{pmatrix} -3 \\ 0 \\ 1 \end{pmatrix}$

$$\begin{pmatrix} 4 \\ -2 \\ 1 \end{pmatrix} = \begin{pmatrix} \alpha - \beta - 3\mu \\ \alpha + 3\beta \\ \alpha + 2\beta + \mu \end{pmatrix} \Rightarrow \begin{cases} \alpha - \beta - 3\mu = 4 \\ \alpha + 3\beta = -2 \\ \alpha + 2\beta + \mu = 1 \end{cases}$$





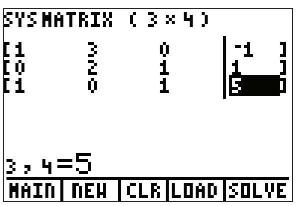


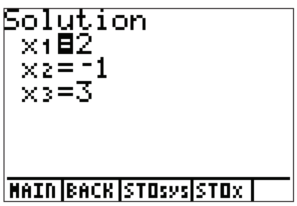


Therefore, $\alpha = \frac{31}{7}$, $\beta = -\frac{15}{7}$, $\mu = \frac{6}{7}$.

21 We have to solve a vector equation: $\begin{pmatrix} -1 \\ 1 \\ 5 \end{pmatrix} = \alpha\begin{pmatrix} 1 \\ 0 \\ 1 \end{pmatrix} + \beta\begin{pmatrix} 3 \\ 2 \\ 0 \end{pmatrix} + \mu\begin{pmatrix} 0 \\ 1 \\ 1 \end{pmatrix}$

$$\begin{pmatrix} -1 \\ 1 \\ 5 \end{pmatrix} = \begin{pmatrix} \alpha + 3\beta \\ 2\beta + \mu \\ \alpha + \mu \end{pmatrix} \Rightarrow \begin{cases} \alpha + 3\beta = -1 \\ 2\beta + \mu = 1 \\ \alpha + \mu = 5 \end{cases}$$

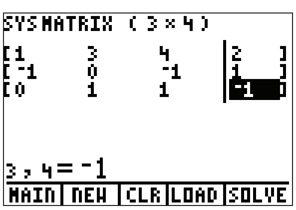


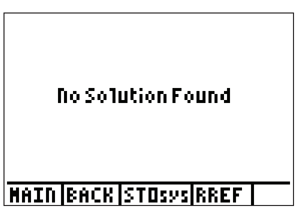


Therefore, $\alpha = 2$, $\beta = -1$, $\mu = 3$.

22 We have to solve a vector equation: $\begin{pmatrix} 2 \\ 1 \\ -1 \end{pmatrix} = \alpha\begin{pmatrix} 1 \\ -1 \\ 0 \end{pmatrix} + \beta\begin{pmatrix} 3 \\ 0 \\ 1 \end{pmatrix} + \mu\begin{pmatrix} 4 \\ -1 \\ 1 \end{pmatrix}$

$$\begin{pmatrix} 2 \\ 1 \\ -1 \end{pmatrix} = \begin{pmatrix} \alpha + 3\beta + 4\mu \\ -\alpha - \mu \\ \beta + \mu \end{pmatrix} \Rightarrow \begin{cases} \alpha + 3\beta + 4\mu = 2 \\ -\alpha - \mu = 1 \\ \beta + \mu = -1 \end{cases}$$





Hence, there are no such scalars α, β, μ.

23 a) $\mathbf{u} - \mathbf{v}$ and $\mathbf{u} + \mathbf{v}$ are diagonals of a parallelogram. So, the parallelogram has diagonals of the same length; hence, it is a rectangle.

b) $$\mathbf{u} - \mathbf{v} = \begin{pmatrix} v_1 - u_1 \\ v_2 - u_2 \\ v_3 - u_3 \end{pmatrix} \Rightarrow |\mathbf{u} - \mathbf{v}| = \sqrt{(v_1 - u_1)^2 + (v_2 - u_2)^2 + (v_3 - u_3)^2}$$

$$\mathbf{u} + \mathbf{v} = \begin{pmatrix} v_1 + u_1 \\ v_2 + u_2 \\ v_3 + u_3 \end{pmatrix} \Rightarrow |\mathbf{u} + \mathbf{v}| = \sqrt{(v_1 + u_1)^2 + (v_2 + u_2)^2 + (v_3 + u_3)^2}$$

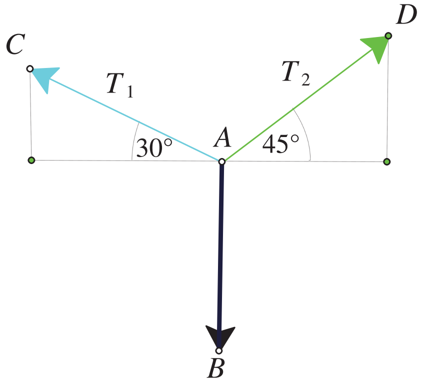
Hence, $\sqrt{(v_1 - u_1)^2 + (v_2 - u_2)^2 + (v_3 - u_3)^2} = \sqrt{(v_1 + u_1)^2 + (v_2 + u_2)^2 + (v_3 + u_3)^2}$.

$(v_1 - u_1)^2 + (v_2 - u_2)^2 + (v_3 - u_3)^2 = (v_1 + u_1)^2 + (v_2 + u_2)^2 + (v_3 + u_3)^2$

$v_1^2 - 2v_1u_1 + u_1^2 + v_2^2 - 2v_2u_2 + u_2^2 + v_3^2 - 2v_3u_3 + u_3^2 = v_1^2 + 2v_1u_1 + u_1^2 + v_2^2 + 2v_2u_2 + u_2^2 + v_3^2 + 2v_3u_3 + u_3^2$

So: $0 = 4v_1u_1 + 4v_2u_2 + 4v_3u_3 \Rightarrow v_1u_1 + v_2u_2 + v_3u_3 = 0$

24 We will introduce notation as on the diagram below:



If the traffic light is in equilibrium: $\overrightarrow{AB} + \overrightarrow{AC} + \overrightarrow{AD} = \vec{0}$.

We will express the vectors in component form:

$$\overrightarrow{AB} = \begin{pmatrix} 0 \\ -125 \end{pmatrix}$$

Vector $\overrightarrow{AC}$ is parallel to the unit vector $\begin{pmatrix} -\cos 30^\circ \\ \sin 30^\circ \end{pmatrix} = \begin{pmatrix} -\frac{\sqrt{3}}{2} \\ \frac{1}{2} \end{pmatrix}$ and its magnitude is T_1; hence,

$$\overrightarrow{AC} = T_1 \begin{pmatrix} -\frac{\sqrt{3}}{2} \\ \frac{1}{2} \end{pmatrix}.$$

Vector $\overrightarrow{AD}$ is parallel to the unit vector $\begin{pmatrix} \cos 45^\circ \\ \sin 45^\circ \end{pmatrix} = \begin{pmatrix} \frac{\sqrt{2}}{2} \\ \frac{\sqrt{2}}{2} \end{pmatrix}$ and its magnitude is T_2; hence,

$$\overrightarrow{AD} = T_2 \begin{pmatrix} \frac{\sqrt{2}}{2} \\ \frac{\sqrt{2}}{2} \end{pmatrix}.$$

Now, we have: $\begin{pmatrix} 0 \\ -125 \end{pmatrix} + T_1 \begin{pmatrix} -\frac{\sqrt{3}}{2} \\ \frac{1}{2} \end{pmatrix} + T_2 \begin{pmatrix} \frac{\sqrt{2}}{2} \\ \frac{\sqrt{2}}{2} \end{pmatrix} = \vec{0} \Rightarrow \begin{pmatrix} -T_1 \frac{\sqrt{3}}{2} + T_2 \frac{\sqrt{2}}{2} \\ -125 + T_1 \frac{1}{2} + T_2 \frac{\sqrt{2}}{2} \end{pmatrix} = \vec{0}$

So, $-T_1 \frac{\sqrt{3}}{2} + T_2 \frac{\sqrt{2}}{2} = 0 \Rightarrow -T_1\sqrt{3} + T_2\sqrt{2} \Rightarrow T_1 = T_2 \frac{\sqrt{2}}{\sqrt{3}}$

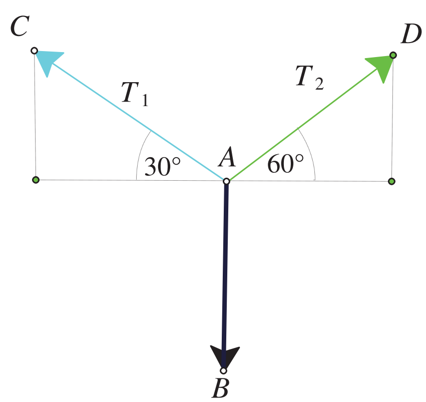
$-125 + T_1 \frac{1}{2} + T_2 \frac{\sqrt{2}}{2} = 0 \Rightarrow -125 + T_2 \frac{\sqrt{2}}{\sqrt{3}} \frac{1}{2} + T_2 \frac{\sqrt{2}}{2} = 0$

$\Rightarrow -250\sqrt{3} + \sqrt{2}T_2 + \sqrt{6}T_2 = 0 \Rightarrow T_2 = \frac{250\sqrt{3}}{\sqrt{2}+\sqrt{6}} = \frac{125\left(3\sqrt{2}-\sqrt{6}\right)}{2}$

And: $T_1 = T_2 \frac{\sqrt{2}}{\sqrt{3}} = \frac{125\left(3\sqrt{2}-\sqrt{6}\right)}{2} \frac{\sqrt{2}}{\sqrt{3}} = \frac{125\sqrt{2}\sqrt{3}\left(\sqrt{3}-1\right)\sqrt{2}}{2\sqrt{3}} = 125\left(\sqrt{3}-1\right)$

Therefore, the cable tensions are $T_1 = 125\left(\sqrt{3}-1\right)$ N and $T_2 = \frac{125\left(3\sqrt{2}-\sqrt{6}\right)}{2}$ N.

25 We will introduce notation as on the diagram below:



For vectors it holds: $\overrightarrow{AB} + \overrightarrow{AC} + \overrightarrow{AD} = \vec{0}$.

We will express the vectors in component form:

$\overrightarrow{AB} = \begin{pmatrix} 0 \\ -300 \end{pmatrix}$

Vector $\overrightarrow{AC}$ is parallel to the unit vector $\begin{pmatrix} -\cos 30^\circ \\ \sin 30^\circ \end{pmatrix} = \begin{pmatrix} -\frac{\sqrt{3}}{2} \\ \frac{1}{2} \end{pmatrix}$ and its magnitude is T_1; hence,

$\overrightarrow{AC} = T_1 \begin{pmatrix} -\frac{\sqrt{3}}{2} \\ \frac{1}{2} \end{pmatrix}$.

Vector $\overrightarrow{AD}$ is parallel to the unit vector $\begin{pmatrix} \cos 60^\circ \\ \sin 60^\circ \end{pmatrix} = \begin{pmatrix} \frac{1}{2} \\ \frac{\sqrt{3}}{2} \end{pmatrix}$ and its magnitude is T_2; hence

$\overrightarrow{AD} = T_2 \begin{pmatrix} \frac{1}{2} \\ \frac{\sqrt{3}}{2} \end{pmatrix}$.

Now, we have: $\begin{pmatrix} 0 \\ -300 \end{pmatrix} + T_1 \begin{pmatrix} -\frac{\sqrt{3}}{2} \\ \frac{1}{2} \end{pmatrix} + T_2 \begin{pmatrix} \frac{1}{2} \\ \frac{\sqrt{3}}{2} \end{pmatrix} = \vec{0} \Rightarrow \begin{pmatrix} -T_1 \frac{\sqrt{3}}{2} + T_2 \frac{1}{2} \\ -300 + T_1 \frac{1}{2} + T_2 \frac{\sqrt{3}}{2} \end{pmatrix} = \vec{0}$

So, $-T_1 \frac{\sqrt{3}}{2} + T_2 \frac{1}{2} = 0 \Rightarrow -T_1\sqrt{3} + T_2 \Rightarrow T_2 = T_1\sqrt{3}$

$-300 + T_1 \frac{1}{2} + T_2 \frac{\sqrt{3}}{2} = 0 \Rightarrow -300 + T_1 \frac{1}{2} + T_1\sqrt{3}\frac{\sqrt{3}}{2} = 0$

$\Rightarrow -300 + 2T_1 = 0 \Rightarrow T_1 = 150$

And: $T_2 = T_1\sqrt{3} = 150\sqrt{3}$

Therefore, the cable tensions are $T_1 = 150$ N and $T_2 = 150\sqrt{3}$ N.

Exercise 14.2

1 **a)** $\mathbf{u} \cdot \mathbf{v} = 3 \cdot 2 + (-2) \cdot (-1) + 4 \cdot (-6) = -16$

$\cos\theta = \frac{-16}{\sqrt{3^2 + (-2)^2 + 4^2}\sqrt{2^2 + (-1)^2 + (-6)^2}} = \frac{-16}{\sqrt{29}\sqrt{41}} \Rightarrow \theta \approx 117.65^\circ$

b) $\mathbf{u} \cdot \mathbf{v} = 2 \cdot (-1) + (-6) \cdot 3 + 0 \cdot 5 = -20$

$\cos\theta = \frac{-20}{\sqrt{2^2 + (-6)^2 + 0^2}\sqrt{(-1)^2 + (3)^2 + 5^2}} = \frac{-20}{\sqrt{40}\sqrt{35}} \Rightarrow \theta \approx 122.31^\circ$

c) $\mathbf{u} \cdot \mathbf{v} = 3 \cdot 5 + (-1) \cdot 2 = 13$

$\cos\theta = \frac{13}{\sqrt{3^2 + (-1)^2}\sqrt{5^2 + 2^2}} = \frac{13}{\sqrt{10}\sqrt{29}} \Rightarrow \theta \approx 40.24^\circ$

d) $\mathbf{u} \cdot \mathbf{v} = 1 \cdot 0 + (-3) \cdot 5 + 0 \cdot 2 = -15$

$\cos\theta = \frac{-15}{\sqrt{1^2 + (-3)^2 + 0^2}\sqrt{0^2 + 5^2 + (-2)^2}} = \frac{-15}{\sqrt{10}\sqrt{29}} \Rightarrow \theta \approx 151.74^\circ$

e) $\mathbf{u} \cdot \mathbf{v} = |\mathbf{u}|\,|\mathbf{v}| \cos\theta = 3 \cdot 4 \cdot \cos\frac{\pi}{3} = 3 \cdot 4 \cdot \frac{1}{2} = 6$; angle is $\theta = 60^\circ$

f) $\mathbf{u} \cdot \mathbf{v} = |\mathbf{u}|\,|\mathbf{v}| \cos\theta = 3 \cdot 4 \cdot \cos\frac{2\pi}{3} = 3 \cdot 4 \cdot \frac{-1}{2} = -6$; angle is $\theta = 120^\circ$

2 **a)** $\mathbf{u} \cdot \mathbf{v} = 2 \cdot (-1) + (-6) \cdot 3 + 4 \cdot 5 = 0$. The dot product is zero; hence, the vectors are orthogonal.

b) $\mathbf{u} \cdot \mathbf{v} = 3 \cdot 5 + (-7) \cdot 2 = 1$. The dot product is positive; hence, the angle is acute.

c) $\mathbf{u} \cdot \mathbf{v} = 1 \cdot 0 + (-3) \cdot 6 + 6 \cdot 3 = 0$. The dot product is zero; hence, the vectors are orthogonal.

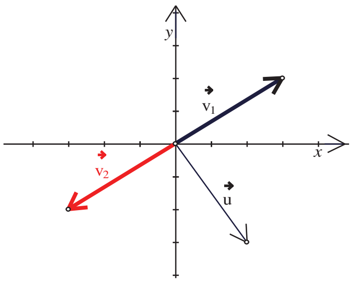
3 **a)** $\mathbf{v} \cdot \mathbf{u} = -y \cdot x + x \cdot y = 0$. The dot product is zero; hence, **v** is orthogonal to **u**.

$\mathbf{w} \cdot \mathbf{u} = y \cdot x + (-x) \cdot y = 0$. The dot product is zero; hence, **w** is orthogonal to **u**.

Pay attention to the relationship between the coordinates of a two-dimensional vector and a vector that is perpendicular to it.

b) The vectors perpendicular to $\mathbf{u} = 2\mathbf{i} - 3\mathbf{j}$ are $3\mathbf{i} + 2\mathbf{j}$ and $-3\mathbf{i} - 2\mathbf{j}$.

Unit vectors in the direction of those vectors are: $\mathbf{v}_1 = \frac{1}{\sqrt{13}}(3\mathbf{i} + 2\mathbf{j})$ and $\mathbf{v}_2 = -\frac{1}{\sqrt{13}}(3\mathbf{i} + 2\mathbf{j})$.



4 a) i) $|\mathbf{v}| = \sqrt{2^2 + (-3)^2 + 1^2} = \sqrt{14}$

$\mathbf{i} \cdot \mathbf{v} = 1 \cdot 2 + 0 \cdot (-3) + 0 \cdot 1 = 2 \Rightarrow \cos\alpha = \frac{2}{\sqrt{14}}$

$\mathbf{j} \cdot \mathbf{v} = 0 \cdot 2 + 1 \cdot (-3) + 0 \cdot 1 = -3 \Rightarrow \cos\beta = \frac{-3}{\sqrt{14}}$

$\mathbf{k} \cdot \mathbf{v} = 0 \cdot 2 + 0 \cdot (-3) + 1 \cdot 1 = 1 \Rightarrow \cos\gamma = \frac{1}{\sqrt{14}}$

ii) $\cos^2\alpha + \cos^2\beta + \cos^2\gamma = \left(\frac{2}{\sqrt{14}}\right)^2 + \left(\frac{-3}{\sqrt{14}}\right)^2 + \left(\frac{1}{\sqrt{14}}\right)^2 = \frac{4+9+1}{14} = 1$

iii) $\alpha = \cos^{-1}\left(\frac{2}{\sqrt{14}}\right) = 57.6884...^\circ \approx 58^\circ$, $\beta = \cos^{-1}\left(\frac{-3}{\sqrt{14}}\right) = 143.300\,77...^\circ \approx 143^\circ$,

$\gamma = \cos^{-1}\left(\frac{1}{\sqrt{14}}\right) = 74.4986...^\circ \approx 74^\circ$

b) i) $|\mathbf{v}| = \sqrt{1^2 + (-2)^2 + 1^2} = \sqrt{6}$

$\mathbf{i} \cdot \mathbf{v} = 1 \cdot 1 + 0 \cdot (-2) + 0 \cdot 1 = 1 \Rightarrow \cos\alpha = \frac{1}{\sqrt{6}}$

$\mathbf{j} \cdot \mathbf{v} = 0 \cdot 1 + 1 \cdot (-2) + 0 \cdot 1 = -2 \Rightarrow \cos\beta = \frac{-2}{\sqrt{6}}$

$\mathbf{k} \cdot \mathbf{v} = 0 \cdot 1 + 0 \cdot (-2) + 1 \cdot 1 = 1 \Rightarrow \cos\gamma = \frac{1}{\sqrt{6}}$

ii) $\cos^2\alpha + \cos^2\beta + \cos^2\gamma = \left(\frac{1}{\sqrt{6}}\right)^2 + \left(\frac{-2}{\sqrt{6}}\right)^2 + \left(\frac{1}{\sqrt{6}}\right)^2 = \frac{1+4+1}{6} = 1$

iii) $\alpha = \cos^{-1}\left(\frac{1}{\sqrt{6}}\right) = 65.9051...^\circ \approx 66^\circ$, $\beta = \cos^{-1}\left(\frac{-2}{\sqrt{6}}\right) = 144.7356...^\circ \approx 145^\circ$,

$\gamma = \cos^{-1}\left(\frac{1}{\sqrt{6}}\right) = 65.9051...^\circ \approx 66^\circ$

c) i) $|\mathbf{v}| = \sqrt{3^2 + (-2)^2 + 1^2} = \sqrt{14}$

$\mathbf{i} \cdot \mathbf{v} = 1 \cdot 3 + 0 \cdot (-2) + 0 \cdot 1 = 3 \Rightarrow \cos\alpha = \frac{3}{\sqrt{14}}$

$\mathbf{j} \cdot \mathbf{v} = 0 \cdot 3 + 1 \cdot (-2) + 0 \cdot 1 = -2 \Rightarrow \cos\beta = \dfrac{-2}{\sqrt{14}}$

$\mathbf{k} \cdot \mathbf{v} = 0 \cdot 3 + 0 \cdot (-2) + 1 \cdot 1 = 1 \Rightarrow \cos\gamma = \dfrac{1}{\sqrt{14}}$

ii) $\cos^2\alpha + \cos^2\beta + \cos^2\gamma = \left(\dfrac{3}{\sqrt{14}}\right)^2 + \left(\dfrac{-2}{\sqrt{14}}\right)^2 + \left(\dfrac{1}{\sqrt{14}}\right)^2 = \dfrac{9+4+1}{14} = 1$

iii) $\alpha = \cos^{-1}\left(\dfrac{3}{\sqrt{14}}\right) = 36.6992...^\circ \approx 37^\circ$, $\beta = \cos^{-1}\left(\dfrac{-2}{\sqrt{14}}\right) = 122.3115...^\circ \approx 122^\circ$,

$\gamma = \cos^{-1}\left(\dfrac{1}{\sqrt{14}}\right) = 74.4986...^\circ \approx 74^\circ$

d) **i)** $|\mathbf{v}| = \sqrt{3^2 + 0^2 + (-4)^2} = 5$

$\mathbf{i} \cdot \mathbf{v} = 1 \cdot 3 + 0 \cdot 0 + 0 \cdot (-4) = 3 \Rightarrow \cos\alpha = \dfrac{3}{5}$

$\mathbf{j} \cdot \mathbf{v} = 0 \cdot 3 + 1 \cdot 0 + 0 \cdot (-4) = 0 \Rightarrow \cos\beta = 0$

$\mathbf{k} \cdot \mathbf{v} = 0 \cdot 3 + 0 \cdot 0 + 1 \cdot (-4) = -4 \Rightarrow \cos\gamma = \dfrac{-4}{5}$

ii) $\cos^2\alpha + \cos^2\beta + \cos^2\gamma = \left(\dfrac{3}{5}\right)^2 + 0^2 + \left(\dfrac{-4}{5}\right)^2 = \dfrac{9+16}{25} = 1$

iii) $\alpha = \cos^{-1}\left(\dfrac{3}{5}\right) = 53.1301...^\circ \approx 53^\circ$, $\beta = \cos^{-1} 0 = 90^\circ$, $\gamma = \cos^{-1}\left(\dfrac{-4}{5}\right) = 143.1301...^\circ \approx 143^\circ$

5 $\begin{pmatrix} \cos\dfrac{\pi}{3} \\ \cos\dfrac{\pi}{4} \\ \cos\dfrac{2\pi}{3} \end{pmatrix} = \begin{pmatrix} \dfrac{1}{2} \\ \dfrac{\sqrt{2}}{2} \\ -\dfrac{1}{2} \end{pmatrix}$

6 $3\begin{pmatrix} \cos\dfrac{\pi}{4} \\ \cos\dfrac{\pi}{4} \\ \cos\dfrac{\pi}{2} \end{pmatrix} = 3\begin{pmatrix} \dfrac{\sqrt{2}}{2} \\ \dfrac{\sqrt{2}}{2} \\ 0 \end{pmatrix} = \begin{pmatrix} \dfrac{3\sqrt{2}}{2} \\ \dfrac{3\sqrt{2}}{2} \\ 0 \end{pmatrix}$

7 **a)** $\mathbf{u} \cdot \mathbf{v} = 3 \cdot (m-2) + 5 \cdot (m+3) + 0 \cdot 0 = 8m + 9$

Vectors are perpendicular if their dot product is zero; therefore: $8m + 9 = 0 \Rightarrow m = -\dfrac{9}{8}$

b) $\mathbf{u} \cdot \mathbf{v} = (2m) \cdot (m-1) + (m-1) \cdot m + (m+1) \cdot (m-1) = 4m^2 - 3m - 1$

Vectors are perpendicular if their dot product is zero; therefore:

$4m^2 - 3m - 1 = 0 \Rightarrow m = 1, m = -\dfrac{1}{4}$

8 $\mathbf{u} \cdot \mathbf{w} = \mathbf{u} \cdot (\mathbf{u} + m\mathbf{v}) = \mathbf{u} \cdot \mathbf{u} + m\mathbf{u} \cdot \mathbf{v} = (-3)^2 + 1^2 + 2^2 + m((-3) \cdot 1 + 1 \cdot 2 + 2 \cdot 1) = 14 + m$

Vectors are orthogonal if their dot product is zero; therefore:

$14 + m = 0 \Rightarrow m = -14$

9 **a)** $\cos\theta = \frac{\mathbf{u}\cdot\mathbf{v}}{|\mathbf{u}|\,|\mathbf{v}|} = \frac{(-2)\cdot 6 + 5\cdot(-3) + 4\cdot 0}{\sqrt{(-2)^2+5^2+4^2}\sqrt{6^2+(-3)^2+0^2}} = \frac{-27}{\sqrt{45}\sqrt{45}} = -\frac{27}{45}$

$\Rightarrow \theta = \cos^{-1}\left(-\frac{27}{45}\right) = 126.8698...^\circ \approx 127^\circ$

b) $\mathbf{u}+\mathbf{v} = (-2, 5, 4) + (6, -3, 0) = (4, 2, 4)$

$\cos\theta = \frac{\mathbf{u}\cdot(\mathbf{u}+\mathbf{v})}{|\mathbf{u}|\,|\mathbf{u}+\mathbf{v}|} = \frac{(-2)\cdot 4 + 5\cdot 2 + 4\cdot 4}{\sqrt{45}\sqrt{4^2+2^2+4^2}} = \frac{18}{\sqrt{45}\sqrt{36}} = \frac{1}{\sqrt{5}} \Rightarrow \theta = \cos^{-1}\left(\frac{1}{\sqrt{5}}\right) = 63.4349...^\circ \approx 63^\circ$

c) $\cos\theta = \frac{\mathbf{v}\cdot(\mathbf{u}+\mathbf{v})}{|\mathbf{v}|\,|\mathbf{u}+\mathbf{v}|} = \frac{6\cdot 4 + (-3)\cdot 2 + 0\cdot 4}{\sqrt{45}\cdot 6} = \frac{18}{\sqrt{45}\cdot 6} = \frac{1}{\sqrt{5}} \Rightarrow \theta = \cos^{-1}\left(\frac{1}{\sqrt{5}}\right) = 63.4349...^\circ \approx 63^\circ$

10 $\overrightarrow{AB} = (3-1, 5-2, -2-(-3)) = (2, 3, 1)$

$\overrightarrow{AC} = (m-1, 1-2, -10m-(-3)) = (m-1, -1, -10m+3)$

a) The points A, B and C are collinear if $\overrightarrow{AC}$ is parallel to $\overrightarrow{AB}$. Given this collinearity, $\overrightarrow{AC} = t\,\overrightarrow{AB} \Rightarrow (2, 3, 1) = t\,(m-1, -1, -10m+3)$. From the second set of coordinates, we can determine t: $3 = -t \Rightarrow t = -3$. Hence: $2 = -3\cdot(m-1) \Rightarrow m = \frac{1}{3}$. Checking with the third set of coordinates: $1 = -3\left(-10\cdot\frac{1}{3}+3\right) \Rightarrow 1 = -3\cdot\frac{-1}{3}$; so, it fits and for $m = \frac{1}{3}$ the points are collinear.

b) $\overrightarrow{AC}\cdot\overrightarrow{AB} \Rightarrow 2\cdot(m-1) + 3\cdot(-1) + 1\cdot(-10m+3) = -8m-2$

Vectors are perpendicular if their dot product is zero; therefore:

$\overrightarrow{AC}\cdot\overrightarrow{AB} = 0 \Rightarrow -8m-2 = 0 \Rightarrow m = -\frac{1}{4}$.

11 The vector equation of the line is an equation of the form: $\mathbf{r} = \mathbf{r}_0 + t\mathbf{v}$, where $\mathbf{r}_0$ is the position vector of any point on the line and the direction vector $\mathbf{v}$ is a vector parallel to the line.

For the median through A, we can take the position vector of point A for $\mathbf{r}_0$ and the vector from A to the midpoint of BC for $\mathbf{v}$. So:

$\mathbf{r}_0 = (4, -2, -1),\ m_{BC} = \left(\frac{3+3}{2}, \frac{-5+1}{2}, \frac{-1+2}{2}\right) = \left(3, -2, \frac{1}{2}\right)$

$\mathbf{v} = \left(3-4, -2-(-2), \frac{1}{2}-(-1)\right) = \left(-1, 0, \frac{3}{2}\right)$. Therefore:

$m_A : \mathbf{r} = (4, -2, -1) + m\left(-1, 0, \frac{3}{2}\right)$.

For the median through B, we can take the position vector of point B for $\mathbf{r}_0$ and the vector from B to the midpoint of AC for $\mathbf{v}$. So:

$\mathbf{r}_0 = (3, -5, -1),\ m_{AC} = \left(\frac{4+3}{2}, \frac{-2+1}{2}, \frac{-1+2}{2}\right) = \left(\frac{7}{2}, -\frac{1}{2}, \frac{1}{2}\right)$

$\mathbf{v} = \left(\frac{7}{2}-3, -\frac{1}{2}-(-5), \frac{1}{2}-(-1)\right) = \left(\frac{1}{2}, \frac{9}{2}, \frac{3}{2}\right)$. Therefore:

$m_B : \mathbf{r} = (3, -5, -1) + n\left(\frac{1}{2}, \frac{9}{2}, \frac{3}{2}\right)$.

For the median through C, we can take the position vector of point C for $\mathbf{r}_0$ and the vector from C to the midpoint of AB for $\mathbf{v}$. So:

$$\mathbf{r}_0 = (3, 1, 2),\ m_{AB} = \left(\frac{4+3}{2}, \frac{-2+(-5)}{2}, \frac{-1+(-1)}{2}\right) = \left(\frac{7}{2}, -\frac{7}{2}, -1\right)$$

$$\mathbf{v} = \left(\frac{7}{2} - 3, -\frac{7}{2} - 1, -1 - 2\right) = \left(\frac{1}{2}, -\frac{9}{2}, -3\right).\ \text{Therefore:}$$

$$m_C : \mathbf{r} = (3, 1, 2) + k\left(\frac{1}{2}, -\frac{9}{2}, -3\right).$$

The centroid is the point where all the medians meet. We will find the intersection point of two lines, and then check that this point is also on the third line.

If m_A and m_B intersect, then:

$$(4, -2, -1) + m\left(-1, 0, \frac{3}{2}\right) = (3, -5, -1) + n\left(\frac{1}{2}, \frac{9}{2}, \frac{3}{2}\right).\ \text{Therefore, we have:}$$

$$\begin{cases} 4 - m = 3 + \frac{1}{2}n \\ \left.\begin{aligned} -2 + 0m &= -5 + \frac{9}{2}n \Rightarrow n = \frac{2}{3} \\ -1 + \frac{3}{2}m &= -1 + \frac{3}{2}n \end{aligned}\right\} \Rightarrow m = \frac{2}{3} \end{cases}$$

Putting $m = n = \frac{2}{3}$ we can see that it fits the equation, so the point of intersection of m_A and m_B is:

$(4, -2, -1) + \frac{2}{3}\left(-1, 0, \frac{3}{2}\right) = \left(\frac{10}{3}, -2, 0\right)$. Now, we have to check that $\left(\frac{10}{3}, -2, 0\right)$ is on the third line as well: $\left(\frac{10}{3}, -2, 0\right) = (3, 1, 2) + \frac{2}{3}\cdot\left(\frac{1}{2}, -\frac{9}{2}, -3\right)$. Hence, we can see that the centroid is $\left(\frac{10}{3}, -2, 0\right)$.

Note: For the centroid, it holds that: $\left(\frac{10}{3}, -2, 0\right) = \left(\frac{4+3+3}{3}, \frac{-2-5+1}{3}, \frac{-1-1+2}{3}\right)$. The formula $\left(\frac{x_A + x_B + x_C}{3}, \frac{y_A + y_B + y_C}{3}, \frac{z_A + z_B + z_C}{3}\right)$ holds, in general, for a triangle with vertices A, B, C.

12 $\overrightarrow{AB} = (-3-1, 2-2, 1-3) = (-4, 0, -2);\ |\overrightarrow{AB}| = \sqrt{20}$

$\overrightarrow{AC} = (1-1, -4-2, 3-3) = (0, -6, 0);\ |\overrightarrow{AC}| = 6$

$\overrightarrow{AD} = (3-1, 2-2, -3-3) = (2, 0, -6);\ |\overrightarrow{AD}| = \sqrt{40}$

$\overrightarrow{BC} = (1-(-3), -4-2, 3-1) = (4, -6, 2);\ |\overrightarrow{BC}| = \sqrt{56}$

$\overrightarrow{BD} = (3-(-3), 2-2, -3-1) = (6, 0, -4);\ |\overrightarrow{BD}| = \sqrt{52}$

$\overrightarrow{CD} = (3-1, 2-(-4), -3-3) = (2, 6, -6);\ |\overrightarrow{CD}| = \sqrt{76}$

We will calculate the angles by finding the dot product and using the cosine angle formula:

$\overrightarrow{AB}\cdot\overrightarrow{AC} = 0 \Rightarrow \text{angle} = 90^\circ$

$\overrightarrow{AB}\cdot\overrightarrow{AD} = -4 \Rightarrow \cos^{-1}\left(\frac{-4}{\sqrt{20}\sqrt{40}}\right) = 98.1301...^\circ \Rightarrow \text{angle} = 180^\circ - 98.1301...^\circ \approx 82^\circ$

$\overrightarrow{AB}\cdot\overrightarrow{BD} = -16 \Rightarrow \cos^{-1}\left(\frac{-16}{\sqrt{20}\sqrt{52}}\right) = 119.7448...^\circ \Rightarrow \text{angle} = 180^\circ - 119.7448...^\circ \approx 60^\circ$

$\overrightarrow{AB}\cdot\overrightarrow{BC} = -20 \Rightarrow \cos^{-1}\left(\dfrac{-20}{\sqrt{20}\sqrt{56}}\right) = 126.6992...^\circ \Rightarrow \text{angle} = 180^\circ - 126.6992...^\circ \approx 53^\circ$

$\overrightarrow{AC}\cdot\overrightarrow{AD} = 0 \Rightarrow \text{angle} = 90^\circ$

$\overrightarrow{AC}\cdot\overrightarrow{BC} = 36 \Rightarrow \cos^{-1}\left(\dfrac{36}{6\sqrt{56}}\right) = 36.6992...^\circ \Rightarrow \text{angle} \approx 37^\circ$

$\overrightarrow{AC}\cdot\overrightarrow{CD} = -36 \Rightarrow \cos^{-1}\left(\dfrac{-36}{6\sqrt{76}}\right) = 133.4915...^\circ \Rightarrow \text{angle} = 180^\circ - 133.4915...^\circ \approx 47^\circ$

$\overrightarrow{AD}\cdot\overrightarrow{BD} = 36 \Rightarrow \cos^{-1}\left(\dfrac{36}{\sqrt{40}\sqrt{52}}\right) = 37.8749...^\circ \Rightarrow \text{angle} \approx 38^\circ$

$\overrightarrow{AD}\cdot\overrightarrow{CD} = 40 \Rightarrow \cos^{-1}\left(\dfrac{40}{\sqrt{40}\sqrt{76}}\right) = 43.4915...^\circ \Rightarrow \text{angle} \approx 43^\circ$

$\overrightarrow{BC}\cdot\overrightarrow{BD} = 16 \Rightarrow \cos^{-1}\left(\dfrac{16}{\sqrt{56}\sqrt{52}}\right) = 72.7525...^\circ \Rightarrow \text{angle} \approx 73^\circ$

$\overrightarrow{BC}\cdot\overrightarrow{CD} = -40 \Rightarrow \cos^{-1}\left(\dfrac{-40}{\sqrt{56}\sqrt{76}}\right) = 127.8168...^\circ \Rightarrow \text{angle} = 180^\circ - 127.8168...^\circ \approx 52^\circ$

$\overrightarrow{BD}\cdot\overrightarrow{CD} = 36 \Rightarrow \cos^{-1}\left(\dfrac{36}{\sqrt{52}\sqrt{76}}\right) = 55.0643...^\circ \Rightarrow \text{angle} \approx 55^\circ$

13 Total surface area consists of triangles *ABC, ABD, ACD,* and *BCD*:

$A_{ABC} = \dfrac{1}{2}\left|\overrightarrow{AB}\right|\left|\overrightarrow{AC}\right| \sin 90^\circ \approx \dfrac{1}{2}\sqrt{20}\cdot 6\cdot 1 = 6\sqrt{5} \approx 13.4164$

(**Note:** $\overrightarrow{AB}$ and $\overrightarrow{AC}$ are perpendicular.)

$A_{ABD} = \dfrac{1}{2}\left|\overrightarrow{AB}\right|\left|\overrightarrow{AD}\right| \sin\theta \approx \dfrac{1}{2}\sqrt{20}\cdot\sqrt{40}\cdot\sin 82^\circ \approx 14.0045$

(**Note:** $\theta = \angle\overrightarrow{AB}, \overrightarrow{AD} \approx 82^\circ$)

$A_{ACD} = \dfrac{1}{2}\left|\overrightarrow{AC}\right|\left|\overrightarrow{CD}\right| \sin\theta \approx \dfrac{1}{2}6\cdot\sqrt{76}\cdot\sin 47^\circ \approx 19.1274$

(**Note:** θ is the angle between the vectors $\angle\overrightarrow{CA}, \overrightarrow{CD} = 180^\circ - \angle\overrightarrow{AC}, \overrightarrow{CD} \approx 47^\circ$.)

$A_{BCD} = \dfrac{1}{2}\left|\overrightarrow{BC}\right|\left|\overrightarrow{CD}\right| \sin\theta \approx \dfrac{1}{2}\sqrt{56}\cdot\sqrt{76}\cdot\sin 52^\circ \approx 25.7041$

(**Note:** θ is the angle between the vectors $\angle\overrightarrow{CB}, \overrightarrow{CD} = 180^\circ - \angle\overrightarrow{AC}, \overrightarrow{CD} \approx 52^\circ$.)

Therefore, $A \approx 72.25$.

14 $\overrightarrow{CD} = (2, 6, -6) \Rightarrow \overrightarrow{DC} = (-2, -6, 6)$

$\vec{i}\cdot\overrightarrow{DC} = -2 \Rightarrow \cos^{-1}\left(\dfrac{-2}{\sqrt{76}}\right) \approx 103.3^\circ$

$\vec{j}\cdot\overrightarrow{DC} = -6 \Rightarrow \cos^{-1}\left(\dfrac{-6}{\sqrt{76}}\right) \approx 133.5^\circ$

$\vec{k}\cdot\overrightarrow{DC} = 6 \Rightarrow \cos^{-1}\left(\dfrac{6}{\sqrt{76}}\right) \approx 46.5^\circ$

15 Given $\overrightarrow{AD} = (2, 0, -6) \Rightarrow \overrightarrow{DA} = (-2, 0, 6)$; $\overrightarrow{BD} = (6, 0, -4) \Rightarrow \overrightarrow{DB} = (-6, 0, 4)$; $\overrightarrow{AC} = (0, -6, 0)$:

$\left(\overrightarrow{DA} - \overrightarrow{DB}\right)\cdot\overrightarrow{AC} = ((-2, 0, 6) - (-6, 0, 4))\cdot(0, -6, 0) = (4, 0, 2)\cdot(0, -6, 0) = 0$

16 $\cos\frac{\pi}{3} = \frac{1}{2}$

$$\cos\theta = \frac{\begin{pmatrix}3\\-k\\-1\end{pmatrix}\begin{pmatrix}1\\-3\\k\end{pmatrix}}{\sqrt{9+k^2+1}\sqrt{1+9+k^2}} = \frac{3+3k-k}{k^2+10}$$

Therefore: $\frac{1}{2} = \frac{3+2k}{k^2+10} \Rightarrow k^2+10 = 6+4k \Rightarrow k^2-4k+4=0 \Rightarrow k=2$

17 $\cos\frac{\pi}{3} = \frac{1}{2}$

$$\cos\theta = \frac{\begin{pmatrix}k\\1\\1\end{pmatrix}\begin{pmatrix}1\\k\\1\end{pmatrix}}{\sqrt{k^2+1+1}\sqrt{1+k^2+1}} = \frac{k+k+1}{k^2+2}$$

Therefore: $\frac{1}{2} = \frac{1+2k}{k^2+2} \Rightarrow k^2+2 = 2+4k \Rightarrow k^2-4k=0 \Rightarrow k=0, k=4$

18 $\begin{pmatrix}2\\x\\y\end{pmatrix}\cdot\begin{pmatrix}3\\1\\-1\end{pmatrix} = 6+x-y=0$

$\begin{pmatrix}2\\x\\y\end{pmatrix}\cdot\begin{pmatrix}4\\-1\\2\end{pmatrix} = 8-x+2y=0$

Hence, we have to solve the system of equations:

$$\begin{cases}6+x-y=0\\8-x+2y=0\end{cases}$$

Adding the equations: $14+y=0 \Rightarrow y=-14$ and $x=-20$

19 Two vectors are parallel if t exists such that $\mathbf{u} = t\mathbf{v}$. Therefore:

$$\begin{pmatrix}1-x\\2x-2\\3+x\end{pmatrix} = t\begin{pmatrix}2-x\\1+x\\1+x\end{pmatrix}\text{; and }\begin{cases}1-x=t(2-x)\\2x-2=t(1+x)\\3+x=t(1+x)\end{cases}$$

From the last two equations, we can see that $3+x=2x-2 \Rightarrow x=5$; hence, $3+5=t(1+5) \Rightarrow t=\frac{4}{3}$. Verifying using the first equation, we can see that $x=5$ is the solution.

20 $A\hat{B}C = \angle\overrightarrow{BA}, \overrightarrow{BC}$

So: $\overrightarrow{BA} = \overrightarrow{OA} - \overrightarrow{OB} = \begin{pmatrix}2\\3\\1\end{pmatrix} - \begin{pmatrix}3\\5\\4\end{pmatrix} = \begin{pmatrix}-1\\-2\\-3\end{pmatrix}$

$$\cos A\hat{B}C = \frac{\begin{pmatrix}-1\\-2\\-3\end{pmatrix}\begin{pmatrix}-1\\4\\0\end{pmatrix}}{\sqrt{1+4+9}\sqrt{1+16}} = \frac{1-8+0}{\sqrt{14\cdot17}} \Rightarrow A\hat{B}C = \cos^{-1}\frac{-7}{\sqrt{14\cdot17}} \approx 117^\circ$$

$$\overrightarrow{AC} = \overrightarrow{AB} + \overrightarrow{BC} = -\overrightarrow{BA} + \overrightarrow{BC} = \begin{pmatrix} 1 \\ 2 \\ 3 \end{pmatrix} + \begin{pmatrix} -1 \\ 4 \\ 0 \end{pmatrix} = \begin{pmatrix} 0 \\ 6 \\ 3 \end{pmatrix}$$

$$B\hat{A}C = \angle \overrightarrow{AB}, \overrightarrow{AC}$$

$$\cos B\hat{A}C = \frac{\begin{pmatrix} 1 \\ 2 \\ 3 \end{pmatrix}\begin{pmatrix} 0 \\ 6 \\ 3 \end{pmatrix}}{\sqrt{1+4+9}\sqrt{36+9}} = \frac{0+12+9}{\sqrt{14 \cdot 45}} \Rightarrow B\hat{A}C = \cos^{-1}\frac{21}{\sqrt{14 \cdot 45}} \approx 33^\circ$$

21 **a)** $(b, 3, 2) \cdot (1, b, 1) = b + 3b + 2 = 4b + 2 = 0$

Vectors are orthogonal if their dot product is zero; therefore: $4b + 2 = 0 \Rightarrow b = -\frac{1}{2}$

b) $(4, -2, 7) \cdot (b^2, b, 0) = 4b^2 - 2b + 7 \cdot 0 = 4b^2 - 2b$

Vectors are orthogonal if their dot product is zero; therefore: $4b^2 - 2b = 0 \Rightarrow b = 0, \frac{1}{2}$

For $b = 0$, the vector $(b^2, b, 0)$ is a zero vector. A zero vector has no direction; therefore, it is not orthogonal to any vector. Hence, the vectors are only orthogonal for $b = \frac{1}{2}$.

c) Vectors are orthogonal if their dot product is zero; therefore:

$2b^2 - 11b + 15 = 0 \Rightarrow b = \frac{5}{2}, b = 3$

d) Vectors are orthogonal if their dot product is zero; therefore:

$12 + 20 - 2b^2 = 0 \Rightarrow b^2 = 16 \Rightarrow b = \pm 4$

22 To determine the angle between two vectors, we are going to find their dot product: $(\mathbf{p} + \mathbf{q})(\mathbf{p} - \mathbf{q}) = \mathbf{p}^2 - \mathbf{q}^2$.

Since, for any vector: $\mathbf{v}^2 = |\mathbf{v}|\,|\mathbf{v}| \cos 0^\circ = |\mathbf{v}|^2 \cdot 1 = |\mathbf{v}|^2$, we have: $(\mathbf{p} + \mathbf{q})(\mathbf{p} - \mathbf{q}) = \mathbf{p}^2 - \mathbf{q}^2 = |\mathbf{p}|^2 - |\mathbf{q}|^2 = 0$; therefore, the vectors are perpendicular.

23 We can find the z-component by transforming 300 m/min into km/h:
300 m/min = 0.3 km/min = $0.3 \cdot 60$ km/h = 18 km/h

The velocity vector in the xy-plane is parallel to the vector $(-1, 1)$. The unit vector in this direction is $\frac{1}{\sqrt{2}}(-1, 1)$. Since the airspeed is 200 km/h, and its vertical component is 18 km/h, then the velocity of the xy-component is: $\sqrt{200^2 - 18^2} = \sqrt{39\,676} = 199.188... \approx 199$ km/h. So, the velocity vector in the xy-plane is: $\frac{\sqrt{39\,676}}{\sqrt{2}}(-1, 1) = \sqrt{19\,838}\,(-1, 1) \approx (-140.8, 140.8)$. Hence, the velocity vector is

$(-140.8, 140.8, 18)$.

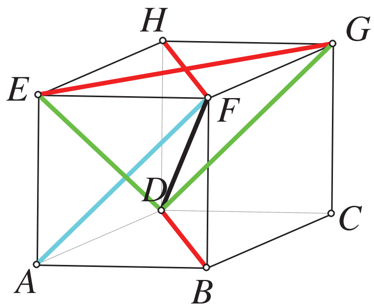
24 Vectors are perpendicular if their dot product is zero; therefore: $2t + 4t - 10 - t = 0 \Rightarrow t = 2$

25 Vectors are perpendicular if their dot product is zero; therefore: $t + 3t + 2 = 0 \Rightarrow t = -\frac{1}{2}$

26 Vectors are perpendicular if their dot product is zero; therefore: $4t^2 - 2t + 0 = 0 \Rightarrow t = 0, t = \frac{1}{2}$

Note: For $t = 0$, the second vector is a zero vector, but, according to the definition, a zero vector is perpendicular to all vectors.

27 Some of the diagonals are shown in the diagram below. Using the properties of symmetry, we can see that most of the angles are the same.



We will use the component form of the diagonals:

$$\overrightarrow{DF} = \begin{pmatrix} a \\ a \\ a \end{pmatrix}, \overrightarrow{AF} = \overrightarrow{DG} = \begin{pmatrix} 0 \\ a \\ a \end{pmatrix}, \overrightarrow{DE} = \overrightarrow{CF} = \begin{pmatrix} a \\ 0 \\ a \end{pmatrix}, \overrightarrow{DB} = \overrightarrow{HF} = \begin{pmatrix} a \\ a \\ 0 \end{pmatrix},$$

$$\overrightarrow{HA} = \overrightarrow{GB} = \begin{pmatrix} a \\ 0 \\ -a \end{pmatrix}, \overrightarrow{AC} = \overrightarrow{EG} = \begin{pmatrix} -a \\ a \\ 0 \end{pmatrix}, \overrightarrow{EB} = \overrightarrow{HC} = \begin{pmatrix} 0 \\ a \\ -a \end{pmatrix}$$

$$\cos\left(\overrightarrow{DF}, \overrightarrow{AF}\right) = \frac{2a^2}{\sqrt{3a^2}\sqrt{2a^2}} = \frac{2}{\sqrt{6}} \qquad \cos\left(\overrightarrow{DF}, \overrightarrow{HA}\right) = \frac{0}{\sqrt{3a^2}\sqrt{2a^2}} = 0$$

$$\cos\left(\overrightarrow{DF}, \overrightarrow{DE}\right) = \frac{2a^2}{\sqrt{3a^2}\sqrt{2a^2}} = \frac{2}{\sqrt{6}} \qquad \cos\left(\overrightarrow{DF}, \overrightarrow{AC}\right) = \frac{0}{\sqrt{3a^2}\sqrt{2a^2}} = 0$$

$$\cos\left(\overrightarrow{DF}, \overrightarrow{DB}\right) = \frac{2a^2}{\sqrt{3a^2}\sqrt{2a^2}} = \frac{2}{\sqrt{6}} \qquad \cos\left(\overrightarrow{DF}, \overrightarrow{EB}\right) = \frac{0}{\sqrt{3a^2}\sqrt{2a^2}} = 0$$

Hence, the angle is either 90° or $\cos^{-1}\left(\frac{2}{\sqrt{6}}\right)$.

28 To simplify the notation, denote the vector by $\mathbf{v} = |\mathbf{a}|\,\mathbf{b} + |\mathbf{b}|\,\mathbf{a}$. We will determine the angles between $\mathbf{v}$ and $\mathbf{a}$, and $\mathbf{v}$ and $\mathbf{b}$.

Using the fact that $\mathbf{a}^2 = |\mathbf{a}|^2$ and $\mathbf{b}^2 = |\mathbf{b}|^2$, we have:

$$\cos\alpha = \frac{(|\mathbf{a}|\,\mathbf{b} + |\mathbf{b}|\,\mathbf{a})\,\mathbf{a}}{|\mathbf{v}|\,|\mathbf{a}|} = \frac{|\mathbf{a}|\,\mathbf{ba} + |\mathbf{b}|\,\mathbf{a}^2}{|\mathbf{v}|\,|\mathbf{a}|} = \frac{\mathbf{ba}}{|\mathbf{v}|} + \frac{|\mathbf{b}|\,|\mathbf{a}|}{|\mathbf{v}|}$$

$$\cos\beta = \frac{(|\mathbf{a}|\,\mathbf{b} + |\mathbf{b}|\,\mathbf{a})\,\mathbf{b}}{|\mathbf{v}|\,|\mathbf{b}|} = \frac{|\mathbf{a}|\,\mathbf{b}^2 + |\mathbf{b}|\,\mathbf{ab}}{|\mathbf{v}|\,|\mathbf{b}|} = \frac{|\mathbf{b}|\,|\mathbf{a}|}{|\mathbf{v}|} + \frac{\mathbf{ab}}{|\mathbf{v}|}$$

Since the cosines of the angles are the same, and the angles are from 0° to 180°, then $\alpha = \beta$, and vector $\mathbf{v}$ is the angle bisector.

29 The scalar product should be 0, so: $2 - m + n = 0 \Rightarrow m = 2 + n$

From the magnitudes, we have: $\sqrt{1 + m^2 + 1} = \sqrt{4 + 1 + n^2} \Rightarrow m^2 = 3 + n^2$

Hence, $(2 + n)^2 = 3 + n^2 \Rightarrow 4n = -1 \Rightarrow n = -\frac{1}{4}$ and $m = 2 - \frac{1}{4} = \frac{7}{4}$

30 $\cos^2\frac{\pi}{4}+\cos^2\frac{\pi}{6}+\cos^2\frac{2\pi}{3}=\left(\frac{\sqrt{2}}{2}\right)^2+\left(\frac{\sqrt{3}}{2}\right)^2+\left(-\frac{1}{2}\right)^2=\frac{1}{2}+\frac{3}{4}+\frac{1}{4}=\frac{3}{2}\neq 1$; hence, they cannot be the direction angles of one vector.

31 $\cos^2\alpha+\cos^2\beta+\cos^2\gamma=\left(\frac{1}{2}\right)^2+\left(\frac{\sqrt{2}}{2}\right)^2+\cos^2\gamma=\frac{3}{4}+\cos^2\gamma=1\Rightarrow\cos\gamma=\pm\frac{1}{2}$

Hence, $\gamma=\frac{\pi}{3}$ or $\gamma=\frac{2\pi}{3}$.

32 If $\alpha=\beta=\gamma\Rightarrow 3\cos^2\alpha=1\Rightarrow\cos\alpha=\pm\sqrt{\frac{1}{3}}$

Hence, the angles are $\cos^{-1}\left(\pm\frac{\sqrt{3}}{3}\right)$.

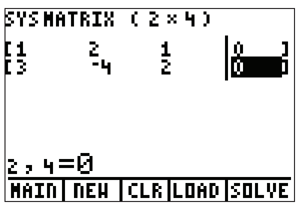
33 We can write $\mathbf{u}=\begin{pmatrix}|\mathbf{u}|\cos\alpha\\|\mathbf{u}|\cos\beta\\|\mathbf{u}|\cos\gamma\end{pmatrix}\Rightarrow-\mathbf{u}=\begin{pmatrix}|\mathbf{u}|(-\cos\alpha)\\|\mathbf{u}|(-\cos\beta)\\|\mathbf{u}|(-\cos\gamma)\end{pmatrix}=\begin{pmatrix}|\mathbf{u}|\cos(\pi-\alpha)\\|\mathbf{u}|\cos(\pi-\beta)\\|\mathbf{u}|\cos(\pi-\gamma)\end{pmatrix}$. Hence, the direction vectors are: $\pi-\alpha,\pi-\beta,\pi-\gamma$.

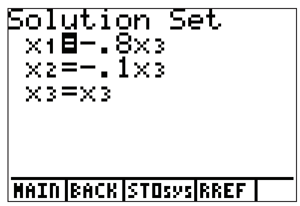
34 Let the vector be $\mathbf{u}=\begin{pmatrix}x\\y\\z\end{pmatrix}$. Then, $x^2+y^2+z^2=1$ and $\begin{cases}x+2y+z=0\\3x-4y+2z=0\end{cases}$.

Solution of the system 2 equation with two unknowns is: $x=-\frac{4}{5}z,\ y=-\frac{1}{10}z$; hence:

$\frac{16}{25}z^2+\frac{1}{100}z^2+z^2=1\Rightarrow z^2=\frac{165}{100}$. So, the vectors are:

$$\mathbf{u}=\pm\frac{\sqrt{165}}{10}\begin{pmatrix}-\frac{4}{5}\\-\frac{1}{10}\\1\end{pmatrix}=\pm\frac{\sqrt{165}}{100}\begin{pmatrix}-8\\-1\\10\end{pmatrix}$$





Exercise 14.3

1 a) $\mathbf{i}\times(\mathbf{i}+\mathbf{j}+\mathbf{k})=\begin{vmatrix}\mathbf{i}&\mathbf{j}&\mathbf{k}\\1&0&0\\1&1&1\end{vmatrix}=\begin{vmatrix}0&0\\1&1\end{vmatrix}\mathbf{i}-\begin{vmatrix}1&0\\1&1\end{vmatrix}\mathbf{j}+\begin{vmatrix}1&0\\1&1\end{vmatrix}\mathbf{k}=-\mathbf{j}+\mathbf{k}$

b) $\mathbf{i}\times\mathbf{i}+\mathbf{i}\times\mathbf{j}+\mathbf{i}\times\mathbf{k}=\begin{vmatrix}\mathbf{i}&\mathbf{j}&\mathbf{k}\\1&0&0\\1&0&0\end{vmatrix}+\underbrace{\begin{vmatrix}\mathbf{i}&\mathbf{j}&\mathbf{k}\\1&0&0\\0&1&0\end{vmatrix}}_{\begin{vmatrix}1&0\\1&1\end{vmatrix}\mathbf{k}}+\underbrace{\begin{vmatrix}\mathbf{i}&\mathbf{j}&\mathbf{k}\\1&0&0\\0&0&1\end{vmatrix}}_{\begin{vmatrix}1&0\\1&1\end{vmatrix}\mathbf{j}}=\vec{0}+\begin{vmatrix}1&0\\1&1\end{vmatrix}\mathbf{k}-\begin{vmatrix}1&0\\1&1\end{vmatrix}\mathbf{j}=\mathbf{k}-\mathbf{j}$

The results are the same.

2 a) $\mathbf{j}\times(\mathbf{i}+\mathbf{j}+\mathbf{k})=\begin{vmatrix}\mathbf{i}&\mathbf{j}&\mathbf{k}\\0&1&0\\1&1&1\end{vmatrix}=\begin{vmatrix}1&0\\1&1\end{vmatrix}\mathbf{i}-\begin{vmatrix}0&0\\1&1\end{vmatrix}\mathbf{j}+\begin{vmatrix}0&1\\1&1\end{vmatrix}\mathbf{k}=\mathbf{i}-\mathbf{k}$

b) $\mathbf{j}\times\mathbf{i}+\mathbf{j}\times\mathbf{j}+\mathbf{j}\times\mathbf{k}=\begin{vmatrix}\mathbf{i}&\mathbf{j}&\mathbf{k}\\0&1&0\\1&0&0\end{vmatrix}+\begin{vmatrix}\mathbf{i}&\mathbf{j}&\mathbf{k}\\0&1&0\\0&1&0\end{vmatrix}+\begin{vmatrix}\mathbf{i}&\mathbf{j}&\mathbf{k}\\0&1&0\\0&0&1\end{vmatrix}=\begin{vmatrix}0&1\\1&0\end{vmatrix}\mathbf{k}+\vec{0}+\begin{vmatrix}1&0\\0&1\end{vmatrix}\mathbf{i}=-\mathbf{k}+\mathbf{i}$

The results are the same.

3 a) $\mathbf{k}\times(\mathbf{i}+\mathbf{j}+\mathbf{k})=\begin{vmatrix}\mathbf{i}&\mathbf{j}&\mathbf{k}\\0&0&1\\1&1&1\end{vmatrix}=\begin{vmatrix}0&1\\1&1\end{vmatrix}\mathbf{i}-\begin{vmatrix}0&1\\1&1\end{vmatrix}\mathbf{j}+\begin{vmatrix}0&0\\1&1\end{vmatrix}\mathbf{k}=-\mathbf{i}+\mathbf{j}$

b) $\mathbf{k}\times\mathbf{i}+\mathbf{k}\times\mathbf{j}+\mathbf{k}\times\mathbf{k}=\begin{vmatrix}\mathbf{i}&\mathbf{j}&\mathbf{k}\\0&0&1\\1&0&0\end{vmatrix}+\begin{vmatrix}\mathbf{i}&\mathbf{j}&\mathbf{k}\\0&0&1\\0&1&0\end{vmatrix}+\begin{vmatrix}\mathbf{i}&\mathbf{j}&\mathbf{k}\\0&0&1\\0&0&1\end{vmatrix}=-\begin{vmatrix}0&1\\1&0\end{vmatrix}\mathbf{j}+\begin{vmatrix}0&1\\1&0\end{vmatrix}\mathbf{i}+\vec{0}=\mathbf{j}-\mathbf{i}$

The results are the same.

4 $\mathbf{u}\cdot(\mathbf{v}+\mathbf{w})=\begin{vmatrix}\mathbf{i}&\mathbf{j}&\mathbf{k}\\u_1&u_2&u_3\\v_1+w_1&v_2+w_2&v_3+w_3\end{vmatrix}$

$$=\begin{vmatrix}u_2&u_3\\v_2+w_2&v_3+w_3\end{vmatrix}\mathbf{i}-\begin{vmatrix}u_1&u_3\\v_1+w_1&v_3+w_3\end{vmatrix}\mathbf{j}+\begin{vmatrix}u_1&u_2\\v_1+w_1&v_2+w_2\end{vmatrix}\mathbf{k}$$

$$=\begin{pmatrix}u_2(v_3+w_3)-u_3(v_2+w_2)\\-u_1(v_3+w_3)+u_3(v_1+w_1)\\u_1(v_2+w_2)-u_2(v_1+w_1)\end{pmatrix}$$

$\mathbf{u}\cdot\mathbf{v}+\mathbf{u}\cdot\mathbf{w}=\begin{vmatrix}\mathbf{i}&\mathbf{j}&\mathbf{k}\\u_1&u_2&u_3\\v_1&v_2&v_3\end{vmatrix}+\begin{vmatrix}\mathbf{i}&\mathbf{j}&\mathbf{k}\\u_1&u_2&u_3\\w_1&w_2&w_3\end{vmatrix}$

$$=\begin{vmatrix}u_2&u_3\\v_2&v_3\end{vmatrix}\mathbf{i}-\begin{vmatrix}u_1&u_3\\v_1&v_3\end{vmatrix}\mathbf{j}+\begin{vmatrix}u_1&u_2\\v_1&v_2\end{vmatrix}\mathbf{k}+\begin{vmatrix}u_2&u_3\\w_2&w_3\end{vmatrix}\mathbf{i}-\begin{vmatrix}u_1&u_3\\w_1&w_3\end{vmatrix}\mathbf{j}+\begin{vmatrix}u_1&u_2\\w_1&w_2\end{vmatrix}\mathbf{k}$$

$$=\begin{pmatrix}u_2v_3-u_3v_2\\-u_1v_3+u_3v_1\\u_1v_2-u_2v_1\end{pmatrix}+\begin{pmatrix}u_2w_3-u_3w_2\\-u_1w_3+u_3w_1\\u_1w_2-u_2w_1\end{pmatrix}=\begin{pmatrix}u_2(v_3+w_3)-u_3(v_2+w_2)\\-u_1(v_3+w_3)+u_3(v_1+w_1)\\u_1(v_2+w_2)-u_2(v_1+w_1)\end{pmatrix}$$

Hence, $\mathbf{u}\cdot(\mathbf{v}+\mathbf{w})=\mathbf{u}\cdot(\mathbf{v}+\mathbf{w})$.

$\mathbf{u}\times(\mathbf{v}-\mathbf{w})=\begin{vmatrix}\mathbf{i}&\mathbf{j}&\mathbf{k}\\u_1&u_2&u_3\\v_1-w_1&v_2-w_2&v_3-w_3\end{vmatrix}$

$$=\begin{vmatrix}u_2&u_3\\v_2-w_2&v_3-w_3\end{vmatrix}\mathbf{i}-\begin{vmatrix}u_1&u_3\\v_1-w_1&v_3-w_3\end{vmatrix}\mathbf{j}+\begin{vmatrix}u_1&u_2\\v_1-w_1&v_2-w_2\end{vmatrix}\mathbf{k}$$

$$= \begin{pmatrix} u_2(v_3 - w_3) - u_3(v_2 - w_2) \\ -u_1(v_3 - w_3) + u_3(v_1 - w_1) \\ u_1(v_2 - w_2) - u_2(v_1 - w_1) \end{pmatrix}$$

$$\mathbf{u} \times \mathbf{v} - \mathbf{u} \times \mathbf{w} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ u_1 & u_2 & u_3 \\ v_1 & v_2 & v_3 \end{vmatrix} - \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ u_1 & u_2 & u_3 \\ w_1 & w_2 & w_3 \end{vmatrix}$$

$$= \begin{vmatrix} u_2 & u_3 \\ v_2 & v_3 \end{vmatrix}\mathbf{i} - \begin{vmatrix} u_1 & u_3 \\ v_1 & v_3 \end{vmatrix}\mathbf{j} + \begin{vmatrix} u_1 & u_2 \\ v_1 & v_2 \end{vmatrix}\mathbf{k} - \left(\begin{vmatrix} u_2 & u_3 \\ w_2 & w_3 \end{vmatrix}\mathbf{i} - \begin{vmatrix} u_1 & u_3 \\ w_1 & w_3 \end{vmatrix}\mathbf{j} + \begin{vmatrix} u_1 & u_2 \\ w_1 & w_2 \end{vmatrix}\mathbf{k} \right)$$

$$= \begin{pmatrix} u_2v_3 - u_3v_2 \\ -u_1v_3 + u_3v_1 \\ u_1v_2 - u_2v_1 \end{pmatrix} - \begin{pmatrix} u_2w_3 - u_3w_2 \\ -u_1w_3 + u_3w_1 \\ u_1w_2 - u_2w_1 \end{pmatrix} = \begin{pmatrix} u_2(v_3 - w_3) - u_3(v_2 - w_2) \\ -u_1(v_3 - w_3) + u_3(v_1 - w_1) \\ u_1(v_2 - w_2) - u_2(v_1 - w_1) \end{pmatrix}$$

Hence, $\mathbf{u} \times (\mathbf{v} - \mathbf{w}) = \mathbf{u} \times (\mathbf{v} - \mathbf{w})$.

5 $\begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ 2 & 3 & -2 \\ -3 & 2 & 3 \end{vmatrix} = \begin{vmatrix} 3 & -2 \\ 2 & 3 \end{vmatrix}\mathbf{i} - \begin{vmatrix} 2 & -2 \\ -3 & 3 \end{vmatrix}\mathbf{j} + \begin{vmatrix} 2 & 3 \\ -3 & 2 \end{vmatrix}\mathbf{k} = 13\mathbf{i} + 13\mathbf{k}$

$(13\mathbf{i} + 13\mathbf{k})(2\mathbf{i} + 3\mathbf{j} - 2\mathbf{k}) = 26 + 0 - 26 = 0$

$(13\mathbf{i} + 13\mathbf{k})(-3\mathbf{i} + 2\mathbf{j} + 3\mathbf{k}) = -39 + 0 + 39 = 0$

6 $\begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ 4 & 3 & 0 \\ 0 & -2 & 2 \end{vmatrix} = \begin{vmatrix} 3 & 0 \\ -2 & 2 \end{vmatrix}\mathbf{i} - \begin{vmatrix} 4 & 0 \\ 0 & 2 \end{vmatrix}\mathbf{j} + \begin{vmatrix} 4 & 3 \\ 0 & 2 \end{vmatrix}\mathbf{k} = 6\mathbf{i} - 8\mathbf{j} - 8\mathbf{k}$

$(6\mathbf{i} - 8\mathbf{j} - 8\mathbf{k})(4\mathbf{i} + 3\mathbf{j}) = 24 - 24 + 0 = 0$

$(6\mathbf{i} - 8\mathbf{j} - 8\mathbf{k})(-2\mathbf{j} + 2\mathbf{k}) = 0 + 16 - 16 = 0$

7 $\begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ 1 & 2 & -1 \\ 4 & 1 & -3 \end{vmatrix} = \begin{vmatrix} 2 & -1 \\ 1 & -3 \end{vmatrix}\mathbf{i} - \begin{vmatrix} 1 & -1 \\ 4 & -3 \end{vmatrix}\mathbf{j} + \begin{vmatrix} 1 & 2 \\ 4 & 1 \end{vmatrix}\mathbf{k} = \begin{pmatrix} -5 \\ -1 \\ -7 \end{pmatrix}$

$\begin{pmatrix} -5 \\ -1 \\ -7 \end{pmatrix}\begin{pmatrix} 1 \\ 2 \\ -1 \end{pmatrix} = -5 - 2 + 7 = 0$

$\begin{pmatrix} -5 \\ -1 \\ -7 \end{pmatrix}\begin{pmatrix} 4 \\ 1 \\ -3 \end{pmatrix} = -20 - 1 + 21 = 0$

8 $\begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ 5 & 1 & 2 \\ 3 & 0 & 1 \end{vmatrix} = \begin{vmatrix} 1 & 2 \\ 0 & 1 \end{vmatrix}\mathbf{i} - \begin{vmatrix} 5 & 2 \\ 3 & 1 \end{vmatrix}\mathbf{j} + \begin{vmatrix} 5 & 1 \\ 3 & 0 \end{vmatrix}\mathbf{k} = \mathbf{i} + \mathbf{j} - 3\mathbf{k}$

$(\mathbf{i} + \mathbf{j} - 3\mathbf{k})(5\mathbf{i} + \mathbf{j} + 2\mathbf{k}) = 5 + 1 - 6 = 0$

$(\mathbf{i} + \mathbf{j} - 3\mathbf{k})(3\mathbf{i} + \mathbf{k}) = 3 + 0 - 3 = 0$

9 **a)** $\begin{pmatrix}2\\1\\m\end{pmatrix}\begin{vmatrix}\mathbf{i}&\mathbf{j}&\mathbf{k}\\3&2&3\\m&2&1\end{vmatrix}=\begin{pmatrix}2\\1\\m\end{pmatrix}\left(\begin{vmatrix}2&3\\2&1\end{vmatrix}\mathbf{i}-\begin{vmatrix}3&3\\m&1\end{vmatrix}\mathbf{j}+\begin{vmatrix}3&2\\m&2\end{vmatrix}\mathbf{k}\right)=\begin{pmatrix}2\\1\\m\end{pmatrix}\begin{pmatrix}-4\\-3+3m\\6-2m\end{pmatrix}$

$$=-8-3+3m+6m-2m^2=-2m^2+9m-11$$

b) $\begin{pmatrix}m\\2\\1\end{pmatrix}\begin{vmatrix}\mathbf{i}&\mathbf{j}&\mathbf{k}\\2&1&m\\3&2&3\end{vmatrix}=\begin{pmatrix}m\\2\\1\end{pmatrix}\left(\begin{vmatrix}1&m\\2&3\end{vmatrix}\mathbf{i}-\begin{vmatrix}2&m\\3&3\end{vmatrix}\mathbf{j}+\begin{vmatrix}2&1\\3&2\end{vmatrix}\mathbf{k}\right)=\begin{pmatrix}m\\2\\1\end{pmatrix}\begin{pmatrix}3-2m\\-6+3m\\1\end{pmatrix}$

$$=3m-2m^2-12+6m+1=-2m^2+9m-11$$

c) $\begin{pmatrix}3\\2\\3\end{pmatrix}\begin{vmatrix}\mathbf{i}&\mathbf{j}&\mathbf{k}\\m&2&1\\2&1&m\end{vmatrix}=\begin{pmatrix}3\\2\\3\end{pmatrix}\left(\begin{vmatrix}2&1\\1&m\end{vmatrix}\mathbf{i}-\begin{vmatrix}m&1\\2&m\end{vmatrix}\mathbf{j}+\begin{vmatrix}m&2\\2&1\end{vmatrix}\mathbf{k}\right)=\begin{pmatrix}3\\2\\3\end{pmatrix}\begin{pmatrix}2m-1\\-m^2+2\\m-4\end{pmatrix}$

$$=6m-3-2m^2+4+3m-12=-2m^2+9m-11$$

10 **a)** $\mathbf{u}\times(\mathbf{v}\times\mathbf{w})=\begin{pmatrix}3\\0\\4\end{pmatrix}\times\begin{pmatrix}-28\\10\\1\end{pmatrix}=\begin{pmatrix}-40\\-115\\30\end{pmatrix}$

b) $(\mathbf{u}\times\mathbf{v})\times\mathbf{w}=\begin{pmatrix}-8\\-20\\6\end{pmatrix}\times\begin{pmatrix}2\\5\\6\end{pmatrix}=\begin{pmatrix}-150\\60\\0\end{pmatrix}$

c) $(\mathbf{u}\times\mathbf{v})\times(\mathbf{v}\times\mathbf{w})=\begin{pmatrix}-8\\-20\\6\end{pmatrix}\times\begin{pmatrix}-28\\10\\1\end{pmatrix}=\begin{pmatrix}-80\\-160\\-640\end{pmatrix}$

d) $(\mathbf{v}\times\mathbf{w})\times(\mathbf{u}\times\mathbf{v})=\begin{pmatrix}-28\\10\\1\end{pmatrix}\times\begin{pmatrix}-8\\-20\\6\end{pmatrix}=\begin{pmatrix}80\\160\\640\end{pmatrix}$

e) $(\mathbf{uw})\,\mathbf{v}-(\mathbf{uv})\,\mathbf{w}=(6+0+24)\begin{pmatrix}1\\2\\8\end{pmatrix}-(3+0+32)\begin{pmatrix}2\\5\\6\end{pmatrix}=\begin{pmatrix}30\\60\\240\end{pmatrix}-\begin{pmatrix}70\\175\\210\end{pmatrix}=\begin{pmatrix}-40\\-115\\30\end{pmatrix}$

f) $(\mathbf{wu})\,\mathbf{v}-(\mathbf{wv})\,\mathbf{u}=(6+0+24)\begin{pmatrix}1\\2\\8\end{pmatrix}-(2+10+48)\begin{pmatrix}3\\0\\4\end{pmatrix}=\begin{pmatrix}30\\60\\240\end{pmatrix}-\begin{pmatrix}180\\0\\240\end{pmatrix}=\begin{pmatrix}-150\\60\\0\end{pmatrix}$

11 $\mathbf{u}\times\mathbf{v}=\begin{pmatrix}-6\\4\\1\end{pmatrix}\times\begin{pmatrix}3\\1\\5\end{pmatrix}=\begin{pmatrix}19\\33\\-18\end{pmatrix}$, $|\mathbf{u}\cdot\ \mathbf{v}|=\sqrt{19^2+33^2+18^2}=\sqrt{1774}$

There are two unit vectors: $\pm\dfrac{\sqrt{1774}}{1774}\begin{pmatrix}19\\33\\-18\end{pmatrix}$.

12 We need a vector perpendicular to both $\overrightarrow{AB}$ and $\overrightarrow{AC}$.

$$\overrightarrow{AB} \times \overrightarrow{AC} = \begin{pmatrix} 1 \\ 1 \\ -3 \end{pmatrix} \times \begin{pmatrix} -1 \\ 3 \\ -1 \end{pmatrix} = \begin{pmatrix} 8 \\ 4 \\ 4 \end{pmatrix}$$

Vector $\overrightarrow{AB} \times \overrightarrow{AC}$ is parallel to $\begin{pmatrix} 2 \\ 1 \\ 1 \end{pmatrix}$ whose magnitude is $\sqrt{4+1+1} = \sqrt{6}$.

There are two unit vectors: $\pm \frac{\sqrt{6}}{6}\begin{pmatrix} 2 \\ 1 \\ 1 \end{pmatrix}$.

13 $\mathbf{u} \times \mathbf{v} = \begin{pmatrix} 2 \\ 0 \\ 3 \end{pmatrix} \times \begin{pmatrix} 1 \\ 4 \\ 2 \end{pmatrix} = \begin{pmatrix} -12 \\ -1 \\ 8 \end{pmatrix}$, $A = |\mathbf{u} \cdot \ \mathbf{v}| = \sqrt{12^2 + 1^2 + 8^2} = \sqrt{209}$

14 $\mathbf{u} \times \mathbf{v} = \begin{pmatrix} 3 \\ 4 \\ 1 \end{pmatrix} \times \begin{pmatrix} 0 \\ 3 \\ -1 \end{pmatrix} = \begin{pmatrix} -7 \\ 3 \\ 9 \end{pmatrix}$, $A = |\mathbf{u} \cdot \ \mathbf{v}| = \sqrt{7^2 + 3^2 + 9^2} = \sqrt{139}$

15 We will introduce notation for the points: $A(2, -1, 1)$, $B(5, 1, 4)$, $C(0, 1, 1)$, $D(3, 3, 4)$.

Vectors $\overrightarrow{AB} = \begin{pmatrix} 5-2 \\ 1+1 \\ 4-1 \end{pmatrix} = \begin{pmatrix} 3 \\ 2 \\ 3 \end{pmatrix}$ and $\overrightarrow{DC} = \begin{pmatrix} 3-0 \\ 3-1 \\ 4-1 \end{pmatrix} = \begin{pmatrix} 3 \\ 2 \\ 3 \end{pmatrix}$ are the same; hence, $ABCD$ is a parallelogram. Since $A = |\overrightarrow{AB} \cdot \ \overrightarrow{AC}|$, we have to find the vector product of those vectors.

$$\overrightarrow{AB} \times \overrightarrow{AC} = \begin{pmatrix} 3 \\ 2 \\ 3 \end{pmatrix} \times \begin{pmatrix} -2 \\ 2 \\ 0 \end{pmatrix} = \begin{pmatrix} -6 \\ -6 \\ 10 \end{pmatrix}$$

$$A = |\overrightarrow{AB} \cdot \ \overrightarrow{AC}| = \sqrt{36 + 36 + 100} = \sqrt{172} = 2\sqrt{43}$$

16 The points are coplanar if the vectors $\overrightarrow{PQ}$, $\overrightarrow{PR}$, $\overrightarrow{PS}$ are coplanar. That means their scalar triple product must be zero.

$$\overrightarrow{PQ} = \begin{pmatrix} 1 \\ 1 \\ -1 \end{pmatrix}, \overrightarrow{PR} = \begin{pmatrix} 2 \\ 3 \\ -2 \end{pmatrix}, \overrightarrow{PS} = \begin{pmatrix} 4 \\ 5 \\ -4 \end{pmatrix}$$

$$\begin{vmatrix} 1 & 1 & -1 \\ 2 & 3 & -2 \\ 4 & 5 & -4 \end{vmatrix} = 0$$

17 We will find the scalar triple product of the vectors:

$$\overrightarrow{AB} = \begin{pmatrix} 3-m \\ 1 \\ m+2 \end{pmatrix}, \overrightarrow{AC} = \begin{pmatrix} 2-m \\ -3 \\ 0 \end{pmatrix}, \overrightarrow{AD} = \begin{pmatrix} 4-m \\ 5 \\ 6 \end{pmatrix}$$

$$\begin{vmatrix} 3-m & 1 & m+2 \\ 2-m & -3 & 0 \\ 4-m & 5 & 6 \end{vmatrix} = 0 \Rightarrow (3-m)\cdot(-18) - (2-m)6 + (m+2)(10 - 5m + 12 - 3m) = 0 \Rightarrow$$

$$-8m^2 + 30m - 22 = 0 \Rightarrow m_1 = 1, m_2 = \frac{11}{4}$$

18 The area of the triangle is half the area of the parallelogram formed with $\overrightarrow{AB}$ and $\overrightarrow{AC}$.

$$\overrightarrow{AB} \times \overrightarrow{AC} = \begin{pmatrix} -1 \\ -5 \\ 2 \end{pmatrix} \times \begin{pmatrix} 1 \\ -1 \\ 3 \end{pmatrix} = \begin{pmatrix} -13 \\ 5 \\ 6 \end{pmatrix}$$

$$A = \frac{1}{2}\left|\overrightarrow{AB} \cdot \ \overrightarrow{AC}\right| = \frac{1}{2}\sqrt{169 + 25 + 36} = \frac{\sqrt{230}}{2}$$

19 The area of the triangle is half the area of the parallelogram formed with $\overrightarrow{AB}$ and $\overrightarrow{AC}$.

$$\overrightarrow{AB} \times \overrightarrow{AC} = \begin{pmatrix} -1 \\ 4 \\ 8 \end{pmatrix} \times \begin{pmatrix} 3 \\ 0 \\ 10 \end{pmatrix} = \begin{pmatrix} 40 \\ 34 \\ -12 \end{pmatrix}$$

$$A = \frac{1}{2}\left|\overrightarrow{AB} \cdot \ \overrightarrow{AC}\right| = \frac{1}{2}\sqrt{40^2 + 34^2 + 12^2} = \frac{\sqrt{2900}}{2} = 5\sqrt{29}$$

20 $\begin{vmatrix} 3 & -2 & 2 \\ 5 & 2 & -2 \\ 1 & 2 & 6 \end{vmatrix} = 3\begin{vmatrix} 2 & -2 \\ 2 & 6 \end{vmatrix} + 2\begin{vmatrix} 5 & -2 \\ 1 & 6 \end{vmatrix} + 2\begin{vmatrix} 5 & 2 \\ 1 & 2 \end{vmatrix} = 3(16) + 2(32) + 2(8) = 128$

21 $\begin{vmatrix} 2 & -1 & 3 \\ 1 & 4 & 3 \\ -3 & 2 & -2 \end{vmatrix} = 2\begin{vmatrix} 4 & 3 \\ 2 & -2 \end{vmatrix} + \begin{vmatrix} 1 & 3 \\ -3 & -3 \end{vmatrix} + 3\begin{vmatrix} 1 & 4 \\ -5 & 2 \end{vmatrix} = 21$

22 $\begin{vmatrix} 3 & 2 & 1 \\ 1 & -3 & 1 \\ 5 & 1 & 2 \end{vmatrix} = 3\begin{vmatrix} -3 & 1 \\ 1 & 2 \end{vmatrix} - 2\begin{vmatrix} 1 & 1 \\ 5 & 2 \end{vmatrix} + \begin{vmatrix} 1 & -3 \\ 5 & 1 \end{vmatrix} = 1$

In questions 23–24, since $V = |\mathbf{u}\,(\mathbf{v} \cdot \ \mathbf{w})|$, we have to find the scalar triple product.

23 $\begin{vmatrix} 3 & -5 & 3 \\ 1 & 5 & -1 \\ 3 & 2 & -3 \end{vmatrix} = 3\begin{vmatrix} 5 & -1 \\ 2 & -3 \end{vmatrix} + 5\begin{vmatrix} 1 & -1 \\ 3 & -3 \end{vmatrix} + 3\begin{vmatrix} 1 & 5 \\ 3 & 2 \end{vmatrix} = -78$

So, $V = 78$.

24 $\begin{vmatrix} 4 & 2 & 3 \\ 5 & 6 & 2 \\ 2 & 3 & 5 \end{vmatrix} = 63$

So, $V = 63$.

In questions 25–26, we will determine whether the vectors are coplanar by checking if their scalar triple product is zero.

25 $\begin{vmatrix} 2 & -1 & 2 \\ 4 & 1 & -1 \\ 6 & -3 & 1 \end{vmatrix} = -30 \neq 0$

So, they are not coplanar.

26 $\begin{vmatrix} 4 & -2 & -1 \\ 9 & -6 & -1 \\ 6 & -6 & 1 \end{vmatrix} = 0$

So, they are coplanar.

27 Vectors are coplanar if their scalar triple product is zero.

$$\begin{vmatrix} 1 & m & 1 \\ 3 & 0 & m \\ 5 & -4 & 0 \end{vmatrix} = \begin{vmatrix} 0 & m \\ -4 & 0 \end{vmatrix} - m\begin{vmatrix} 3 & m \\ 5 & 0 \end{vmatrix} + \begin{vmatrix} 3 & 0 \\ 5 & -4 \end{vmatrix} = 4m - m(-5m) + (-12) = 5m^2 + 4m - 12$$

Hence, $5m^2 + 4m - 12 = 0 \Rightarrow m_1 = -2, m_2 = \frac{6}{5}$

28 Vectors are coplanar if their scalar triple product is zero.

$$\begin{vmatrix} 2 & -3 & 2m \\ m & -3 & 1 \\ 1 & 3 & -2 \end{vmatrix} = 2\begin{vmatrix} -3 & 1 \\ 3 & -2 \end{vmatrix} + 3\begin{vmatrix} m & 1 \\ 1 & -2 \end{vmatrix} + 2m\begin{vmatrix} m & -3 \\ 1 & 3 \end{vmatrix} = 2(3) + 3(-2m-1) + 2m(3m+3) = 6m^2 + 3$$

Hence, $6m^2 + 3 = 0$ has no solution, so they cannot be coplanar.

29 **a)** Since $V = |\mathbf{u}(\mathbf{v} \cdot \ \mathbf{w})|$, we have to find the scalar triple product:

$$\begin{vmatrix} 1 & 4 & 2 \\ -3 & 2 & 1 \\ -1 & 1 & 4 \end{vmatrix} = 49$$

So, $V = 49$.

b) $\mathbf{u} \times \mathbf{v} = \begin{pmatrix} -3 \\ 2 \\ 1 \end{pmatrix} \times \begin{pmatrix} 1 \\ 4 \\ 2 \end{pmatrix} = \begin{pmatrix} 0 \\ 7 \\ -14 \end{pmatrix}$

The area is: $A = |\mathbf{u} \cdot \ \mathbf{v}| = \sqrt{0 + 7^2 + 14^2} = \sqrt{245} = 7\sqrt{5}$

c) Since $V = Bh \Rightarrow h = \frac{V}{B} = \frac{49}{7\sqrt{5}} = \frac{7\sqrt{5}}{5}$

d) $\cos(\mathbf{w}, \mathbf{u} \times \mathbf{v}) = \dfrac{\begin{pmatrix} -1 \\ 1 \\ 4 \end{pmatrix}\begin{pmatrix} 0 \\ 7 \\ -14 \end{pmatrix}}{\sqrt{18} \cdot 7\sqrt{5}} = \dfrac{-49}{21\sqrt{10}} = -\dfrac{7\sqrt{10}}{30}$

So, the acute angle between the vector **w** and a vector perpendicular to the plane determined by **u** and **v** is $\cos^{-1}\left(\frac{7\sqrt{10}}{30}\right)$. Hence, the angle between **w** and the plane is $90^\circ - \cos^{-1}\left(\frac{7\sqrt{10}}{30}\right)$.

30 **a)** $V_{\text{tetrahedron}} = \frac{1}{3}(7\sqrt{5})\left(\frac{7\sqrt{5}}{5}\right) = \frac{49}{3} = \frac{1}{3}V_{\text{parallelepiped}}$

Hence, $V_{\text{tetrahedron}} = \frac{1}{3}|\mathbf{u}(\mathbf{v} \cdot \ \mathbf{w})|$.

b) $\overrightarrow{AB} = \begin{pmatrix} 3 \\ -1 \\ -3 \end{pmatrix}, \overrightarrow{AC} = \begin{pmatrix} 2 \\ -2 \\ 1 \end{pmatrix}, \overrightarrow{AD} = \begin{pmatrix} 4 \\ -4 \\ 3 \end{pmatrix}$

$$\begin{vmatrix} 3 & -1 & -3 \\ 2 & -2 & 1 \\ 4 & -4 & 3 \end{vmatrix} = -4$$

Hence, $V = \frac{4}{3}$.

31 From the definitions, we have: $\mathbf{uv} = |\mathbf{u}|\,|\mathbf{v}|\cos\theta$, $|\mathbf{u}\times\mathbf{v}| = |\mathbf{u}|\,|\mathbf{v}|\sin\theta$, so:

$|\mathbf{u}|\,|\mathbf{v}|\cos\theta = |\mathbf{u}|\,|\mathbf{v}|\sin\theta \Rightarrow \cos\theta = \sin\theta$. Since $0 \leqslant \theta \leqslant 180^\circ$, it follows that $\theta = 45^\circ$.

32 We will transform the right side of the formula:

$$\sqrt{|\mathbf{u}|^2|\mathbf{v}|^2-(\mathbf{uv})^2} = \sqrt{|\mathbf{u}|^2|\mathbf{v}|^2-|\mathbf{u}|^2|\mathbf{v}|^2\cos^2\theta} = \sqrt{|\mathbf{u}|^2|\mathbf{v}|^2\left(1-\cos^2\theta\right)}$$
$$= \sqrt{|\mathbf{u}|^2|\mathbf{v}|^2\sin^2\theta} = |\mathbf{u}|\,|\mathbf{v}|\sin\theta = |\mathbf{u}\times\mathbf{v}|$$

33 We will transform the right side of the formula:

$$\frac{\left|\overrightarrow{AP}\times\overrightarrow{AB}\right|}{\left|\overrightarrow{AB}\right|} = \frac{\left|\overrightarrow{AP}\right|\left|\overrightarrow{AB}\right|\sin\theta}{\left|\overrightarrow{AB}\right|} = \left|\overrightarrow{AP}\right|\sin\theta = d$$

34 **a)** In this case, the distance will be: $d = \dfrac{\left|\overrightarrow{BA}\cdot\overrightarrow{BC}\right|}{\left|\overrightarrow{BC}\right|}$.

$$\overrightarrow{BA}\times\overrightarrow{BC} = \begin{pmatrix}-4\\0\\2\end{pmatrix}\times\begin{pmatrix}-3\\2\\-4\end{pmatrix} = \begin{pmatrix}-4\\-22\\-8\end{pmatrix}$$

Hence, $d = \dfrac{\sqrt{4^2+22^2+8^2}}{\sqrt{3^2+2^2+4^2}} = \dfrac{\sqrt{564}}{\sqrt{29}} = \sqrt{\dfrac{564}{29}}$.

b) In this case, the distance will be: $d = \dfrac{\left|\overrightarrow{BA}\cdot\overrightarrow{BC}\right|}{\left|\overrightarrow{BC}\right|}$.

$$\overrightarrow{BA}\times\overrightarrow{BC} = \begin{pmatrix}2\\2\\0\end{pmatrix}\times\begin{pmatrix}-2\\1\\0\end{pmatrix} = \begin{pmatrix}0\\0\\6\end{pmatrix}$$

Hence, $d = \dfrac{6}{\sqrt{2^2+1^2}} = \dfrac{6}{\sqrt{5}} = \dfrac{6\sqrt{5}}{5}$.

c) In this case, the distance will be: $d = \dfrac{\left|\overrightarrow{BA}\cdot\overrightarrow{BC}\right|}{\left|\overrightarrow{BC}\right|}$.

$$\overrightarrow{BA}\times\overrightarrow{BC} = \begin{pmatrix}1\\2\\-1\end{pmatrix}\times\begin{pmatrix}2\\2\\0\end{pmatrix} = \begin{pmatrix}2\\-2\\-2\end{pmatrix}$$

Hence, $d = \dfrac{\sqrt{2^2+2^2+2^2}}{\sqrt{2^2+2^2}} = \dfrac{\sqrt{12}}{\sqrt{8}} = \sqrt{\dfrac{3}{2}}$.

In questions 35–37, we will use the distributive property of the vector product and the fact that the vector product of collinear vectors is a zero vector.

35 $(\mathbf{u}+\mathbf{v})\times(\mathbf{v}-\mathbf{u}) = \mathbf{u}\times\mathbf{v}-\mathbf{u}\times\mathbf{u}+\mathbf{v}\times\mathbf{v}-\mathbf{v}\times\mathbf{u} = \mathbf{u}\times\mathbf{v}-\vec{0}+\vec{0}+\mathbf{u}\times\mathbf{v} = 2(\mathbf{u}\times\mathbf{v})$

36 $(2\mathbf{u}+3\mathbf{v})\times(4\mathbf{v}-5\mathbf{u}) = 8\mathbf{u}\times\mathbf{v}-10\mathbf{u}\times\mathbf{u}+12\mathbf{v}\times\mathbf{v}-15\mathbf{v}\times\mathbf{u}$

$$= 8\mathbf{u}\times\mathbf{v}-\vec{0}+\vec{0}+15\mathbf{u}\times\mathbf{v} = 23(\mathbf{u}\times\mathbf{v})$$

37 $(m\mathbf{u}+n\mathbf{v})\times(p\mathbf{v}-q\mathbf{u}) = mp\mathbf{u}\times\mathbf{v}-mq\mathbf{u}\times\mathbf{u}+np\mathbf{v}\times\mathbf{v}-nq\mathbf{v}\times\mathbf{u}$

$$= mp\mathbf{u}\times\mathbf{v}-\vec{0}+\vec{0}+nq\mathbf{u}\times\mathbf{v} = (mp+nq)\,\mathbf{u}\times\mathbf{v}$$

38 a) $\overrightarrow{AB}\times\overrightarrow{AC}=\begin{pmatrix}-a\\b\\0\end{pmatrix}\times\begin{pmatrix}-a\\0\\c\end{pmatrix}=\begin{vmatrix}\mathbf{i}&\mathbf{j}&\mathbf{k}\\-a&b&0\\-a&0&c\end{vmatrix}=\begin{pmatrix}bc\\ac\\ab\end{pmatrix}$

Hence, $o=\frac{1}{2}\left|\overrightarrow{AB}\cdot\ \overrightarrow{AC}\right|=\frac{1}{2}\sqrt{b^2c^2+a^2c^2+a^2b^2}$.

b) The faces are right triangles, so we can find the area using the half product of the legs:

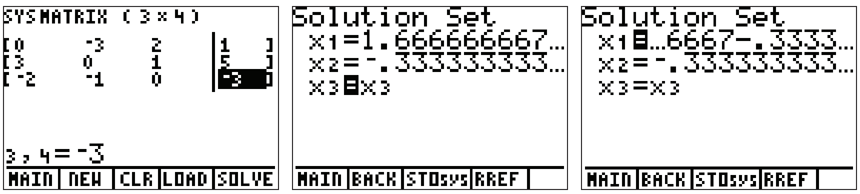
$A_1=\frac{ab}{2}, A_2=\frac{ac}{2}, A_3=\frac{bc}{2}$

c) Hence,

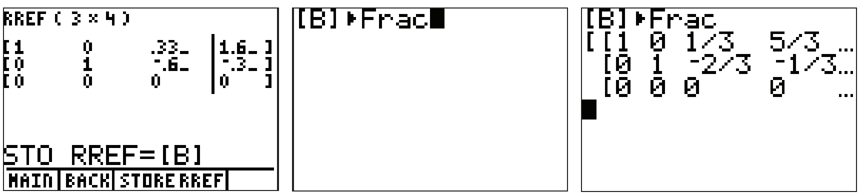
$$(A_1)^2+(A_2)^2+(A_3)^2=\frac{a^2b^2}{4}+\frac{a^2c^2}{4}+\frac{b^2c^2}{4}=\frac{a^2b^2+a^2c^2+b^2c^2}{4}=\left(\frac{\sqrt{a^2b^2+a^2c^2+b^2c^2}}{2}\right)^2=o^2$$

39 $\begin{vmatrix}\mathbf{i}&\mathbf{j}&\mathbf{k}\\-1&2&3\\x&y&z\end{vmatrix}=\mathbf{i}\begin{vmatrix}2&3\\y&z\end{vmatrix}-\mathbf{j}\begin{vmatrix}-1&3\\x&z\end{vmatrix}+\mathbf{k}\begin{vmatrix}-1&2\\x&y\end{vmatrix}=\begin{pmatrix}2z-3y\\z+3x\\-y-2x\end{pmatrix}$

So, $\begin{pmatrix}2z-3y\\z+3x\\-y-2x\end{pmatrix}=\begin{pmatrix}1\\5\\-3\end{pmatrix}\Rightarrow\begin{cases}-3y+2z=1\\3x+z=5\\-2x-y=-3\end{cases}\Rightarrow x=\frac{5}{3}-\frac{1}{3}z,\ y=-\frac{1}{3}+\frac{2}{3}z,\ z=z;\ \mathbf{v}=\begin{pmatrix}\frac{5}{3}-t\\-\frac{1}{3}+2t\\3t\end{pmatrix}, t\in\mathbb{R}$

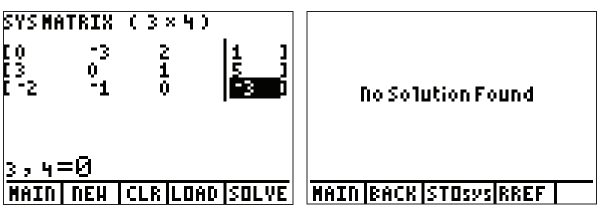


To achieve a better result, we can store the rref in matrix form, and then interpret the result.



40 $\begin{vmatrix}\mathbf{i}&\mathbf{j}&\mathbf{k}\\-1&2&3\\x&y&z\end{vmatrix}=\mathbf{i}\begin{vmatrix}2&3\\y&z\end{vmatrix}-\mathbf{j}\begin{vmatrix}-1&3\\x&z\end{vmatrix}+\mathbf{k}\begin{vmatrix}-1&2\\x&y\end{vmatrix}=\begin{pmatrix}2z-3y\\z+3x\\-y-2x\end{pmatrix}$

So, $\begin{pmatrix}2z-3y\\z+3x\\-y-2x\end{pmatrix}=\begin{pmatrix}1\\5\\0\end{pmatrix}\Rightarrow\begin{cases}-3y+2z=1\\3x+z=5\\-2x-y=0\end{cases}$ $\Rightarrow$ no solutions. Hence, there is no such vector.



Exercise 14.4

1 **a)** In the equation $\mathbf{r} = \mathbf{r}_0 + t\mathbf{u}$, vector $\mathbf{r}_0 = \begin{pmatrix} -1 \\ 0 \\ 2 \end{pmatrix}$, and $\mathbf{u} = \begin{pmatrix} 1 \\ 5 \\ -4 \end{pmatrix}$, so the vector equation of the line is $\mathbf{r} = \begin{pmatrix} -1 \\ 0 \\ 2 \end{pmatrix} + t\begin{pmatrix} 1 \\ 5 \\ -4 \end{pmatrix}$. The parametric equations are: $\begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} -1+t \\ 5t \\ 2-4t \end{pmatrix}$.

b) Substituting $\mathbf{r}_0 = \begin{pmatrix} 3 \\ -1 \\ 2 \end{pmatrix}$ and $\mathbf{u} = \begin{pmatrix} 2 \\ 5 \\ -1 \end{pmatrix}$ into $\mathbf{r} = \mathbf{r}_0 + t\mathbf{u}$, we get the vector equation of the line:

$\mathbf{r} = \begin{pmatrix} 3 \\ -1 \\ 2 \end{pmatrix} + t\begin{pmatrix} 2 \\ 5 \\ -1 \end{pmatrix}$. The parametric equations are: $\begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} 3+2t \\ -1+5t \\ 2-t \end{pmatrix}$.

c) Substituting $\mathbf{r}_0 = \begin{pmatrix} 1 \\ -2 \\ 6 \end{pmatrix}$ and $\mathbf{u} = \begin{pmatrix} 3 \\ 5 \\ -11 \end{pmatrix}$ into $\mathbf{r} = \mathbf{r}_0 + t\mathbf{u}$, we get the vector equation of the line:

$\mathbf{r} = \begin{pmatrix} 1 \\ -2 \\ 6 \end{pmatrix} + t\begin{pmatrix} 3 \\ 5 \\ -11 \end{pmatrix}$. The parametric equations are: $\begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} 1+3t \\ -2+5t \\ 6-11t \end{pmatrix}$.

2 **a)** In the equation $\mathbf{r} = \mathbf{r}_0 + t\mathbf{u}$, vector $\mathbf{r}_0 = \mathbf{r}_A = \begin{pmatrix} -1 \\ 4 \\ 2 \end{pmatrix}$, and $\mathbf{u} = \overrightarrow{AB} = \begin{pmatrix} 7+1 \\ 5-4 \\ 0-2 \end{pmatrix} = \begin{pmatrix} 8 \\ 1 \\ -2 \end{pmatrix}$, so the vector equation of the line is $\mathbf{r} = \begin{pmatrix} -1 \\ 4 \\ 2 \end{pmatrix} + t\begin{pmatrix} 8 \\ 1 \\ -2 \end{pmatrix}$.

Note: For $\mathbf{r}_0$ we can use $\mathbf{r}_B$ and for $\mathbf{u}$ we can use $\overrightarrow{BA}$, or any vector parallel to this vector. Therefore, it is possible to find different, but correct, equations of the line.

b) Substituting $\mathbf{r}_0 = \mathbf{r}_A = \begin{pmatrix} 4 \\ 2 \\ -3 \end{pmatrix}$ and $\mathbf{u} = \overrightarrow{AB} = \begin{pmatrix} 0-4 \\ -2-2 \\ 1+3 \end{pmatrix} = \begin{pmatrix} -4 \\ -4 \\ 4 \end{pmatrix}$ into $\mathbf{r} = \mathbf{r}_0 + t\mathbf{u}$, we get the vector equation of the line: $\mathbf{r} = \begin{pmatrix} 4 \\ 2 \\ -3 \end{pmatrix} + t\begin{pmatrix} -4 \\ -4 \\ 4 \end{pmatrix}$.

Note: Since $\mathbf{u}$ is parallel to $\overrightarrow{AB}$, we can use $\mathbf{u} = -\frac{1}{4}\overrightarrow{AB} = \begin{pmatrix} 1 \\ 1 \\ -1 \end{pmatrix}$, and the vector equation of the line would be $\mathbf{r} = \begin{pmatrix} 4 \\ 2 \\ -3 \end{pmatrix} + s\begin{pmatrix} 1 \\ 1 \\ -1 \end{pmatrix}$.

c) Substituting $\mathbf{r}_0 = \mathbf{r}_A = \begin{pmatrix} 1 \\ 3 \\ -3 \end{pmatrix}$ and $\mathbf{u} = \overrightarrow{AB} = \begin{pmatrix} 5-1 \\ 1-3 \\ 2+3 \end{pmatrix} = \begin{pmatrix} 4 \\ -2 \\ 5 \end{pmatrix}$ into $\mathbf{r} = \mathbf{r}_0 + t\mathbf{u}$, we get the vector equation of the line: $\mathbf{r} = \begin{pmatrix} 1 \\ 3 \\ -3 \end{pmatrix} + t\begin{pmatrix} 4 \\ -2 \\ 5 \end{pmatrix}$.

3 **a)** In the equation $\mathbf{r} = \mathbf{a} + t\mathbf{b}$, we substitute $\mathbf{a} = \mathbf{r}_A = \begin{pmatrix} 3 \\ -2 \end{pmatrix}$ and $\mathbf{u} = \overrightarrow{AB} = \begin{pmatrix} 5-3 \\ 1+2 \end{pmatrix} = \begin{pmatrix} 2 \\ 3 \end{pmatrix}$, so the equation of the line is $\mathbf{r} = \begin{pmatrix} 3 \\ -2 \end{pmatrix} + t\begin{pmatrix} 2 \\ 3 \end{pmatrix}$.

b) In the equation $\mathbf{r} = \mathbf{a} + t\mathbf{b}$, we substitute $\mathbf{a} = \mathbf{r}_A = \begin{pmatrix} 0 \\ -2 \end{pmatrix}$ and $\mathbf{u} = \overrightarrow{AB} = \begin{pmatrix} 5-0 \\ 0+2 \end{pmatrix} = \begin{pmatrix} 5 \\ 2 \end{pmatrix}$, so the equation of the line is $\mathbf{r} = \begin{pmatrix} 0 \\ -2 \end{pmatrix} + t\begin{pmatrix} 5 \\ 2 \end{pmatrix}$.

4 Method I:
To determine the equation of the line in the required form, we need to find two points on the line $\mathbf{r} = (2, 1) + t(3, -2)$. One point is $(2, 1)$. Another point we can find by letting, for example, $t = 1$; therefore, the point is $(5, -1)$. The equation of the line through those two points is:
$\frac{y-1}{x-2} = \frac{-1-1}{5-2} \Rightarrow 3(y-1) = -2(x-2) \Rightarrow 2x + 3y = 7.$

Method II:
We can write the equation of the line $\mathbf{r} = (2, 1) + t(3, -2)$ in parametric form:
$\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 2+3t \\ 1-2t \end{pmatrix}$. From the first row: $x = 2 + 3t \Rightarrow t = \frac{x-2}{3}$. Substituting t into the second row, we get: $y = 1 - 2t = 1 - 2\frac{x-2}{3} \Rightarrow 3(y-1) = -2(x-2) \Rightarrow 2x + 3y = 7.$

Method III:
We can write the equation of the line $\mathbf{r} = (2, 1) + t(3, -2)$ in parametric form:
$\begin{cases} x = 2 + 3t \\ y = 1 - 2t \end{cases}$. Using the elimination method:
$\begin{cases} x = 2 + 3t \ / \cdot 2 \\ y = 1 - 2t \ / \cdot 3 \end{cases} \Rightarrow \begin{cases} 2x = 4 + 6t \\ 3y = 3 - 6t \end{cases} \Rightarrow 2x + 3y = 7$

5 In the equation $\mathbf{r} = \mathbf{r}_0 + t\mathbf{u}$, the vector $\mathbf{r}_0 = \mathbf{r}_A = 2\mathbf{i} - 3\mathbf{j}$ and $\mathbf{u}$ can be the same as the direction vector of the given line; therefore, $\mathbf{u} = 4\mathbf{i} - 3\mathbf{j}$. So: $\mathbf{r} = 2\mathbf{i} - 3\mathbf{j} + \lambda(4\mathbf{i} - 3\mathbf{j})$.

6 In the equation $\mathbf{r} = \mathbf{r}_0 + t\mathbf{u}$, the vector $\mathbf{r}_0 = \mathbf{r}_A = (-2, 1, 4)$ and $\mathbf{u} = (3, -4, 7)$. So, we have:
$\mathbf{r} = (-2, 1, 4) + t(3, -4, 7)$.

Solution Paper 1 type

7 a The lines are not parallel since the direction vectors $(1, 3, 1)$ and $(1, 4, 2)$ are not a scalar multiple of each other. For lines to intersect, there should be some point (x_0, y_0, z_0) which satisfies the equations of both lines, $\mathbf{r} = (2, 2, 3) + t(1, 3, 1)$ and $\mathbf{r} = (2, 3, 4) + s(1, 4, 2)$, for some values of z and s. (Note: We have to change the parameter in one of the equations so that they are not the same.) So:

$x_0 = 2 + t = 2 + s$

$y_0 = 2 + 3t = 3 + 4s$

$z_0 = 3 + t = 4 + 2s$

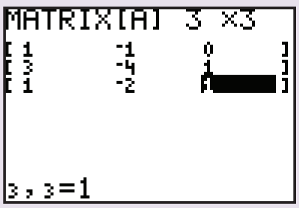
From the first equation, we see that $t = s$. Substituting into the second equation: $2 + 3t = 3 + 4t \Rightarrow t = -1 \Rightarrow t = s = -1$. Finally, substituting these values into the third equation: $3 - 1 = 4 - 2 \Rightarrow -2 = -2$. Hence, the lines intersect, and the point of intersection is: $(2, 2, 3) + (-1)(1, 3, 1) = (1, -1, 2)$.

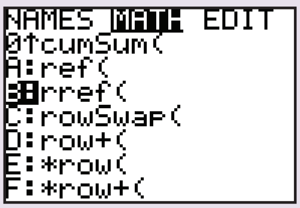
Solution Paper 2 type

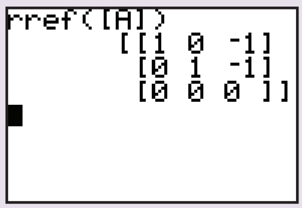
7 a We can solve this system using matrices. Firstly, transform the system of equations:

$$\begin{cases} 2 + t = 2 + s \\ 2 + 3t = 3 + 4s \\ 3 + t = 4 + 2s \end{cases} \Rightarrow \begin{cases} t - s = 0 \\ 3t - 4s = 1 \\ t - 2s = 1 \end{cases}$$

and then use a GDC:

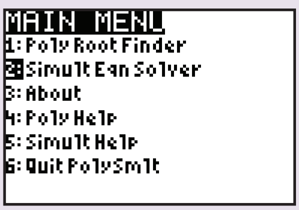


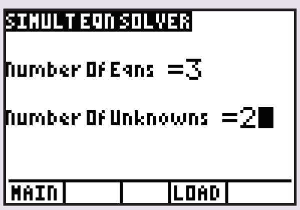


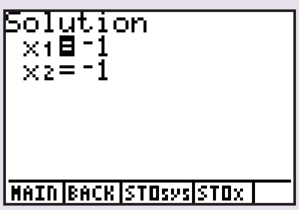


From the result screen, we can see that an intersection exists and that $t = -1$ and $s = -1$.

We can also find the solutions of the system using the Apps menu.







Of course, a GDC will only solve for the parameters – to find the point of intersection, we have to substitute the values of the parameters into the equations of the lines.

Solution Paper 1 type

b The lines are not parallel since the direction vectors (4, 1, 0) and (12, 6, 3) are not a scalar multiple of each other. For lines to intersect, there should be some point (x_0, y_0, z_0) which satisfies the equations of both lines, $\mathbf{r} = (-1, 3, 1) + t\,(4, 1, 0)$ and $\mathbf{r} = (-13, 1, 2) + s\,(12, 6, 3)$, for some values of z and s. (Note: We have to change the parameter in one of the equations so that they are not the same.) So:

$$x_0 = -1 + 4t = -13 + 12s$$
$$y_0 = 3 + t = 1 + 6s$$
$$z_0 = 1 + 0 \cdot t = 2 + 3s$$

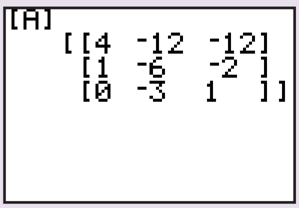
From the last equation, we see that $s = -\frac{1}{3}$. Substituting into the second equation: $3 + t = 1 + 6\left(-\frac{1}{3}\right) \Rightarrow t = -4$.
Finally, substituting these values into the first equation: $-1 - 16 = -13 - 4 \Rightarrow -17 = -17$. Hence, the lines intersect, and the point of intersection is: $\mathbf{r} = (-1, 3, 1) + (-4)(4, 1, 0) = (-17, -1, 1)$.

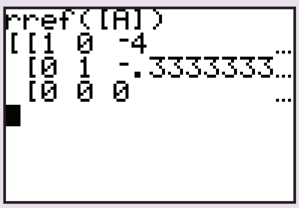
Solution Paper 2 type

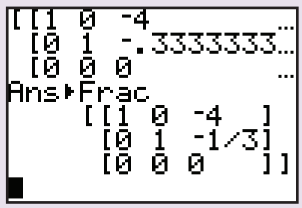
b We can solve this system using matrices. Firstly, transform the system of equations:

$$\begin{cases} -1 + 4t = -13 + 12s \\ 3 + t = 1 + 6s \\ 1 + 0 \cdot t = 2 + 3s \end{cases} \Rightarrow \begin{cases} 4t - 12s = -12 \\ t - 6s = -2 \\ -3s = 1 \end{cases}$$

and then use a GDC:







From the result screen, we can see that an intersection exists, and that $t = -4$ and $s = -\frac{1}{3}$.

Of course, a GDC will only solve for the parameters – to find the point of intersection we have to substitute the values of the parameters into the equations of the lines.

Solution Paper 1 type

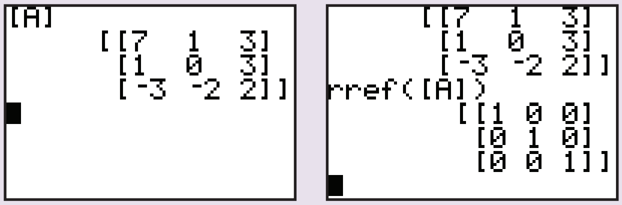
c The lines are not parallel since the direction vectors (7, 1, –3) and (–1, 0, 2) are not a scalar multiple of each other. For lines to intersect, there should be some point (x_0, y_0, z_0) which satisfies the equations of both lines, $\mathbf{r} = (1, 3, 5) + t\,(7, 1, -3)$ and $\mathbf{r} = (4, 6, 7) + s\,(-1, 0, 2)$, for some values of z and s. (Note: We have to change the parameter in one of the equations so that they are not the same.) So:

$$x_0 = 1 + 7t = 4 - s$$
$$y_0 = 3 + t = 6 + 0 \cdot s$$
$$z_0 = 5 - 3 \cdot t = 7 + 2s$$

From the second equation, we can see that $t = 3$. Substituting into the first equation: $1 + 21 = 4 - s \Rightarrow s = -18$. Finally, substituting these values into the last equation: $5 - 3 \cdot 3 = 7 + 2(-18) \Rightarrow -4 \neq -29$. Hence, the lines do not intersect; they are skew.

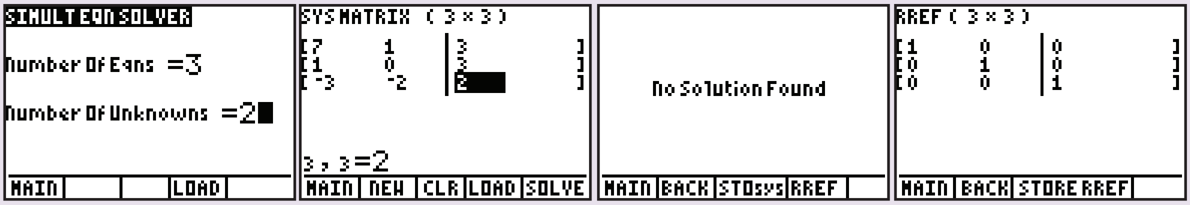
Solution Paper 2 type

c $\begin{cases} 1 + 7t = 4 - s \\ 3 + t = 6 + 0 \cdot s \\ 5 - 3 \cdot t = 7 + 2s \end{cases} \Rightarrow \begin{cases} 7t + s = 3 \\ t + 0 \cdot s = 3 \\ -3t - 2s = 2 \end{cases}$



From the result screen, we can see that an intersection does not exist since we interpret the screen as: $t = 0$, $s = 0$ and $0 = 1$.

Alternatively, solving for parameters using Apps:



From the result screen, we can see that there is no solution.

Solution Paper 1 type

d The lines have parallel direction vectors $\begin{pmatrix} -2 \\ 1 \\ -1 \end{pmatrix}$ and $\begin{pmatrix} -4 \\ 2 \\ -2 \end{pmatrix}$, since $\begin{pmatrix} -4 \\ 2 \\ -2 \end{pmatrix} = 2\begin{pmatrix} -2 \\ 1 \\ -1 \end{pmatrix}$.

To check whether the lines coincide, we examine the point (3, 4, 6), which is on the first line

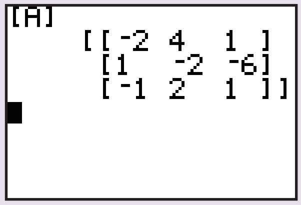
$\begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} 3 \\ 4 \\ 6 \end{pmatrix} + t\begin{pmatrix} -2 \\ 1 \\ -1 \end{pmatrix}$, and see whether it lies on the second line, $\begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} 5 \\ -2 \\ 7 \end{pmatrix} + s\begin{pmatrix} -4 \\ 2 \\ -2 \end{pmatrix}$.

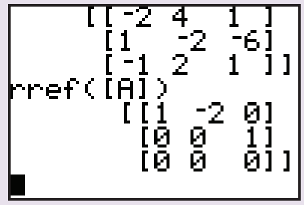
So: $\begin{pmatrix} 3 \\ 4 \\ 6 \end{pmatrix} = \begin{pmatrix} 5 \\ -2 \\ 7 \end{pmatrix} + s\begin{pmatrix} -4 \\ 2 \\ -2 \end{pmatrix} \Rightarrow \begin{cases} 3 = 5 - 4s \Rightarrow s = \frac{1}{2} \\ 4 = -2 + 2s \Rightarrow s = 3 \\ 6 = 7 - 2s \Rightarrow \end{cases}$

We can see that the point is not on the other line, so the lines do not coincide; therefore, the lines are parallel.

Solution Paper 2 type

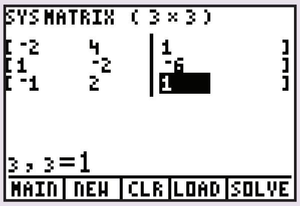
d $\begin{cases} 3-2t=4-4s \\ 4+t=-2+2s \\ 6-1\cdot t=7-2s \end{cases} \Rightarrow \begin{cases} -2t+4s=1 \\ t-2s=-6 \\ -t+2s=1 \end{cases}$

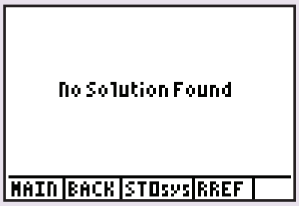




From the result screen, we can see that an intersection does not exist and the lines are parallel. We arrive at this conclusion by interpreting the screen as: $t-2s=0$, $0=1$ and $0=0$.

Alternatively, solving for parameters using Apps:





From the result screen, we can see that there is no solution and that the lines are parallel.

8 **a)** A direction vector is: $(3-2, 2-(-1)) = (1, 3)$; hence, the equations are:

$$\mathbf{r} = (2, -1) + t\,(1, 3) \text{ and } \begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 2+t \\ -1+3t \end{pmatrix}.$$

b) We know a point and a direction vector, so the equations are:

$$\mathbf{r} = (2, -1) + t\,(-3, 7) \text{ and } \begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 2-3t \\ -1+7t \end{pmatrix}.$$

c) For the direction vector, we can use any vector perpendicular to $\begin{pmatrix} -3 \\ 7 \end{pmatrix}$. So, we use vector $\begin{pmatrix} 7 \\ 3 \end{pmatrix}$ as the direction vector of the line, since $\begin{pmatrix} -3 \\ 7 \end{pmatrix}\begin{pmatrix} 7 \\ 3 \end{pmatrix} = -21 + 21 = 0$. Therefore, the equations are: $\mathbf{r} = (2, -1) + t\,(7, 3)$ and $\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 2+7t \\ -1+3t \end{pmatrix}$.

d) We know a point and a direction vector, so the equations are:

$$\mathbf{r} = (0, 2) + t\,(2, -4) \text{ and } \begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 2t \\ 2-4t \end{pmatrix}.$$

9 **a)** Substituting the point $\left(0, \frac{11}{2}, \frac{9}{2}\right)$ into the equation $\begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} 3 \\ 4 \\ 6 \end{pmatrix} + t\begin{pmatrix} -2 \\ 1 \\ -1 \end{pmatrix}$:

$$\begin{pmatrix} 0 \\ \frac{11}{2} \\ \frac{9}{2} \end{pmatrix} = \begin{pmatrix} 3 \\ 4 \\ 6 \end{pmatrix} + t\begin{pmatrix} -2 \\ 1 \\ -1 \end{pmatrix} \Rightarrow \begin{cases} 0 = 3-2t \\ \frac{11}{2} = 4+t \\ \frac{9}{2} = 6-t \end{cases}$$

From the first equation, we can see that $t = \frac{3}{2}$. Checking, using the second equation: $\frac{11}{2} = 4 + \frac{3}{2} \Rightarrow \frac{11}{2} = \frac{11}{2}$, and the third equation: $\frac{9}{2} = 6 - \frac{3}{2} \Rightarrow \frac{9}{2} = \frac{9}{2}$. So, the point is on the line when $t = \frac{3}{2}$.

b) To check whether the point is on the line, we have to find whether the system of equations has a solution:

$$\begin{pmatrix} -1 \\ 4 \\ 6 \end{pmatrix} = \begin{pmatrix} 3 \\ 4 \\ 6 \end{pmatrix} + t\begin{pmatrix} -2 \\ 1 \\ -1 \end{pmatrix} \Rightarrow \begin{cases} -1 = 3 - 2t \\ 4 = 4 + t \\ 6 = 6 - t \end{cases}$$

From the last two equations, we can see that $t = 0$, but this will not satisfy the first equation; hence, there is **no** solution to the system and the point does not lie on the line.

c) We have to solve the system of equations:

$$\begin{pmatrix} \frac{1-2m}{2} \\ \frac{2m}{3} \end{pmatrix} = \begin{pmatrix} 3 \\ 4 \\ 6 \end{pmatrix} + t\begin{pmatrix} -2 \\ 1 \\ -1 \end{pmatrix} \Rightarrow \begin{cases} \frac{1-2m}{2} = 3 - 2t \\ 2m = 4 + t \\ 3 = 6 - t \end{cases}$$

From the last equation, we can see that $t = 3$. Substituting into the second equation:

$2m = 7 \Rightarrow m = \frac{7}{2}$. Checking, using the first equation: $\frac{1 - 2\frac{7}{2}}{2} = 3 - 6 \Rightarrow \frac{-6}{2} = -3$. Therefore, the point will be on the line when $m = \frac{7}{2}$.

10 a) i) The starting position is when $t = 0$, so the point is $(3, -4)$.

ii) The velocity vector is $\mathbf{v} = \begin{pmatrix} 7 \\ 24 \end{pmatrix}$.

iii) The speed is $|\mathbf{v}| = \sqrt{7^2 + 24^2} = 25$.

b) i) The starting position is when $t = 0$, so the point is $(-3, 1)$.

ii) The velocity vector is $\mathbf{v} = \begin{pmatrix} 5 \\ -12 \end{pmatrix}$.

iii) The speed is $|\mathbf{v}| = \sqrt{5^2 + (-12)^2} = 13$.

c) i) The starting position is when $t = 0$, so the point is $(5, -2)$.

ii) The velocity vector is $\mathbf{v} = (24, -7)$.

iii) The speed is $|\mathbf{v}| = \sqrt{24^2 + (-7)^2} = 25$.

11 a) The direction of the velocity vector is given by the unit vector: $\frac{1}{\sqrt{(-3)^2 + 4^2}}\begin{pmatrix} -3 \\ 4 \end{pmatrix} = \frac{1}{5}\begin{pmatrix} -3 \\ 4 \end{pmatrix}$.

So, the velocity vector is: $160 \cdot \frac{1}{5}\begin{pmatrix} -3 \\ 4 \end{pmatrix} = 32\begin{pmatrix} -3 \\ 4 \end{pmatrix} = \begin{pmatrix} -96 \\ 128 \end{pmatrix}$.

b) The direction of the velocity vector is given by the unit vector: $\frac{1}{\sqrt{12^2 + (-5)^2}}\begin{pmatrix} 12 \\ -5 \end{pmatrix} = \frac{1}{13}\begin{pmatrix} 12 \\ -5 \end{pmatrix}$.

So, the velocity vector is: $170 \cdot \frac{1}{13}\begin{pmatrix}12\\-5\end{pmatrix} = \begin{pmatrix}\frac{2040}{13}\\-\frac{850}{13}\end{pmatrix}$.

12 a) The car is travelling from the point (3, 2) to (7, 5), so the direction vector of the velocity vector as given by the unit vector: $\frac{1}{|\mathbf{v}|}\mathbf{v}$, where $\mathbf{v} = (7-3, 5-2) = (4, 3)$. Therefore, the unit vector is: $\frac{1}{\sqrt{4^2+3^2}}(4,3) = \frac{1}{5}(4,3)$; and the velocity vector is: $30 \cdot \frac{1}{5}(4,3) = (24,18)$.

b) The starting point is (3, 2) and the direction vector of the line is (24, 18), so the equation of the position of the car after t hours is $\mathbf{r} = (3,2) + t(24,18)$.

c) We have to determine the parameter of the point (7, 5).

$(7,5) = (3,2) + t(24,18) \Rightarrow (4,3) = t(24,18) \Rightarrow t = \frac{1}{6}$

Therefore, in $\frac{1}{6}$ of an hour, i.e. 10 minutes, the car will reach the traffic light.

13 a) To be perpendicular to the vectors, both dot products have to be zero.

$$\begin{pmatrix}1\\a\\b\end{pmatrix}\begin{pmatrix}1\\-3\\2\end{pmatrix} = 1 - 3a + 2b = 0$$

$$\begin{pmatrix}1\\a\\b\end{pmatrix}\begin{pmatrix}-2\\1\\-1\end{pmatrix} = -2 + a - b = 0$$

So, we have to solve the system: $\begin{cases}-3a + 2b = -1\\ a - b = 2\end{cases} \Rightarrow a = -3, b = -5$

b) $$\cos\theta = \frac{\mathbf{v}\cdot\mathbf{w}}{|\mathbf{v}|\,|\mathbf{w}|} = \frac{\begin{pmatrix}1\\-3\\2\end{pmatrix}\begin{pmatrix}-2\\1\\-1\end{pmatrix}}{\sqrt{1^2+(-3)^2+2^2}\sqrt{(-2)^2+1^2+(-1)^2}} = \frac{-2-3-2}{\sqrt{14}\sqrt{6}}$$

$$= \frac{-7}{2\sqrt{21}} = \frac{-7\sqrt{21}}{2\cdot 21} = -\frac{\sqrt{21}}{6}$$

c) Using the Pythagorean identity for sine, $\sin^2\theta = 1 - \cos^2\theta$, and the fact that sine is positive for angles from $0 - 180^\circ$ we have: $\sin\theta = +\sqrt{1-\cos^2\theta} = \sqrt{1-\frac{21}{36}} = \sqrt{\frac{15}{36}} = \frac{\sqrt{15}}{6}$.

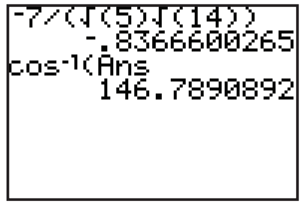
Area of triangle OPQ is: $A = \frac{1}{2}|OP|\,|OQ|\sin P\hat{O}Q = \frac{1}{2}|\mathbf{v}|\,|\mathbf{w}|\sin(\angle\mathbf{v},\mathbf{w})$.

So, $A = \frac{1}{2}\sqrt{14}\sqrt{6}\frac{\sqrt{15}}{6} = \frac{1}{2}\frac{\sqrt{2\cdot7}\sqrt{2\cdot3}\sqrt{3\cdot5}}{6} = \frac{1}{2}\frac{2\sqrt{7}\cdot3\cdot\sqrt{5}}{6} = \frac{\sqrt{35}}{2}$.

14 a) Firstly, we have to determine vectors $\overrightarrow{AB}$ and $\overrightarrow{AC}$:

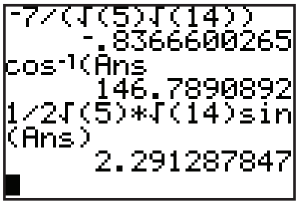
$$\overrightarrow{AB} = \begin{pmatrix}-1-(-1)\\3-2\\5-3\end{pmatrix} = \begin{pmatrix}0\\1\\2\end{pmatrix},\ \overrightarrow{AC} = \begin{pmatrix}0-(-1)\\-1-2\\1-3\end{pmatrix} = \begin{pmatrix}1\\-3\\-2\end{pmatrix}$$

$$\cos\theta = \frac{\begin{pmatrix}0\\1\\2\end{pmatrix}\begin{pmatrix}1\\-3\\-2\end{pmatrix}}{\sqrt{0^2+1^2+2^2}\sqrt{1^2+(-3)^2+(-2)^2}} = \frac{0-3-4}{\sqrt{5}\sqrt{14}} = \frac{-7}{\sqrt{5}\sqrt{14}}$$



Therefore, $\theta = \cos^{-1}\dfrac{-7}{\sqrt{5}\sqrt{14}} \Rightarrow \theta \approx 147^\circ$.

b) The area of the triangle is: $A = \dfrac{1}{2}\left|\overrightarrow{AB}\right|\left|\overrightarrow{AC}\right|\sin\theta = \dfrac{1}{2}\sqrt{5}\sqrt{14}\sin\theta \approx 2.29$.



c) i) Line L_1 goes through the point $(2, -1, 0)$ and its direction vector is $\overrightarrow{AB} = \begin{pmatrix}0\\1\\2\end{pmatrix}$, so its equation is: $\mathbf{r} = \begin{pmatrix}2\\-1\\0\end{pmatrix} + t\begin{pmatrix}0\\1\\2\end{pmatrix}$.

Line L_2 goes through the point $(-1, 1, 1)$ and its direction vector is $\overrightarrow{AC} = \begin{pmatrix}1\\-3\\-2\end{pmatrix}$, so its equation is: $\mathbf{r} = \begin{pmatrix}-1\\1\\1\end{pmatrix} + s\begin{pmatrix}1\\-3\\-2\end{pmatrix}$.

ii) We have to solve the system of equations:

$$\begin{pmatrix}2\\-1\\0\end{pmatrix} + t\begin{pmatrix}0\\1\\2\end{pmatrix} = \begin{pmatrix}-1\\1\\1\end{pmatrix} + s\begin{pmatrix}1\\-3\\-2\end{pmatrix} \Rightarrow \begin{cases} 2 = -1 + s \\ -1 + t = 1 - 3s \\ 2t = 1 - 2s \end{cases}$$

From the first equation, we have $s = 3$, and from the second $t = -7$. Substituting these values into the third equation: $2(-7) = 1 - 2\cdot 3 \Rightarrow -14 \neq -5$. So, there is no point of intersection.

15 a) Let the direction vector be a vector parallel to $\overrightarrow{AB}$: $\overrightarrow{AB} = \begin{pmatrix}6-1\\-7-3\\8-(-17)\end{pmatrix} = \begin{pmatrix}5\\-10\\25\end{pmatrix} = 5\begin{pmatrix}1\\-2\\5\end{pmatrix}$

Thus, we can use the vector $\begin{pmatrix}1\\-2\\5\end{pmatrix}$ as the direction vector. Therefore, the parametric equations of the line are: $\begin{pmatrix}x\\y\\z\end{pmatrix} = \begin{pmatrix}1+t\\3-2t\\-17+5t\end{pmatrix}$.

b) If point P is on the line, then vector $\overrightarrow{OP} = \begin{pmatrix}1+t\\3-2t\\-17+5t\end{pmatrix}$. If $\overrightarrow{OP}$ is perpendicular to the line, then $\overrightarrow{OP}$ and the direction vector of the line are perpendicular, and their dot product is zero.

$$0 = \overrightarrow{OP}\begin{pmatrix}1\\2\\5\end{pmatrix} = \begin{pmatrix}1+t\\3-2t\\-17+5t\end{pmatrix}\begin{pmatrix}1\\-2\\5\end{pmatrix} = 1 + t + -6 + 4t + -85 + 25t = 30t - 90 \Rightarrow t = 3$$

So, $\overrightarrow{OP} = \begin{pmatrix} 1+3 \\ 3-2\cdot 3 \\ -17+5\cdot 3 \end{pmatrix} = \begin{pmatrix} 4 \\ -3 \\ -2 \end{pmatrix}$ and $P(4, -3, -2)$.

16 a) We will find a point and a vector on the line.

Let $y = 0$, then $x = \frac{p}{m}$; hence, point $\left(\frac{p}{m}, 0\right)$ is on the line.

Let $x = 0$, then $y = \frac{p}{n}$; hence, point $\left(0, \frac{p}{n}\right)$ is on the line.

Therefore, the vector $\begin{pmatrix} \frac{p}{m} - 0 \\ 0 - \frac{p}{n} \end{pmatrix} = \begin{pmatrix} \frac{p}{m} \\ -\frac{p}{n} \end{pmatrix}$ is on the line. This vector is parallel to the vector

$\frac{mn}{p} \cdot \begin{pmatrix} \frac{p}{m} \\ -\frac{p}{n} \end{pmatrix} = \begin{pmatrix} n \\ -m \end{pmatrix}$. So, a vector equation of the line is: $\mathbf{r} = \begin{pmatrix} \frac{p}{m} \\ 0 \end{pmatrix} + t\begin{pmatrix} n \\ -m \end{pmatrix}$.

b) i) We already have one point on the line. To determine another point on the line $\mathbf{r} = \begin{pmatrix} x_0 \\ y_0 \end{pmatrix} + t\begin{pmatrix} a \\ b \end{pmatrix}$

we can let, for example, $t = 1$; therefore, the point is: $(x_0 + a, y_0 + b)$. The equation of the line through those two points is: $\frac{y - y_0}{x - x_0} = \frac{y_0 + b - y_0}{x_0 + a - x_0} \Rightarrow a(y - y_0) = b(x - x_0)$. Hence, an equation of the line is: $bx - ay = bx_0 - ay_0$.

ii) We will write the equation of the line in slope-intercept form:

$ay = bx - bx_0 + ay_0 \Rightarrow y = \frac{b}{a}x - \frac{b}{a}x_0 + y_0$. Hence, the slope of the line is $\frac{b}{a}$.

17 Parameterization of a segment: $\mathbf{r}(t) = (1-t)\overrightarrow{OA} + t\,\overrightarrow{OB}, 0 \leqslant t \leqslant 1$

a) $\mathbf{r}(t) = (1-t)\begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix} + t\begin{pmatrix} 1 \\ 1 \\ 3 \end{pmatrix}, 0 \leqslant t \leqslant 1$; hence, $\mathbf{r}(t) = t\begin{pmatrix} 1 \\ 1 \\ 3 \end{pmatrix}, 0 \leqslant t \leqslant 1$

b) $\mathbf{r}(t) = (1-t)\begin{pmatrix} -1 \\ 0 \\ 1 \end{pmatrix} + t\begin{pmatrix} 1 \\ 1 \\ -2 \end{pmatrix} = \begin{pmatrix} -1+t+t \\ t \\ 1-t-2t \end{pmatrix}, 0 \leqslant t \leqslant 1$; hence, $\mathbf{r}(t) = \begin{pmatrix} -1+2t \\ t \\ 1-3t \end{pmatrix}, 0 \leqslant t \leqslant 1$

c) $\mathbf{r}(t) = (1-t)\begin{pmatrix} 1 \\ 0 \\ -1 \end{pmatrix} + t\begin{pmatrix} 0 \\ 3 \\ 0 \end{pmatrix} = \begin{pmatrix} 1-t \\ 3t \\ -1+t \end{pmatrix}, 0 \leqslant t \leqslant 1$; hence, $\mathbf{r}(t) = \begin{pmatrix} 1-t \\ 3t \\ -1+t \end{pmatrix}, 0 \leqslant t \leqslant 1$

18 A direction vector of the parallel line is $2\mathbf{k}$; hence, a vector equation of the line whose equation we have to find is: $\mathbf{r} = (2\mathbf{j} + 3\mathbf{k}) + t(2\mathbf{k})$. The parametric equations are: $\begin{cases} x = 0 \\ y = 2 \\ z = 3 + 2t \end{cases}$

Note: We can write a vector equation in the form: $\mathbf{r} = 2\mathbf{j} + (3 + 2t)\mathbf{k}$.

19 A direction vector of the parallel line is $2\mathbf{i}-3\mathbf{j}+\mathbf{k}$; hence a vector equation of the line whose equation we have to find is: $\mathbf{r}=(\mathbf{i}+2\mathbf{j}-\mathbf{k})+t(2\mathbf{i}-3\mathbf{j}+\mathbf{k})$. The parametric equations are: $\begin{cases} x=1+2t \\ y=2-3t \\ z=-1+t \end{cases}$

Note: We can write a vector equation in the form: $\mathbf{r}=(1+2t)\mathbf{i}+(2-3t)\,\mathbf{j}+(-1+t)\,\mathbf{k}$.

20 A direction vector of the line is $x_0\mathbf{i}+y_0\mathbf{j}+z_0\mathbf{k}$; hence, a vector equation of the line is $\mathbf{r}=0+t(x_0\mathbf{i}+y_0\mathbf{j}+z_0\mathbf{k})$, and the parametric equations are: $\begin{cases} x=tx_0 \\ y=ty_0 \\ z=tz_0 \end{cases}$

21 **a)** A direction vector of the line is a vector perpendicular to the xz-plane; hence, $\mathbf{j}$. Therefore, a vector equation of the line is $\mathbf{r}=(3\mathbf{i}+2\mathbf{j}-3\mathbf{k})+t\mathbf{j}$, and the parametric equations are: $\begin{cases} x=3 \\ y=2+t \\ z=-3 \end{cases}$

b) A direction vector of the line is a vector perpendicular to the yz-plane; hence, $\mathbf{i}$. Therefore, a vector equation of the line is $\mathbf{r}=(3\mathbf{i}+2\mathbf{j}-3\mathbf{k})+t\mathbf{i}$, and the parametric equations are: $\begin{cases} x=3+t \\ y=2 \\ z=-3 \end{cases}$

22 A direction vector of the line is $x_0\mathbf{i}+y_0\mathbf{j}+z_0\mathbf{k}$. Hence, the symmetric (Cartesian) equations of the lines are: $\frac{x-0}{x_0}=\frac{y-0}{y_0}=\frac{z-0}{z_0}\Rightarrow\frac{x}{x_0}=\frac{y}{y_0}=\frac{z}{z_0}$.

Note: We took the origin as a point on the line. We can take point A, then the equation will be: $\frac{x-x_0}{x_0}=\frac{y-y_0}{y_0}=\frac{z-z_0}{z_0}$.

23 The lines are not parallel since the direction vectors $\begin{pmatrix}1\\-1\\2\end{pmatrix}$ and $\begin{pmatrix}0\\1\\1\end{pmatrix}$ are not a scalar multiple of each other.

For the lines to intersect, there should be a point which satisfies the equations of both lines.

We will write the equations in parametric form: $\begin{cases} x=3+t \\ y=1-t \\ z=5+2t \end{cases}$, $\begin{cases} x=1 \\ y=4+\lambda \\ z=2+\lambda \end{cases}$ and solve the system:

$3+t=1$
$1-t=4+\lambda$
$5+2t=2+\lambda$

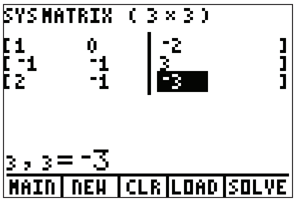
From the first equation, we can see that $t=-2$. From the second, $1+2=4+\lambda\Rightarrow\lambda=-1$; and, finally, substituting those values into the third equation, we have: $5+2(-2)=2+(-1)\Rightarrow 1=1$. Hence, the lines intersect, and the point of intersection is:

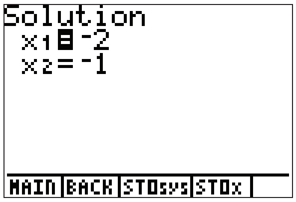
$\begin{cases} x=3-2=1 \\ y=1+2=3 \\ z=5-4=1 \end{cases}$ $(1, 3, 1)$

Note: If it is a Paper 2 question, we can solve the system using matrices, or PolySmlt. Firstly, transform the system of equations:

$$\begin{cases} 3+t=1 \\ 1-t=4+\lambda \\ 5+2t=2+\lambda \end{cases} \Rightarrow \begin{cases} t=-2 \\ -t-\lambda=3 \\ 2t-\lambda=-3 \end{cases}$$

and then use a GDC:





Then substitute back into one of the equations to find the intersection point. Notice that it is easier to solve the system without a GDC since, while preparing the system for the GDC, we would find the solutions. (This note also applies to questions 24–27.)

24 The lines are parallel since for the direction vectors $\begin{pmatrix} 1 \\ -3 \\ 2 \end{pmatrix}$ and $\begin{pmatrix} -2 \\ 6 \\ -4 \end{pmatrix}$ it holds: $\begin{pmatrix} -2 \\ 6 \\ -4 \end{pmatrix} = -2\begin{pmatrix} 1 \\ -3 \\ 2 \end{pmatrix}$. To check whether the lines coincide, we examine the point $(-1, 2, 1)$, which is on the first line, and see whether it lies on the second line, $\begin{cases} x=2-2m \\ y=-1+6m \\ z=-4m \end{cases}$, too.

$$\text{So, } \begin{cases} -1=2-2m \\ 2=-1+6m \\ 1=-4m \end{cases} \Rightarrow \begin{cases} 3=2m \Rightarrow m=\frac{3}{2} \\ 3=6m \Rightarrow m=\frac{3}{6} \\ 1=-4m \Rightarrow \end{cases}$$

We can see that the point is not on the other line, so the lines do not coincide; therefore, the lines are parallel.

25 The lines are not parallel since the direction vectors $\begin{pmatrix} 2 \\ 4 \\ -1 \end{pmatrix}$ and $\begin{pmatrix} 2 \\ 1 \\ 2 \end{pmatrix}$ are not a scalar multiple of each other. For the lines to intersect, there should be a point which satisfies the equations of both lines.

We will write the equations in parametric form: $\begin{cases} x=3+2t \\ y=-1+4t \\ z=2-t \end{cases}$, $\begin{cases} x=3+2\lambda \\ y=2+\lambda \\ z=-2+2\lambda \end{cases}$ and solve the system:

$3+2t=3+2\lambda$

$-1+4t=2+\lambda$

$2-t=-2+2\lambda$

From the first equation, we can see that $t=\lambda$. From the second, $-1+4t=2+t \Rightarrow t=1$; and, finally, substituting those values into the third equation, we have: $2-1=-2+2 \Rightarrow 1 \neq 0$. Hence, the lines do not intersect. They are skew.

26 The lines are not parallel since the direction vectors $\begin{pmatrix}1\\3\\2\end{pmatrix}$ and $\begin{pmatrix}-1\\-1\\2\end{pmatrix}$ are not a scalar multiple of each other. For the lines to intersect, there should be a point which satisfies the equations of both lines.

We will write the equations in parametric form: $\begin{cases} x = 1 - t \\ y = 1 + 3t \\ z = -4 + 2t \end{cases}$, $\begin{cases} x = 1 - \lambda \\ y = -1 - \lambda \\ z = 2\lambda \end{cases}$ and solve the system:

$$1 - t = 1 - \lambda$$
$$1 + 3t = -1 - \lambda$$
$$-4 + 2t = 2\lambda$$

From the first equation, we can see that $t = \lambda$. From the second, $1 + 3t = -1 - t \Rightarrow t = -\frac{1}{2}$; and, finally, substituting those values into the third equation, we have: $-4 + 2\left(-\frac{1}{2}\right) = 2\left(-\frac{1}{2}\right) \Rightarrow -5 \neq -1$. Hence, the lines do not intersect. They are skew.

27 The lines are parallel since for the direction vectors $\begin{pmatrix}-6\\9\\-3\end{pmatrix}$ and $\begin{pmatrix}2\\-3\\1\end{pmatrix}$, it holds: $\begin{pmatrix}-6\\9\\-3\end{pmatrix} = -3\begin{pmatrix}2\\-3\\1\end{pmatrix}$. To check whether the lines coincide, we examine the point (1, 2, 0), which is on the first line, and see whether it lies on the second line, $\begin{cases} x = 2 + 2m \\ y = 3 - 3m \\ z = m \end{cases}$, too.

So, $\begin{cases} 1 = 2 + 2m \\ 2 = 3 - 3m \\ 0 = m \end{cases} \Rightarrow \begin{cases} -1 = 2m \Rightarrow m = -\frac{1}{2} \\ \\ 0 = m \end{cases}$

We can see that the point is not on the other line, so the lines do not coincide; therefore, the lines are parallel.

28 The lines are not parallel since the direction vectors $\begin{pmatrix}5\\1\\3\end{pmatrix}$ and $\begin{pmatrix}3\\-3\\-4\end{pmatrix}$ are not a scalar multiple of each other. For the lines to intersect, there should be a point which satisfies the equations of both lines.

We will write the equations in parametric form: $\begin{cases} x = 2 + 5t \\ y = 1 + t \\ z = 2 + 3t \end{cases}$, $\begin{cases} x = -4 + 3\lambda \\ y = 7 - 3\lambda \\ z = 10 - 4\lambda \end{cases}$ and solve the system:

$$2 + 5t = -4 + 3\lambda$$
$$1 + t = 7 - 3\lambda$$
$$2 + 3t = 10 - 4\lambda$$

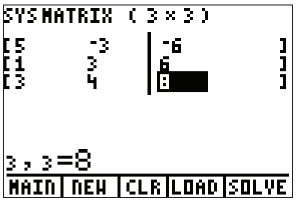
From the second equation, we can see that $t = 6 - 3\lambda$. From the first, $2 + 5(6 - 3\lambda) = -4 + 3\lambda \Rightarrow 18\lambda = 36 \Rightarrow \lambda = 2$; and, finally, substituting the values $\lambda = 2$, $t = 6 - 3 \cdot 2 = 0$ into the third equation, we have: $2 + 3 \cdot 0 = 10 - 4 \cdot 2 \Rightarrow 2 = 2$.

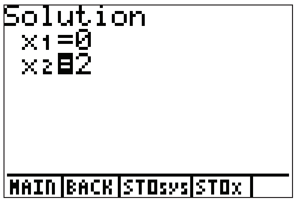
Hence, the lines intersect, and the point of intersection is: $\begin{cases} x = 2+5\cdot 0 \\ y = 1+0 \\ z = 2+3\cdot 0 \end{cases}$ $(2, 1, 2)$

Note: If it is a Paper 2 question, we can solve the system using matrices, or PolySmlt. Firstly, transform the system of equations:

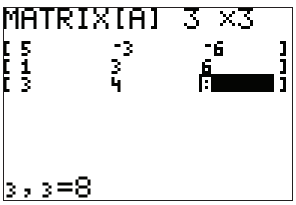
$$\begin{cases} 2+5t = -4+3\lambda \\ 1+t = 7-3\lambda \\ 2+3t = 10-4\lambda \end{cases} \Rightarrow \begin{cases} 5t-3\lambda = -6 \\ t+3\lambda = 6 \\ 3t+4\lambda = 8 \end{cases}$$

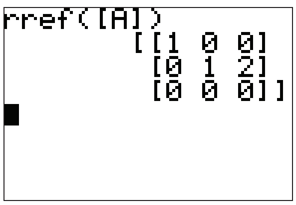
and then use a GDC:





Or input into a matrix:





So, we have $t = 0$ and $\lambda = 2$. Now, we have to substitute back into one of the equations to find the intersection point, $(2, 1, 2)$.

29 The lines are not parallel since the direction vectors $\begin{pmatrix} 1 \\ -2 \\ 1 \end{pmatrix}$ and $\begin{pmatrix} 2 \\ -9 \\ 6 \end{pmatrix}$ are not a scalar multiple of each other. For the lines to intersect, there should be a point which satisfies the equations of both lines. We have to change the name of the parameter in one of the equations and then solve the system:

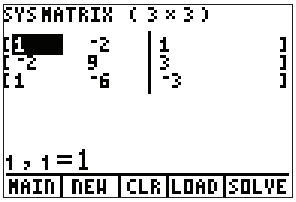
$$1+t = 2+2\lambda$$
$$2-2t = 5-9\lambda$$
$$t+5 = 2+6\lambda$$

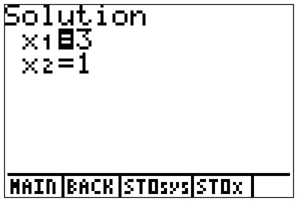
From the first equation, we can see that $t = 1+2\lambda$. From the third, $1+2\lambda+5 = 2+6\lambda \Rightarrow 4\lambda = 4 \Rightarrow \lambda = 1$; and, finally, substituting the values $\lambda = 1, t = 1+2\cdot 1 = 3$ into the second equation, we have: $2-2\cdot 3 = 5-9\cdot 1 \Rightarrow -4 = -4$. Hence, the lines intersect, and the point of intersection is: $\begin{cases} x = 1+3 \\ y = 2-2\cdot 3 \\ z = 3+5 \end{cases}$ $(4, -4, 8)$

Note: If it is a Paper 2 question, we can solve the system using matrices, or PolySmlt. Firstly, transform the system of equations:

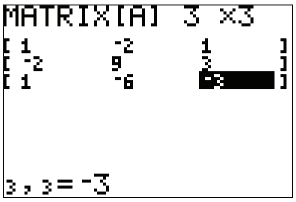
$$\begin{cases} 1+t = 2+2\lambda \\ 2-2t = 5-9\lambda \\ t+5 = 2+6\lambda \end{cases} \Rightarrow \begin{cases} t-2\lambda = 1 \\ -2t+9\lambda = 3 \\ t-6\lambda = -3 \end{cases}$$

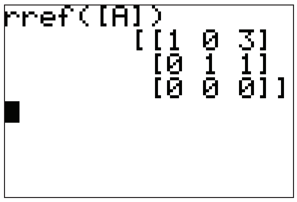
and then use a GDC:





Or input into a matrix:





So, we have $t = 3$ and $\lambda = 1$. Now, we have to substitute back into one of the equations to find the intersection point, $(4, -4, 8)$.

30 The parametric equations of the line are: $\begin{cases} x = 2-3t \\ y = 3+t \\ z = 1+t \end{cases}$

The distance from the origin to a point on the line is:

$d^2 = (2-3t)^2 + (3+t)^2 + (1+t)^2 = 14 - 4t + 11t^2$. The distance is a minimum when:

$$\frac{d\left(14-4t+11t^2\right)}{dt} = 0 \Rightarrow -4+22t = 0 \Rightarrow t = \frac{2}{11}$$

Hence, the point on the line with $t = \dfrac{2}{11}$ is the closest to the origin.

$$\begin{cases} x = 2-3\cdot\dfrac{2}{11} = \dfrac{16}{11} \\ y = 3+\dfrac{2}{11} = \dfrac{35}{11} \\ z = 1+\dfrac{2}{11} = \dfrac{13}{11} \end{cases}, \quad A\left(\frac{16}{11}, \frac{35}{11}, \frac{13}{11}\right)$$

31 Method I:

The parametric equations of the line are: $\begin{cases} x = t \\ y = 4-3t \\ z = 5+t \end{cases}$

The distance from the origin to a point on the line is:

$d^2 = t^2 + (4-3t)^2 + (5+t)^2 = 41 - 14t + 11t^2$. The distance is a minimum when:

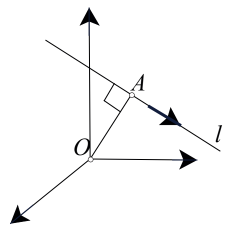
$$\frac{d\left(41-14t+11t^2\right)}{dt} = 0 \Rightarrow -14+22t = 0 \Rightarrow t = \frac{7}{11}$$

Hence, the point on the line with $t = \frac{7}{11}$ is the closest to the origin.

$$\begin{cases} x = \frac{7}{11} \\ y = 4 - 3\cdot\frac{7}{11} = \frac{23}{11} \\ z = 5 + \frac{7}{11} = \frac{62}{11} \end{cases}, \quad A\left(\frac{7}{11}, \frac{23}{11}, \frac{62}{11}\right)$$

Method II:

Notice, point A on the line is closest to the origin if $\overrightarrow{OA}$ is perpendicular to the direction vector of the line:



Since $\overrightarrow{OA} = \begin{pmatrix} t \\ 4-3t \\ 5+t \end{pmatrix}$ and the direction vector of the line is $\mathbf{d} = \begin{pmatrix} 1 \\ -3 \\ 1 \end{pmatrix}$,

then $\overrightarrow{OA}\cdot\mathbf{d} = 0 \Rightarrow \begin{pmatrix} t \\ 4-3t \\ 5+t \end{pmatrix}\begin{pmatrix} 1 \\ -3 \\ 1 \end{pmatrix} = 11t - 7 = 0 \Rightarrow t = \frac{7}{11}$, and the point is $A\left(\frac{7}{11}, \frac{23}{11}, \frac{62}{11}\right)$.

32 Method I:

The parametric equations of the line are: $\begin{cases} x = 5 + t \\ y = 2 - 3t \\ z = 1 + t \end{cases}$

The distance from the point $(-1, 4, 1)$ to a point on the line is:

$d^2 = (5 + t + 1)^2 + (2 - 3t - 4)^2 + (1 + t - 1)^2 = 40 + 24t + 11t^2$. The distance is a minimum when:

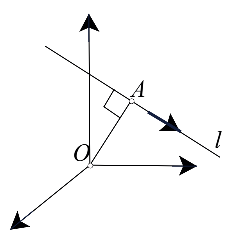
$$\frac{d\left(40 + 24t + 11t^2\right)}{dt} = 0 \Rightarrow 24 + 22t = 0 \Rightarrow t = -\frac{12}{11}$$

Hence, the point on the line with $t = -\frac{12}{11}$ is the closest to the origin.

$$\begin{cases} x = 5 - \frac{12}{11} = \frac{43}{11} \\ y = 2 - 3\cdot\left(-\frac{12}{11}\right) = \frac{58}{11} \\ z = 1 - \frac{12}{11} = -\frac{1}{11} \end{cases}, \quad A\left(\frac{43}{11}, \frac{58}{11}, -\frac{1}{11}\right)$$

Method II:

Notice, point A on the line is closest to the point $P(-1, 4, 1)$ if $\overrightarrow{PA}$ is perpendicular to the direction vector of the line:



Since $\overrightarrow{PA} = \begin{pmatrix} 5+t+1 \\ 2-3t-4 \\ 1+t-1 \end{pmatrix} = \begin{pmatrix} 6+t \\ -2-3t \\ t \end{pmatrix}$ and the direction vector of the line is $\mathbf{d} = \begin{pmatrix} 1 \\ -3 \\ 1 \end{pmatrix}$,

then $\overrightarrow{PA}\cdot\mathbf{d} = 0 \Rightarrow \begin{pmatrix} 6+t \\ -2-3t \\ t \end{pmatrix}\begin{pmatrix} 1 \\ -3 \\ 1 \end{pmatrix} = 11t + 12 = 0 \Rightarrow t = -\frac{12}{11}$, and the point is $A\left(\frac{43}{11}, \frac{58}{11}, -\frac{1}{11}\right)$.

Exercise 14.5

1 For A: $3\cdot 3+2\cdot(-2)-3\cdot(-1)=8\neq 11$; hence, A does not lie in the plane.

For B: $3\cdot 2+2\cdot 1-3\cdot(-1)=11$; hence, B does lie in the plane.

For C: $3\cdot 1+2\cdot 4-3\cdot 0=11$; hence, C does lie in the plane.

2 For A: $(\mathbf{i}-3\mathbf{j}+\mathbf{k})(3\mathbf{i}+2\mathbf{j}-3\mathbf{k})=3-6-3=-6$; hence, A does lie in the plane.

For B: $(\mathbf{i}-3\mathbf{j}+\mathbf{k})(2\mathbf{i}+\mathbf{j}-2\mathbf{k})=2-3-2=-3\neq-6$; hence, B does not lie in the plane.

For C: $(\mathbf{i}-3\mathbf{j}+\mathbf{k})(\mathbf{i}+4\mathbf{j}+0\mathbf{k})=1-12=-11\neq-6$; hence, A does not lie in the plane.

3 A Cartesian equation for the plane is:

$2(x-3)-4(y+2)+3(z-4)=0\Rightarrow 2x-4y+3z=6+8+12\Rightarrow 2x-4y+3z=26$

A vector equation for the plane is:

$$\begin{pmatrix}2\\-4\\3\end{pmatrix}\begin{pmatrix}x\\y\\z\end{pmatrix}=\begin{pmatrix}2\\-4\\3\end{pmatrix}\begin{pmatrix}3\\-2\\4\end{pmatrix}\Rightarrow\begin{pmatrix}2\\-4\\3\end{pmatrix}\begin{pmatrix}x\\y\\z\end{pmatrix}=26$$

4 A Cartesian equation for the plane is:

$2(x+3)+0(y-2)+3(z-1)=0\Rightarrow 2x+3z=-6+3\Rightarrow 2x+3z=-3$

A vector equation for the plane is:

$$\begin{pmatrix}2\\0\\3\end{pmatrix}\begin{pmatrix}x\\y\\z\end{pmatrix}=\begin{pmatrix}2\\0\\3\end{pmatrix}\begin{pmatrix}-3\\2\\1\end{pmatrix}\Rightarrow\begin{pmatrix}2\\0\\3\end{pmatrix}\begin{pmatrix}x\\y\\z\end{pmatrix}=-3$$

5 A Cartesian equation for the plane is:

$0(x-0)+0(y-3)+3(z-1)=0\Rightarrow 3z=3$

A vector equation for the plane is:

$$\begin{pmatrix}0\\0\\3\end{pmatrix}\begin{pmatrix}x\\y\\z\end{pmatrix}=\begin{pmatrix}0\\0\\3\end{pmatrix}\begin{pmatrix}0\\3\\1\end{pmatrix}\Rightarrow\begin{pmatrix}0\\0\\3\end{pmatrix}\begin{pmatrix}x\\y\\z\end{pmatrix}=3$$

6 A vector perpendicular to the plane is the same as a vector perpendicular to the parallel plane; hence, it is $\begin{pmatrix}5\\1\\-2\end{pmatrix}$.

So, a Cartesian equation for the plane is:

$5(x-3)+1(y+2)-2(z-4)=0\Rightarrow 5x+y-2z=15-2-8\Rightarrow 5x+y-2z=5$

A vector equation for the plane is:

$$\begin{pmatrix}5\\1\\-2\end{pmatrix}\begin{pmatrix}x\\y\\z\end{pmatrix}=\begin{pmatrix}5\\1\\-2\end{pmatrix}\begin{pmatrix}3\\-2\\4\end{pmatrix}\Rightarrow\begin{pmatrix}5\\1\\-2\end{pmatrix}\begin{pmatrix}x\\y\\z\end{pmatrix}=5$$

7 A vector perpendicular to the plane is the same as a vector perpendicular to the parallel plane; hence, it is $\begin{pmatrix} 0 \\ 1 \\ -2 \end{pmatrix}$.

So, a Cartesian equation for the plane is:

$0(x-3)+1(y-0)-2(z-1)=0 \Rightarrow y-2z=-2$

A vector equation for the plane is:

$$\begin{pmatrix} 0 \\ 1 \\ -2 \end{pmatrix}\begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} 0 \\ 1 \\ -2 \end{pmatrix}\begin{pmatrix} 3 \\ 0 \\ 1 \end{pmatrix} \Rightarrow \begin{pmatrix} 0 \\ 1 \\ -2 \end{pmatrix}\begin{pmatrix} x \\ y \\ z \end{pmatrix} = -2$$

In questions 8–10 and 12–16, other vectors parallel to the plane, or other points on the plane, can be found; hence, different parametric and vector equations may be obtained.

8 The plane is parallel to a direction vector of the line $\begin{pmatrix} 2 \\ 1 \\ 2 \end{pmatrix}$ and the vector $\begin{pmatrix} 3 \\ -2 \\ 4 \end{pmatrix} - \begin{pmatrix} 1 \\ -2 \\ 5 \end{pmatrix} = \begin{pmatrix} 2 \\ 0 \\ -1 \end{pmatrix}$. So, a parametric equation for the plane is: $\mathbf{r} = \begin{pmatrix} 3 \\ -2 \\ 4 \end{pmatrix} + \lambda\begin{pmatrix} 2 \\ 1 \\ 2 \end{pmatrix} + \mu\begin{pmatrix} 2 \\ 0 \\ -1 \end{pmatrix}$.

From vectors parallel to the plane, we can find a vector perpendicular to the plane:

$\begin{pmatrix} 2 \\ 0 \\ -1 \end{pmatrix} \times \begin{pmatrix} 2 \\ 1 \\ 2 \end{pmatrix} = \begin{pmatrix} 1 \\ -6 \\ 2 \end{pmatrix}$. Therefore, a vector equation for the plane is:

$$\begin{pmatrix} 1 \\ -6 \\ 2 \end{pmatrix}\begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} 1 \\ -6 \\ 2 \end{pmatrix}\begin{pmatrix} 3 \\ -2 \\ 4 \end{pmatrix} \Rightarrow \begin{pmatrix} 1 \\ -6 \\ 2 \end{pmatrix}\begin{pmatrix} x \\ y \\ z \end{pmatrix} = 23.$$

A Cartesian equation for the plane is: $x-6y+2z=23$.

9 The plane is parallel to the direction vectors of the lines $\begin{pmatrix} 2 \\ 1 \\ 2 \end{pmatrix}$ and $\begin{pmatrix} 3 \\ 2 \\ 2 \end{pmatrix}$. So, a parametric equation for the plane is: $\mathbf{r} = \begin{pmatrix} 1 \\ -2 \\ 5 \end{pmatrix} + \lambda\begin{pmatrix} 2 \\ 1 \\ 2 \end{pmatrix} + \mu\begin{pmatrix} 3 \\ 2 \\ 2 \end{pmatrix}$.

From vectors parallel to the plane, we can find a vector perpendicular to the plane:

$\begin{pmatrix} 2 \\ 1 \\ 2 \end{pmatrix} \times \begin{pmatrix} 3 \\ 2 \\ 2 \end{pmatrix} = \begin{pmatrix} -2 \\ 2 \\ 1 \end{pmatrix}$. Therefore, a vector equation for the plane is:

$$\begin{pmatrix} -2 \\ 2 \\ 1 \end{pmatrix}\begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} -2 \\ 2 \\ 1 \end{pmatrix}\begin{pmatrix} 1 \\ -2 \\ 5 \end{pmatrix} \Rightarrow \begin{pmatrix} -2 \\ 2 \\ 1 \end{pmatrix}\begin{pmatrix} x \\ y \\ z \end{pmatrix} = -1.$$

A Cartesian equation for the plane is: $-2x+2y+z=-1$.

10 The plane is parallel to a direction vector of the line $\begin{pmatrix} 2 \\ 1 \\ 3 \end{pmatrix}$ and the vector $\begin{pmatrix} 1 \\ -3 \\ 2 \end{pmatrix} - \begin{pmatrix} 0 \\ 2 \\ -1 \end{pmatrix} = \begin{pmatrix} 1 \\ -5 \\ 3 \end{pmatrix}$. So, a parametric equation for the plane is: $\mathbf{r} = \begin{pmatrix} 1 \\ -3 \\ 2 \end{pmatrix} + \lambda \begin{pmatrix} 2 \\ 1 \\ 3 \end{pmatrix} + \mu \begin{pmatrix} 1 \\ -5 \\ 3 \end{pmatrix}$.

From vectors parallel to the plane, we can find a vector perpendicular to the plane:

$\begin{pmatrix} 2 \\ 1 \\ 3 \end{pmatrix} \times \begin{pmatrix} 1 \\ -5 \\ 3 \end{pmatrix} = \begin{pmatrix} 18 \\ -3 \\ -11 \end{pmatrix}$. Therefore, a vector equation for the plane is:

$$\begin{pmatrix} 18 \\ -3 \\ -11 \end{pmatrix}\begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} 18 \\ -3 \\ -11 \end{pmatrix}\begin{pmatrix} 1 \\ -3 \\ 2 \end{pmatrix} \Rightarrow \begin{pmatrix} 18 \\ -3 \\ -11 \end{pmatrix}\begin{pmatrix} x \\ y \\ z \end{pmatrix} = 5.$$

A Cartesian equation for the plane is: $18x - 3y - 11z = 5$.

11 Vector $\overrightarrow{OM} = \begin{pmatrix} p \\ q \\ r \end{pmatrix}$; hence, a Cartesian equation for the plane is:

$$p(x-p) + q(y-q) + r(z-r) = 0 \Rightarrow px + qy + rz = p^2 + q^2 + r^2$$

A vector equation for the plane is:

$$\begin{pmatrix} p \\ q \\ r \end{pmatrix}\begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} p \\ q \\ r \end{pmatrix}\begin{pmatrix} p \\ q \\ r \end{pmatrix} \Rightarrow \begin{pmatrix} p \\ q \\ r \end{pmatrix}\begin{pmatrix} x \\ y \\ z \end{pmatrix} = p^2 + q^2 + r^2$$

12 The plane is parallel to vectors $\begin{pmatrix} 3-1 \\ -1-2 \\ 0-2 \end{pmatrix} = \begin{pmatrix} 2 \\ -3 \\ -2 \end{pmatrix}$ and $\begin{pmatrix} 7-3 \\ 0+1 \\ -2-0 \end{pmatrix} = \begin{pmatrix} 4 \\ 1 \\ -2 \end{pmatrix}$. So, a parametric equation for the plane is: $\mathbf{r} = \begin{pmatrix} 1 \\ 2 \\ 2 \end{pmatrix} + \lambda \begin{pmatrix} 2 \\ -3 \\ -2 \end{pmatrix} + \mu \begin{pmatrix} 4 \\ 1 \\ -2 \end{pmatrix}$.

From vectors parallel to the plane, we can find a vector perpendicular to the plane:

$\begin{pmatrix} 2 \\ -3 \\ -2 \end{pmatrix} \times \begin{pmatrix} 4 \\ 1 \\ -2 \end{pmatrix} = \begin{pmatrix} 8 \\ -4 \\ 14 \end{pmatrix} = 2\begin{pmatrix} 4 \\ -2 \\ 7 \end{pmatrix}$. Therefore, a vector equation for the plane is:

$$\begin{pmatrix} 4 \\ -2 \\ 7 \end{pmatrix}\begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} 4 \\ -2 \\ 7 \end{pmatrix}\begin{pmatrix} 1 \\ 2 \\ 2 \end{pmatrix} \Rightarrow \begin{pmatrix} 4 \\ -2 \\ 7 \end{pmatrix}\begin{pmatrix} x \\ y \\ z \end{pmatrix} = 14.$$

A Cartesian equation for the plane is: $4x - 2y + 7z = 14$.

13 The plane is parallel to vectors $\begin{pmatrix} 3-2 \\ -1+2 \\ 3+2 \end{pmatrix} = \begin{pmatrix} 1 \\ 1 \\ 5 \end{pmatrix}$ and $\begin{pmatrix} 0-3 \\ 1+1 \\ 5-3 \end{pmatrix} = \begin{pmatrix} -3 \\ 2 \\ 2 \end{pmatrix}$. So, a parametric equation for the plane is: $\mathbf{r} = \begin{pmatrix} 2 \\ -2 \\ -2 \end{pmatrix} + \lambda \begin{pmatrix} 1 \\ 1 \\ 5 \end{pmatrix} + \mu \begin{pmatrix} -3 \\ 2 \\ 2 \end{pmatrix}$.

From vectors parallel to the plane, we can find a vector perpendicular to the plane:

$\begin{pmatrix} 1 \\ 1 \\ 5 \end{pmatrix} \times \begin{pmatrix} -3 \\ 2 \\ 2 \end{pmatrix} = \begin{pmatrix} -8 \\ -17 \\ 5 \end{pmatrix} = -1 \begin{pmatrix} 8 \\ 17 \\ -5 \end{pmatrix}$. Therefore, a vector equation for the plane is:

$\begin{pmatrix} 8 \\ 17 \\ -5 \end{pmatrix} \begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} 8 \\ 17 \\ -5 \end{pmatrix} \begin{pmatrix} 2 \\ -2 \\ -2 \end{pmatrix} \Rightarrow \begin{pmatrix} 8 \\ 17 \\ -5 \end{pmatrix} \begin{pmatrix} x \\ y \\ z \end{pmatrix} = -8.$

A Cartesian equation for the plane is: $8x + 17y - 5z = -8$.

14 The plane is parallel to a direction vector of the line $\begin{pmatrix} 1 \\ 1 \\ 3 \end{pmatrix}$ and the vector $\begin{pmatrix} 2 \\ -1 \\ 5 \end{pmatrix} - \begin{pmatrix} 1 \\ -2 \\ 3 \end{pmatrix} = \begin{pmatrix} 1 \\ 1 \\ 2 \end{pmatrix}$. So, a parametric equation for the plane is: $\mathbf{r} = \begin{pmatrix} 1 \\ -2 \\ 3 \end{pmatrix} + \lambda \begin{pmatrix} 1 \\ 1 \\ 3 \end{pmatrix} + \mu \begin{pmatrix} 1 \\ 1 \\ 2 \end{pmatrix}$.

From vectors parallel to the plane, we can find a vector perpendicular to the plane:

$\begin{pmatrix} 1 \\ 1 \\ 2 \end{pmatrix} \times \begin{pmatrix} 1 \\ 1 \\ 3 \end{pmatrix} = \begin{pmatrix} 1 \\ -1 \\ 0 \end{pmatrix}$. Therefore, a vector equation for the plane is:

$\begin{pmatrix} 1 \\ -1 \\ 0 \end{pmatrix} \begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} 1 \\ -1 \\ 0 \end{pmatrix} \begin{pmatrix} 1 \\ -2 \\ 3 \end{pmatrix} \Rightarrow \begin{pmatrix} 1 \\ -1 \\ 0 \end{pmatrix} \begin{pmatrix} x \\ y \\ z \end{pmatrix} = 3.$

A Cartesian equation for the plane is: $x - y = 3$.

15 The plane is parallel to a direction vector of the line $\begin{pmatrix} 3 \\ 2 \\ 4 \end{pmatrix}$ and the vector $\begin{pmatrix} -3 \\ 4 \\ 0 \end{pmatrix} - \begin{pmatrix} 1 \\ -1 \\ 5 \end{pmatrix} = \begin{pmatrix} -4 \\ 5 \\ -5 \end{pmatrix}$. So, a parametric equation for the plane is: $\mathbf{r} = \begin{pmatrix} 1 \\ -1 \\ 5 \end{pmatrix} + \lambda \begin{pmatrix} -4 \\ 5 \\ -5 \end{pmatrix} + \mu \begin{pmatrix} 3 \\ 2 \\ 4 \end{pmatrix}$.

From vectors parallel to the plane, we can find a vector perpendicular to the plane:

$\begin{pmatrix} -4 \\ 5 \\ -5 \end{pmatrix} \times \begin{pmatrix} 3 \\ 2 \\ 4 \end{pmatrix} = \begin{pmatrix} 30 \\ 1 \\ -23 \end{pmatrix}$. Therefore, a vector equation for the plane is:

$$\begin{pmatrix}30\\1\\-23\end{pmatrix}\begin{pmatrix}x\\y\\z\end{pmatrix}=\begin{pmatrix}30\\1\\-23\end{pmatrix}\begin{pmatrix}1\\-1\\5\end{pmatrix}\Rightarrow\begin{pmatrix}30\\1\\-23\end{pmatrix}\begin{pmatrix}x\\y\\z\end{pmatrix}=-86.$$

A Cartesian equation for the plane is: $30x + y - 23z = -86$.

16 The plane is parallel to the direction vectors of the lines $\begin{pmatrix}1\\0\\1\end{pmatrix}$ and $\begin{pmatrix}1\\-1\\1\end{pmatrix}$. So, a parametric equation for the plane is: $\mathbf{r}=\begin{pmatrix}1\\1\\0\end{pmatrix}+\lambda\begin{pmatrix}1\\0\\1\end{pmatrix}+\mu\begin{pmatrix}1\\-1\\1\end{pmatrix}$.

From vectors parallel to the plane, we can find a vector perpendicular to the plane:

$\begin{pmatrix}1\\0\\1\end{pmatrix}\times\begin{pmatrix}1\\-1\\1\end{pmatrix}=\begin{pmatrix}1\\0\\-1\end{pmatrix}$. Therefore, a vector equation for the plane is:

$$\begin{pmatrix}1\\0\\-1\end{pmatrix}\begin{pmatrix}x\\y\\z\end{pmatrix}=\begin{pmatrix}1\\0\\-1\end{pmatrix}\begin{pmatrix}1\\1\\0\end{pmatrix}\Rightarrow\begin{pmatrix}1\\0\\-1\end{pmatrix}\begin{pmatrix}x\\y\\z\end{pmatrix}=1.$$

A Cartesian equation for the plane is: $x - z = 1$.

17 The angle between the normals is:

$$\cos\theta_1=\frac{\begin{pmatrix}3\\4\\-1\end{pmatrix}\begin{pmatrix}1\\-2\\0\end{pmatrix}}{\sqrt{9+16+1}\sqrt{1+4}}=\frac{-5}{\sqrt{26\cdot 5}}$$

Hence, the angle between the planes is: $\cos\theta=\frac{5}{\sqrt{26\cdot 5}}\Rightarrow\theta=63.98...^\circ\approx 64.0^\circ$.

18 The angle between the normals is:

$$\cos\theta_1=\frac{\begin{pmatrix}4\\-7\\1\end{pmatrix}\begin{pmatrix}3\\2\\2\end{pmatrix}}{\sqrt{16+49+1}\sqrt{9+4+4}}=\frac{0}{\sqrt{66\cdot 17}}$$

Hence, the angle between the planes is: $\cos\theta=0\Rightarrow\theta=90^\circ$.

19 The angle between the normals is

$$\cos\theta_1=\frac{\begin{pmatrix}1\\0\\0\end{pmatrix}\begin{pmatrix}1\\0\\1\end{pmatrix}}{\sqrt{1}\sqrt{1+1}}=\frac{1}{\sqrt{2}}$$

Hence, the angle between the planes is: $\cos\theta=\frac{1}{\sqrt{2}}\Rightarrow\theta=45^\circ$.

20 For the angle between the normal and the direction line, it holds:

$$\cos\theta_1 = \frac{\begin{pmatrix}1\\-2\\2\end{pmatrix}\begin{pmatrix}-6\\3\\-2\end{pmatrix}}{\sqrt{1+4+4}\sqrt{36+9+4}} = \frac{-16}{3\cdot 7}$$

Hence, the angle between the plane and the line is: $\sin\theta = \frac{16}{21} \Rightarrow \theta = 49.6324... \approx 49.6^\circ$.

21 For the angle between the normal and the direction line, it holds:

$$\cos\theta_1 = \frac{\begin{pmatrix}3\\0\\-1\end{pmatrix}\begin{pmatrix}-1\\2\\-1\end{pmatrix}}{\sqrt{9+1}\sqrt{1+4+1}} = \frac{-2}{\sqrt{60}}$$

Hence, the angle between the plane and the line is: $\sin\theta = \frac{2}{\sqrt{60}} \Rightarrow \theta = 14.9632... \approx 15.0^\circ$.

22 The angle between the normals is:

$$\cos\theta_1 = \frac{\begin{pmatrix}1\\1\\1\end{pmatrix}\begin{pmatrix}0\\0\\1\end{pmatrix}}{\sqrt{1+1+1}\sqrt{1}} = \frac{1}{\sqrt{3}}$$

Hence, the angle between the planes is: $\cos\theta = \frac{1}{\sqrt{3}} \Rightarrow \theta = 54.7356...^\circ \approx 54.7^\circ$.

23 Parametric equations of the line are:

$$\begin{cases} x = 5 + l \\ y = -3l \\ z = -2 + 4l \end{cases}$$

A Cartesian equation of the plane is: $x - 3y + 2z = -35$

For the intersection, it holds: $(5 + l) - 3(-3l) + 2(-2 + 4l) = -35 \Rightarrow 18l = -36 \Rightarrow l = -2$

Hence, the point is: $\begin{cases} x = 5 - 2 = 3 \\ y = -3(-2) = 6 \\ z = -2 + 4(-2) = -10 \end{cases}$ $(3, 6, -10)$

24 Parametric equations of the line are:

$$\begin{cases} x = 2 \\ y = 4 - 3\mu \\ z = 3\mu \end{cases}$$

For the intersection, it holds: $4(2) - 2(4 - 3\mu) + 3(3\mu) - 30 = 0 \Rightarrow 15\mu = 30 \Rightarrow \mu = 2$

Hence, the point is: $\begin{cases} x = 2 \\ y = 4 - 3(2) = -2 \\ z = 3(2) = 6 \end{cases}$ $(2, -2, 6)$

25 The direction vector of the line and the normal of the plane are perpendicular (since $\begin{pmatrix}1\\5\\3\end{pmatrix}\begin{pmatrix}2\\-4\\6\end{pmatrix} = 2-20+18=0$); hence, the line and the plane are either parallel or the line is in the plane. Since the point (3, 4, 6) is not on the plane, they are parallel and there is no intersection.

Note: If we solve the system for intersection:

Parametric equations of the line are:

$$\begin{cases} x = 3+t \\ y = 4+5t \\ z = 6+3t \end{cases}$$

A Cartesian equation of the plane is: $2x-4y+6z=5$

For the intersection, it holds: $2(3+t)-4(4+5t)+6(6+3t)=5 \Rightarrow 26=5$; hence, no intersection.

26 The direction vector of the line and the normal of the plane are perpendicular (since $\begin{pmatrix}1\\-\frac{1}{3}\\-\frac{5}{3}\end{pmatrix}\begin{pmatrix}3\\-1\\2\end{pmatrix} = 3+\frac{1}{3}-\frac{10}{3}=0$); hence, the line and the plane are either parallel or the line is in the plane. Since the point (0, 4, 5) is on the plane, the line is in the plane.

Note: If we solve the system for intersection:

For the intersection, it holds: $3(t)-\left(4-\frac{1}{3}t\right)+2\left(5-\frac{5}{3}t\right)=6 \Rightarrow -4+10=6$; hence, all points from the line are on the plane.

27 Solving the system:

$$\begin{cases} x = 10 \\ x+y+z=3 \Rightarrow 10+y+z=3 \Rightarrow y=-7-y \end{cases}$$

Hence, parametric equations of the line of intersection are:

$$\begin{cases} x = 10 \\ y = -7-t \\ z = t \end{cases}$$

and the vector equation is $\mathbf{r} = \begin{pmatrix}10\\-7\\0\end{pmatrix} + t\begin{pmatrix}0\\-1\\1\end{pmatrix}$.

28 We have to solve the system:

$\begin{cases} 2x-y+z=5 \\ x+y-z=4 \end{cases}$. Adding the equations, we have: $3x=9 \Rightarrow x=3$; therefore: $\begin{cases} -y+z=-1 \\ y-z=1 \end{cases} \Rightarrow y=1+z$

Hence, parametric equations of the line of intersection are:

$$\begin{cases} x = 3 \\ y = 1+t \\ z = t \end{cases}$$

and the vector equation is $\mathbf{r} = \begin{pmatrix}3\\1\\0\end{pmatrix} + t\begin{pmatrix}0\\1\\1\end{pmatrix}$.

29 A Cartesian equation of the first line is: $x - y - 2z = 1$, so the planes are parallel and there is no intersection.

Note: If we solve the system for intersection, we will have:

$\begin{cases} x - y - 2z = 1 \\ x - y - 2z = 5 \end{cases}$, and this is obviously inconsistent.

30 A vector perpendicular to the first plane is:

$\begin{pmatrix} 1 \\ -2 \\ 0 \end{pmatrix} \times \begin{pmatrix} 3 \\ 2 \\ -8 \end{pmatrix} = \begin{pmatrix} 16 \\ 8 \\ 8 \end{pmatrix} = 8\begin{pmatrix} 2 \\ 1 \\ 1 \end{pmatrix}$; hence, a Cartesian equation of the plane is:

$2(x-1) + (y-0) + (z-2) = 0 \Rightarrow 2x + y + z = 4.$

Now, we have to solve the system:

$\begin{cases} 2x + y + z = 4 \\ 3x - y - z = 3 \end{cases}$. Adding the equations, we have: $5x = 7 \Rightarrow x = \frac{7}{5}$; therefore:

$$\begin{cases} 2\left(\frac{7}{5}\right) + y + z = 4 \Rightarrow y + z = \frac{6}{5} \\ 3\left(\frac{7}{5}\right) - y - z = 3 \Rightarrow -y - z = -\frac{6}{5} \end{cases} \Rightarrow y = \frac{6}{5} - z$$

Hence, parametric equations of the line of intersection are:

$$\begin{cases} x = \frac{7}{5} \\ y = \frac{6}{5} - t \\ z = t \end{cases}$$

and the vector equation is $\mathbf{r} = \begin{pmatrix} \frac{7}{5} \\ \frac{6}{5} \\ 0 \end{pmatrix} + t\begin{pmatrix} 0 \\ -1 \\ 1 \end{pmatrix}$.

31 A direction vector of the line is perpendicular to the normals of both planes:

$\begin{pmatrix} 2 \\ 1 \\ -1 \end{pmatrix} \times \begin{pmatrix} 1 \\ 2 \\ 1 \end{pmatrix} = \begin{pmatrix} 3 \\ -3 \\ 3 \end{pmatrix} = 3\begin{pmatrix} 1 \\ -1 \\ 1 \end{pmatrix}$. Hence, the normal of the plane is $\begin{pmatrix} 1 \\ -1 \\ 1 \end{pmatrix}$, and a vector equation for the plane is:

$$\begin{pmatrix} 1 \\ -1 \\ 1 \end{pmatrix}\begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} 1 \\ -1 \\ 1 \end{pmatrix}\begin{pmatrix} 2 \\ 1 \\ -1 \end{pmatrix} \Rightarrow \begin{pmatrix} 1 \\ -1 \\ 1 \end{pmatrix}\begin{pmatrix} x \\ y \\ z \end{pmatrix} = 0.$$

A Cartesian equation of the plane is: $x - y + z = 0$.

32 A normal to the plane is perpendicular to the vector $\overrightarrow{AB}$ and the direction vector of the given plane:

$$\begin{pmatrix} 3-1 \\ 2-2 \\ 1-3 \end{pmatrix} \times \begin{pmatrix} 4 \\ -1 \\ 2 \end{pmatrix} = \begin{pmatrix} 2 \\ 0 \\ -2 \end{pmatrix} \times \begin{pmatrix} 4 \\ -1 \\ 2 \end{pmatrix} = \begin{pmatrix} -2 \\ -12 \\ -2 \end{pmatrix} = -2\begin{pmatrix} 1 \\ 6 \\ 1 \end{pmatrix}$$

. Therefore, a vector equation for the plane is:

$$\begin{pmatrix} 1 \\ 6 \\ 1 \end{pmatrix} \times \begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} 1 \\ 6 \\ 1 \end{pmatrix}\begin{pmatrix} 1 \\ 2 \\ 3 \end{pmatrix} \Rightarrow \begin{pmatrix} 1 \\ 6 \\ 1 \end{pmatrix}\begin{pmatrix} x \\ y \\ z \end{pmatrix} = 16.$$

A Cartesian equation of the plane is: $x + 6y + z = 16$.

33 A Cartesian equation of the line is:

$$\frac{x-1}{3-1} = \frac{y-2}{1-2} = \frac{z-5}{1-5} \Rightarrow \frac{x-1}{2} = \frac{y-2}{-1} = \frac{z-5}{-4}$$

; hence, parametric equations of the line are:

$$\begin{cases} x = 1 + 2t \\ y = 2 - t \\ z = 5 - 4t \end{cases}$$

The distance from $(2, -1, 5)$ to a point on the line is:

$d^2 = (1 + 2t - 2)^2 + (2 - t + 1)^2 + (5 - 4t - 5)^2 = 21t^2 - 10t + 10$. The distance is a minimum when:

$$\frac{d\left(21t^2 - 10t + 10\right)}{dt} = 0 \Rightarrow 42t - 10 = 0 \Rightarrow t = \frac{5}{21}$$

Hence, the point on the line with $t = \frac{5}{21}$ is the closest to the origin.

$$\begin{cases} x = 1 + 2\left(\frac{5}{21}\right) \\ y = 2 - \left(\frac{5}{21}\right), \quad A\left(\frac{31}{21}, \frac{37}{21}, \frac{85}{21}\right) \\ z = 5 - 4\left(\frac{5}{21}\right) \end{cases}$$

34 A normal to the plane is perpendicular to the direction vectors of both lines:

$$\begin{pmatrix} 1 \\ -2 \\ 1 \end{pmatrix} \times \begin{pmatrix} 3 \\ 2 \\ 4 \end{pmatrix} = \begin{pmatrix} -10 \\ -1 \\ 8 \end{pmatrix} = -\begin{pmatrix} 10 \\ 1 \\ -8 \end{pmatrix}$$

. Therefore, the plane contains the point $(-1, 2, 3)$ and is perpendicular to the vector $\begin{pmatrix} 10 \\ 1 \\ -8 \end{pmatrix}$; hence, its vector equation is:

$$\begin{pmatrix} 10 \\ 1 \\ -8 \end{pmatrix}\begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} 10 \\ 1 \\ -8 \end{pmatrix}\begin{pmatrix} -1 \\ 2 \\ 3 \end{pmatrix} \Rightarrow \begin{pmatrix} 10 \\ 1 \\ -8 \end{pmatrix}\begin{pmatrix} x \\ y \\ z \end{pmatrix} = -32,$$

and Cartesian:

$10x + y - 8z = -32$.

35 The plane contains points of the first line, so the point (1, 1, 2) is on the plane.

A normal to the plane is perpendicular to the direction vectors of both lines:

$$\begin{pmatrix} 2 \\ 2 \\ -1 \end{pmatrix} \times \begin{pmatrix} 1 \\ 2 \\ 1 \end{pmatrix} = \begin{pmatrix} 4 \\ -3 \\ 2 \end{pmatrix}$$. Therefore, a vector equation for the plane is:

$$\begin{pmatrix} 4 \\ -3 \\ 2 \end{pmatrix} \times \begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} 4 \\ -3 \\ 2 \end{pmatrix}\begin{pmatrix} 1 \\ 1 \\ 2 \end{pmatrix} \Rightarrow \begin{pmatrix} 4 \\ -3 \\ 2 \end{pmatrix}\begin{pmatrix} x \\ y \\ z \end{pmatrix} = 5$$, and Cartesian:

$4x - 3y + 2z = 5.$

36 The equation $\frac{x}{A} + \frac{y}{B} + \frac{z}{C} = 1$ can be written in the form $BCx + ACy + ABz = ABC$ and this is a Cartesian equation of the plane whose normal is vector $\begin{pmatrix} BC \\ AC \\ AB \end{pmatrix}$ and which contains the point $(A, 0, 0)$.

Note: The plane contains the points $(A, 0, 0)$, $(0, B, 0)$ and $(0, 0, C)$.

37 It is easier to write parametric equations of the plane, since we know one point and two non-parallel direction vectors:

$$\begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} 4 \\ -3 \\ -1 \end{pmatrix} + r\begin{pmatrix} 2 \\ -3 \\ 4 \end{pmatrix} + s\begin{pmatrix} 4 \\ 0 \\ -3 \end{pmatrix}$$

Note: If we need another form of the line, we can proceed as follows:

The normals of both planes are parallel to our plane. Hence, their vector product is normal to our plane:

$$\begin{pmatrix} 2 \\ -3 \\ 4 \end{pmatrix} \times \begin{pmatrix} 4 \\ 0 \\ -3 \end{pmatrix} = \begin{pmatrix} 9 \\ 22 \\ 12 \end{pmatrix}$$. Therefore, a vector equation for the plane is:

$$\begin{pmatrix} 9 \\ 22 \\ 12 \end{pmatrix} \times \begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} 9 \\ 22 \\ 12 \end{pmatrix}\begin{pmatrix} 4 \\ -3 \\ -1 \end{pmatrix} \Rightarrow \begin{pmatrix} 9 \\ 22 \\ 12 \end{pmatrix}\begin{pmatrix} x \\ y \\ z \end{pmatrix} = -42$$, and Cartesian:

$9x + 22y + 12z = -42.$

38 It is easier to write parametric equations of the plane, since we know one point and two non-parallel direction vectors (a direction vector of the line and normal of the plane):

$$\begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} 2 \\ 3 \\ 0 \end{pmatrix} + r\begin{pmatrix} 1 \\ -2 \\ 1 \end{pmatrix} + s\begin{pmatrix} 2 \\ -3 \\ 4 \end{pmatrix}$$

Note: If we need another form of the line, we can proceed as follows:

The normal of the plane and the direction vector of the line are parallel to our plane. Hence, their vector product is normal to our plane:

$$\begin{pmatrix} 2 \\ -3 \\ 4 \end{pmatrix} \times \begin{pmatrix} 1 \\ -2 \\ 1 \end{pmatrix} = \begin{pmatrix} 5 \\ 2 \\ -1 \end{pmatrix}$$. Therefore, a vector equation for the plane is:

$$\begin{pmatrix}5\\2\\-1\end{pmatrix}\times\begin{pmatrix}x\\y\\z\end{pmatrix}=\begin{pmatrix}5\\2\\-1\end{pmatrix}\begin{pmatrix}2\\3\\0\end{pmatrix}\Rightarrow\begin{pmatrix}5\\2\\-1\end{pmatrix}\begin{pmatrix}x\\y\\z\end{pmatrix}=16, \text{ and Cartesian:}$$

$5x + 2y - z = 16.$

Review exercise

1 Two vectors are parallel if their components (coordinates) are in proportion. Hence, we can find the ratio of the first coordinates, then of the second coordinates and the third. If all three ratios are the same, the vectors are parallel. We can check for parallelism by solving for constant k such that one is k times the other.

Two vectors are perpendicular if their scalar product is zero.

2 We will find their scalar triple product (hence the determinant of the coordinates of the vectors). If the result is zero, then the vectors are coplanar.

3 We have to find the scalar product of the vectors and their magnitudes, and then use the formula for the cosine of their angle: $\cos\theta = \dfrac{\mathbf{a}\cdot\mathbf{b}}{|\mathbf{a}|\,|\mathbf{b}|}$.

4 For an equation of a line, we need a point from the line and a direction vector, which we use to form an equation.

5 For an equation of a plane, we need a point from the plane and a vector perpendicular to the plane, which we use to form a Cartesian equation of the plane.

If a point and two vectors parallel to the plane are given, we can use the parametric equation of the plane. If we need a Cartesian equation, we can find the vector product of the direction vectors. This vector product will be a normal to the plane, and thus we can find a Cartesian equation of the plane.

6 We can determine the angle between their normals (using scalar product). If this angle is acute, or a right angle, then it is the same as the angle between the planes. If the angle is obtuse, then the angle between the planes is 180° minus the obtained angle.

7 We can determine the angle between a normal of the plane and a direction vector of the line (using scalar product). If this angle is acute, or a right angle, then the angle between the plane and the line is 90° minus the obtained angle. If the angle is obtuse, then the angle between the planes is the obtained angle minus 90°.

8 A vector equation: $\mathbf{r}=\begin{pmatrix}4\\-3\\0\end{pmatrix}+t\begin{pmatrix}1\\2\\1\end{pmatrix}$

Parametric equations: $\begin{cases}x = 4+t\\ y=-3+2t\\ z=t\end{cases}$

A Cartesian equation: $\dfrac{x-4}{1}=\dfrac{y+3}{2}=\dfrac{z}{1}$

9 A direction vector of the line is: $\begin{pmatrix}4+1\\6-1\\-1-4\end{pmatrix}=\begin{pmatrix}5\\5\\-5\end{pmatrix}=5\begin{pmatrix}1\\1\\-1\end{pmatrix}$

A vector equation: $\mathbf{r} = \begin{pmatrix} -1 \\ 1 \\ 4 \end{pmatrix} + t\begin{pmatrix} 1 \\ 1 \\ -1 \end{pmatrix}$

Parametric equations: $\begin{cases} x = -1 + t \\ y = 1 + t \\ z = 4 - t \end{cases}$

A Cartesian equation: $\dfrac{x+1}{1} = \dfrac{y-1}{1} = \dfrac{z-4}{-1} \Rightarrow x + 1 = y - 1 = 4 - z$

10 A direction vector of the line is: $\begin{pmatrix} 0-2 \\ 1-3 \\ 2-0 \end{pmatrix} = \begin{pmatrix} -2 \\ -2 \\ 2 \end{pmatrix} = -2\begin{pmatrix} 1 \\ 1 \\ -1 \end{pmatrix}$

A vector equation: $\mathbf{r} = \begin{pmatrix} 2 \\ 3 \\ 0 \end{pmatrix} + t\begin{pmatrix} 1 \\ 1 \\ -1 \end{pmatrix}$

Parametric equations: $\begin{cases} x = 2 + t \\ y = 3 + t \\ z = -t \end{cases}$

A Cartesian equation: $\dfrac{x-2}{1} = \dfrac{y-3}{1} = \dfrac{z}{-1} \Rightarrow x - 2 = y - 3 = -z$

11 A vector equation: $\mathbf{r} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix} + t\begin{pmatrix} 0 \\ 1 \\ 2 \end{pmatrix} \Rightarrow \mathbf{r} = t\begin{pmatrix} 0 \\ 1 \\ 2 \end{pmatrix}$

Parametric equations: $\begin{cases} x = 0 \\ y = t \\ z = 2t \end{cases}$

A Cartesian equation: $\dfrac{x}{0} = \dfrac{y}{1} = \dfrac{z}{2}$

Note: Usually we say that this line does not have a Cartesian equation, but sometimes we can write zero in the denominator and keep in mind what that means.

12 A direction vector of the line is the same as the direction vector of the parallel line: $\begin{pmatrix} 3 \\ -1 \\ 4 \end{pmatrix}$

A vector equation: $\mathbf{r} = \begin{pmatrix} 4 \\ -1 \\ 2 \end{pmatrix} + t\begin{pmatrix} 3 \\ -1 \\ 4 \end{pmatrix}$

Parametric equations: $\begin{cases} x = 4 + 3t \\ y = -1 - t \\ z = 2 + 4t \end{cases}$

A Cartesian equation: $\dfrac{x-4}{3} = \dfrac{y+1}{-1} = \dfrac{z-2}{4}$

13 If it is parallel to the y-axis, its direction vector could be $\begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix}$.

A vector equation: $\mathbf{r} = \begin{pmatrix} 1 \\ 2 \\ 2 \end{pmatrix} + t\begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix}$

Parametric equations: $\begin{cases} x = 1 \\ y = 2 + t \\ z = 2 \end{cases}$

A Cartesian equation: $\dfrac{x-1}{0} = \dfrac{y-2}{1} = \dfrac{z-2}{0}$

Note: Usually we say that this line does not have a Cartesian equation, but sometimes we can write zero in the denominator and keep in mind what that means.

14 The line is parallel to the normal to the plane; hence, its direction vector is $\begin{pmatrix} 4 \\ -8 \\ 7 \end{pmatrix}$.

A vector equation: $\mathbf{r} = \begin{pmatrix} 3 \\ 5 \\ 6 \end{pmatrix} + t\begin{pmatrix} 4 \\ -8 \\ 7 \end{pmatrix}$

Parametric equations: $\begin{cases} x = 3 + 4t \\ y = 5 - 8t \\ z = 6 + 7t \end{cases}$

A Cartesian equation: $\dfrac{x-3}{4} = \dfrac{y-5}{-8} = \dfrac{z-6}{7}$

15 The line is parallel to their vector product:

$\begin{pmatrix} 2 \\ 3 \\ 4 \end{pmatrix} \times \begin{pmatrix} 4 \\ 5 \\ 6 \end{pmatrix} = \begin{pmatrix} -2 \\ 4 \\ -2 \end{pmatrix} = -2\begin{pmatrix} 1 \\ -2 \\ 1 \end{pmatrix}$

A vector equation: $\mathbf{r} = \begin{pmatrix} 3 \\ 5 \\ 6 \end{pmatrix} + t\begin{pmatrix} 1 \\ -2 \\ 1 \end{pmatrix}$

Parametric equations: $\begin{cases} x = 3 + t \\ y = 5 - 2t \\ z = 6 + t \end{cases}$

A Cartesian equation: $\dfrac{x-3}{1} = \dfrac{y-5}{-2} = \dfrac{z-6}{1}$

16 So, a Cartesian equation for the plane is: $4(x-1) - 1(y-3) + (z-0) = 0 \Rightarrow 4x - y + z = 1$

A vector equation for the plane is: $\begin{pmatrix} 4 \\ -1 \\ 1 \end{pmatrix}\begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} 4 \\ -1 \\ 1 \end{pmatrix}\begin{pmatrix} 1 \\ 3 \\ 0 \end{pmatrix} \Rightarrow \begin{pmatrix} 4 \\ -1 \\ 1 \end{pmatrix}\begin{pmatrix} x \\ y \\ z \end{pmatrix} = 1$

17 A normal to the plane is the same as the normal to the parallel plane: $\begin{pmatrix}2\\3\\-1\end{pmatrix}$

A Cartesian equation for the plane is: $2(x-2)+3(y-0)-(z-5)=0 \Rightarrow 2x+3y-z=-1$

A vector equation for the plane is: $\begin{pmatrix}2\\3\\-1\end{pmatrix}\begin{pmatrix}x\\y\\z\end{pmatrix}=\begin{pmatrix}2\\3\\-1\end{pmatrix}\begin{pmatrix}2\\0\\5\end{pmatrix}\Rightarrow\begin{pmatrix}2\\3\\-1\end{pmatrix}\begin{pmatrix}x\\y\\z\end{pmatrix}=-1$

18 Vectors in the plane are: $\begin{pmatrix}3-2\\-1-1\\0+1\end{pmatrix}=\begin{pmatrix}1\\-2\\1\end{pmatrix}$ and $\begin{pmatrix}1-2\\-1-1\\2+1\end{pmatrix}=\begin{pmatrix}-1\\-2\\3\end{pmatrix}$. Hence, a vector perpendicular to the plane is: $\begin{pmatrix}1\\-2\\1\end{pmatrix}\times\begin{pmatrix}-1\\-2\\3\end{pmatrix}=\begin{pmatrix}-4\\-4\\-4\end{pmatrix}=-4\begin{pmatrix}1\\1\\1\end{pmatrix}$

A Cartesian equation for the plane is: $(x-2)+(y-1)+(z+1)=0 \Rightarrow x+y+z=2$
A vector equation for the plane is:

$$\begin{pmatrix}1\\1\\1\end{pmatrix}\begin{pmatrix}x\\y\\z\end{pmatrix}=\begin{pmatrix}1\\1\\1\end{pmatrix}\begin{pmatrix}2\\1\\-1\end{pmatrix}\Rightarrow\begin{pmatrix}1\\1\\1\end{pmatrix}\begin{pmatrix}x\\y\\z\end{pmatrix}=2$$

19 A direction vector of the line is normal to the plane; hence, a normal is: $\begin{pmatrix}3\\-1\\4\end{pmatrix}$

A Cartesian equation for the plane is: $3(x-2)-(y-5)+4(z-4)=0 \Rightarrow 3x-y+4z=17$
A vector equation for the plane is:

$$\begin{pmatrix}3\\-1\\4\end{pmatrix}\begin{pmatrix}x\\y\\z\end{pmatrix}=\begin{pmatrix}3\\-1\\4\end{pmatrix}\begin{pmatrix}2\\5\\4\end{pmatrix}\Rightarrow\begin{pmatrix}3\\-1\\4\end{pmatrix}\begin{pmatrix}x\\y\\z\end{pmatrix}=17$$

20 A vector perpendicular to the plane is: $\begin{pmatrix}2\\-1\\2\end{pmatrix}$

A Cartesian equation for the plane is: $2(x-2)-(y+1)+2(z-2)=0 \Rightarrow 2x-y+2z=9$

A vector equation for the plane is:

$$\begin{pmatrix}2\\-1\\2\end{pmatrix}\begin{pmatrix}x\\y\\z\end{pmatrix}=\begin{pmatrix}2\\-1\\2\end{pmatrix}\begin{pmatrix}2\\-1\\2\end{pmatrix}\Rightarrow\begin{pmatrix}2\\-1\\2\end{pmatrix}\begin{pmatrix}x\\y\\z\end{pmatrix}=9$$

21 Parametric equations of the first line are: $\begin{cases}x=1+2t\\y=2+3t\\z=3+4t\end{cases}$. Hence, we have to solve the system:

$$\begin{cases}x=1+2t=2+s\\y=2+3t=4+2s\\z=3+4t=-4s-1\end{cases}\Rightarrow\begin{cases}2t-s=1\\3t-2s=2\\4t+4s=-4\end{cases}\Rightarrow s=2t-1\Rightarrow 2t-4t+2=2\Rightarrow t=0,\ s=-1$$

$$\Rightarrow 0-4=-4$$

Therefore, the point of intersection is: $(1, 2, 3)$.

22 A vector perpendicular to the plane is: $\begin{pmatrix} 2 \\ 3 \\ 4 \end{pmatrix} \times \begin{pmatrix} 1 \\ 2 \\ -4 \end{pmatrix} = \begin{pmatrix} -20 \\ 12 \\ 1 \end{pmatrix}$

A Cartesian equation for the plane is: $-20(x-1)+12(y-2)+(z-3)=0 \Rightarrow -20x+12y+z=7$

A vector equation for the plane is:

$$\begin{pmatrix} -20 \\ 12 \\ 1 \end{pmatrix}\begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} -20 \\ 12 \\ 1 \end{pmatrix}\begin{pmatrix} 1 \\ 2 \\ 3 \end{pmatrix} \Rightarrow \begin{pmatrix} -20 \\ 12 \\ 1 \end{pmatrix}\begin{pmatrix} x \\ y \\ z \end{pmatrix} = 7$$

23 Parametric equations of the lines are:

$\begin{cases} x = t \\ y = 2-t \\ z = 1+t \end{cases}$ and $\begin{cases} x = 2+2s \\ y = 3+s \\ z = 6+5s \end{cases}$. Hence, we have to solve the system:

$$\begin{cases} x = t = 2+2s \\ y = 2-t = 3+s \\ z = 1+t = 6+5s \end{cases} \begin{matrix} \\ \Rightarrow 2-2-2s = 3+s \Rightarrow s=-1, t=0 \\ \Rightarrow 1+0 = 6-5 \end{matrix}$$

Therefore, the point of intersection is: $(0, 2, 1)$.

24 A vector perpendicular to the plane is: $\begin{pmatrix} 1 \\ -1 \\ 1 \end{pmatrix} \times \begin{pmatrix} 2 \\ 1 \\ 5 \end{pmatrix} = \begin{pmatrix} -6 \\ -3 \\ 3 \end{pmatrix} = -3\begin{pmatrix} 2 \\ 1 \\ -1 \end{pmatrix}$

A Cartesian equation for the plane is: $2(x-0)+(y-2)-(z-1)=0 \Rightarrow 2x+y-z=1$

A vector equation for the plane is:

$$\begin{pmatrix} 2 \\ 1 \\ -1 \end{pmatrix}\begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} 2 \\ 1 \\ -1 \end{pmatrix}\begin{pmatrix} 0 \\ 2 \\ 1 \end{pmatrix} \Rightarrow \begin{pmatrix} 2 \\ 1 \\ -1 \end{pmatrix}\begin{pmatrix} x \\ y \\ z \end{pmatrix} = 1$$

25 A vector perpendicular to the plane is: $\begin{pmatrix} 1 \\ -2 \\ 1 \end{pmatrix}$

A Cartesian equation for the plane is: $(x-1)-2(y+2)+(z-1)=0 \Rightarrow x-2y+z=6$

A vector equation for the plane is:

$$\begin{pmatrix} 1 \\ -2 \\ 1 \end{pmatrix}\begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} 1 \\ -2 \\ 1 \end{pmatrix}\begin{pmatrix} 1 \\ -2 \\ 1 \end{pmatrix} \Rightarrow \begin{pmatrix} 1 \\ -2 \\ 1 \end{pmatrix}\begin{pmatrix} x \\ y \\ z \end{pmatrix} = 6$$

Chapter 15

Exercise 15.1

1 **a)** $y = (3x-8)^4 \Rightarrow \dfrac{dy}{dx} = 4 \times (3x-8)^3 \times 3 = 12(3x-8)^3$

b) $y = \sqrt{1-x} \Rightarrow \dfrac{dy}{dx} = \dfrac{1}{2} \times (1-x)^{-\frac{1}{2}} \times (-1) = \dfrac{-1}{2\sqrt{1-x}}$

c) $y = \sin x \cos x$

We can find the derivative of this function in two different ways. The first method uses the product rule, whilst the second involves rewriting the original function by using the double angle formula and then differentiating it by applying the chain rule. We will demonstrate both methods.

$y = \sin x \cos x \Rightarrow y' = \cos x \times \cos x + \sin x \times (-\sin x) = \cos^2 x - \sin^2 x \, [= \cos 2x]$

$y = \sin x \cos x = \dfrac{1}{2} \times 2 \sin x \cos x = \dfrac{1}{2} \sin 2x \Rightarrow y' = \dfrac{1}{\cancel{2}} \times \cos 2x \times \cancel{2} = \cos 2x$

d) $y = 2 \sin\left(\dfrac{x}{2}\right) \Rightarrow \dfrac{dy}{dx} = \cancel{2} \times \cos\left(\dfrac{x}{2}\right) \times \dfrac{1}{\cancel{2}} = \cos\left(\dfrac{x}{2}\right)$

e) $y = \left(x^2+4\right)^{-2} \Rightarrow \dfrac{dy}{dx} = -2 \times \left(x^2+4\right)^{-3} \times 2x = -4x\left(x^2+4\right)^{-3}$

f) $y = \dfrac{x+1}{x-1} \Rightarrow y' = \dfrac{1 \times (x-1) - (x+1) \times 1}{(x-1)^2} = \dfrac{\cancel{x} - 1 - \cancel{x} - 1}{(x-1)^2} = \dfrac{-2}{(x-1)^2}$

g) $y = \dfrac{1}{\sqrt{x+2}} = (x+2)^{-\frac{1}{2}} \Rightarrow \dfrac{dy}{dx} = -\dfrac{1}{2}(x+2)^{-\frac{3}{2}} = -\dfrac{1}{2(x+2)\sqrt{x+2}}$

h) $y = \cos^2 x \Rightarrow \dfrac{dy}{dx} = 2 \cos x \times (-\sin x) = -2 \cos x \sin x = -\sin 2x$

i) $y = x\sqrt{1-x} \Rightarrow y' = 1 \times \sqrt{1-x} + x \times \dfrac{-1}{2\sqrt{1-x}} = \dfrac{1-x-x}{2\sqrt{1-x}} = \dfrac{1-2x}{2\sqrt{1-x}}$

j) $y = \dfrac{1}{3x^2-5x+7} = \left(3x^2-5x+7\right)^{-1} \Rightarrow \dfrac{dy}{dx} = -1 \times \left(3x^2-5x+7\right)^{-2} \times (6x-5) = \dfrac{5-6x}{\left(3x^2-5x+7\right)^2}$

k) $y = \sqrt[3]{2x+5} = (2x+5)^{\frac{1}{3}} \Rightarrow \dfrac{dy}{dx} = \dfrac{1}{3} \times (2x+5)^{-\frac{2}{3}} \times 2 = \dfrac{2}{3\sqrt[3]{(2x+5)^2}}$

l) $y = (2x-1)^3\left(x^4+1\right) \Rightarrow y' = 3(2x-1)^2 \times 2 \times \left(x^4+1\right) + (2x-1)^3 \times 4x^3$

$= (2x-1)^2\left(6x^4 + 6 + 8x^4 - 4x^3\right) = 2(2x-1)^2\left(7x^4 - 2x^3 + 3\right)$

m) $y = \dfrac{\sin x}{x} \Rightarrow y' = \dfrac{\cos x \times x - \sin x \times 1}{x^2} = \dfrac{x \cos x - \sin x}{x^2}$

n) $y = \dfrac{x^2}{x+2} \Rightarrow y' = \dfrac{2x(x+2) - x^2 \times 1}{(x+2)^2} = \dfrac{x^2+4x}{(x+2)^2}$

o) $y = \sqrt[3]{x^2}\cos x \Rightarrow y' = \dfrac{2}{3}x^{-\frac{1}{3}}\cos x + x^{\frac{2}{3}} \times (-\sin x) = \dfrac{1}{3\sqrt[3]{x}}(2\cos x - 3x\sin x)$

2 **a)** $x = -1 \Rightarrow y = \left(2(-1)^2 - 1\right)^3 = 1 \Rightarrow P(-1, 1)$

$y' = 3 \times \left(2x^2 - 1\right)^2 \times 4x = 12x\left(2x^2-1\right)^2 \Rightarrow m = y'(-1) = -12$

Equation of tangent line: $y = -12(x+1) + 1 \Rightarrow y = -12x - 11$

b) $x = 3 \Rightarrow y = \sqrt{3 \times 3^2 - 2} = 5 \Rightarrow P(3, 5)$

$y' = \dfrac{1}{\not 2}\left(3x^2 - 2\right)^{-\frac{1}{2}} \times \not 6\, 3x = \dfrac{3x}{\sqrt{3x^2-2}} \Rightarrow m = y'(3) = \dfrac{9}{5}$

Equation of tangent line: $y = \dfrac{9}{5}(x-3) + 5 \Rightarrow y = \dfrac{9}{5}x - \dfrac{2}{5}$

c) $x = \pi \Rightarrow y = \sin 2\pi = 0 \Rightarrow P(\pi, 0)$

$y' = \cos 2x \times 2 = 2\cos 2x \Rightarrow m = y'(\pi) = 2\cos 2\pi = 2$

Equation of tangent line: $y = 2(x - \pi) - 0 \Rightarrow y = 2x - 2\pi$

d) $x = 1 \Rightarrow y = \dfrac{1^3 + 1}{2 \times 1} = 1 \Rightarrow P(1, 1)$

There are several ways to determine the numerical derivative at point P. It would be advisable to rewrite the original function as a sum of the powers of x and then differentiate it.

$$y = \frac{x^3+1}{2x} = \frac{1}{2}\left(x^2 + \frac{1}{x}\right) \Rightarrow y' = \frac{1}{2}\left(2x - \frac{1}{x^2}\right) = x - \frac{1}{2x^2} \Rightarrow m = y'(1) = 1 - \frac{1}{2} = \frac{1}{2}$$

Equation of tangent line: $y = \dfrac{1}{2}(x-1) + 1 \Rightarrow y = \dfrac{1}{2}x + \dfrac{1}{2}$

3 **a)** $s(t) = \cos\left(t^2 - 1\right) \Rightarrow v(t) = s'(t) = -\sin\left(t^2-1\right) \times 2t = -2t\sin\left(t^2-1\right)$

b) At $t = 0: v(0) = -2 \times 0 \times \sin\left(0^2 - 1\right) = 0$

c) $v(t) = 0 \Rightarrow -2t\sin\left(t^2-1\right) = 0 \Rightarrow t = 0$ or $\sin\left(t^2-1\right) = 0$. Now we need to find all the values of t, within the given interval, which satisfy the second equation:

$$\sin\left(t^2-1\right) = 0 \Rightarrow \begin{cases} t^2 - 1 = 0 \Rightarrow t = 1 \\ t^2 - 1 = \pi \Rightarrow t = \sqrt{\pi + 1} \approx 2.04 \end{cases}$$

d) When $0 < t < 1 \Rightarrow t^2 - 1 < 0 \Rightarrow \sin\left(t^2-1\right) < 0 \Rightarrow -2t\sin\left(t^2-1\right) > 0$

t	0	$0 < t < 1$	1	$1 < t < 2.04$	2.04	$2.04 < t < 2.5$
$v(t)$	0	positive	0	negative	0	positive
Description	stopped	moving to the right	stopped	moving to the left	stopped	moving to the right

4 $y = \dfrac{2}{x^2 - 8} \Rightarrow y' = 2 \times (-1) \times \left(x^2 - 8\right)^{-2} \times 2x = -\dfrac{4x}{\left(x^2-8\right)^2}$

a) $m_T = y'(3) = -\dfrac{4 \times 3}{\left(3^2 - 8\right)^2} = -12 \Rightarrow$ Equation of tangent: $y = -12(x-3) + 2 \Rightarrow y = -12x + 38$

b) $m_N = -\dfrac{1}{y'(3)} = \dfrac{\left(3^2-8\right)^2}{4 \times 3} = \dfrac{1}{12} \Rightarrow$ Equation of normal: $y = \dfrac{1}{12}(x-3) + 2 \Rightarrow y = \dfrac{1}{12}x + \dfrac{7}{4}$

5 $y = \sqrt{1+4x} \Rightarrow y' = \dfrac{1}{\not{2}\sqrt{1+4x}} \times \not{4}\,2 = \dfrac{2}{\sqrt{1+4x}}$

a) $m_T = y'(2) = \dfrac{2}{\sqrt{1+4\times 2}} = \dfrac{2}{3} \Rightarrow$ Equation of tangent: $y = \dfrac{2}{3}(x-2)+3 \Rightarrow y = \dfrac{2}{3}x + \dfrac{5}{3}$

b) $m_N = -\dfrac{1}{y'(2)} = -\dfrac{\sqrt{1+4\times 2}}{2} = -\dfrac{3}{2} \Rightarrow$ Equation of normal: $y = -\dfrac{3}{2}(x-2)+3 \Rightarrow y = -\dfrac{3}{2}x + 6$

6 $y = \dfrac{x}{(x+1)} \Rightarrow y' = \dfrac{1\times(x+1) - x\times 1}{(x+1)^2} = \dfrac{1}{(x+1)^2}$

a) $m_T = y'(1) = \dfrac{1}{(1+1)^2} = \dfrac{1}{4} \Rightarrow$ Equation of tangent: $y = \dfrac{1}{4}(x-1) + \dfrac{1}{2} \Rightarrow y = \dfrac{1}{4}x + \dfrac{1}{4}$

b) $m_N = -\dfrac{1}{y'(1)} = -4 \Rightarrow$ Equation of normal: $y = -4(x-1) + \dfrac{1}{2} \Rightarrow y = -4x + \dfrac{9}{2}$

7 **a)** $y = \sin\left(2x - \dfrac{\pi}{2}\right) \Rightarrow \dfrac{dy}{dx} = \cos\left(2x - \dfrac{\pi}{2}\right)\times 2 = 2\cos\left(2x - \dfrac{\pi}{2}\right)$

$$\Rightarrow \frac{d^2y}{dx^2} = 2\left(-\sin\left(2x - \frac{\pi}{2}\right)\right)\times 2 = -4\sin\left(2x - \frac{\pi}{2}\right)$$

b) $\dfrac{d^2y}{dx^2} = 0 \Rightarrow -4\sin\left(2x - \dfrac{\pi}{2}\right) = 0 \Rightarrow \begin{cases} 2x - \dfrac{\pi}{2} = 0 \Rightarrow x = \dfrac{\pi}{4} \Rightarrow y = \sin(0) = 0 \\ 2x - \dfrac{\pi}{2} = \pi \Rightarrow x = \dfrac{3\pi}{4} \Rightarrow y = \sin(\pi) = 0 \end{cases}$

So, the points of inflexion are: $I_1\left(\dfrac{\pi}{4}, 0\right)$ and $I_2\left(\dfrac{3\pi}{4}, 0\right)$.

8 $y = x(x-4)^2$

a) **i)** $y = 0 \Rightarrow x(x-4)^2 = 0 \Rightarrow x_1 = 0$ or $x_{2,3} = 4$.

We notice that 4 is a double zero and therefore the graph of the function will have a stationary point at it.

ii) $y' = 1\times(x-4)^2 + x\times 2(x-4)\times 1 = (x-4)(x-4+2x) = (x-4)(3x-4)$

$y' = 0 \Rightarrow (x-4)(3x-4) = 0 \Rightarrow x = 4$ or $x = \dfrac{4}{3} \Rightarrow (4, 0)$ or $\left(\dfrac{4}{3}, 0\right)$

In order to identify the stationary point as a minimum or maximum, we are going to use the second derivative test: $y''(x_0) < 0$ at the maximum point.

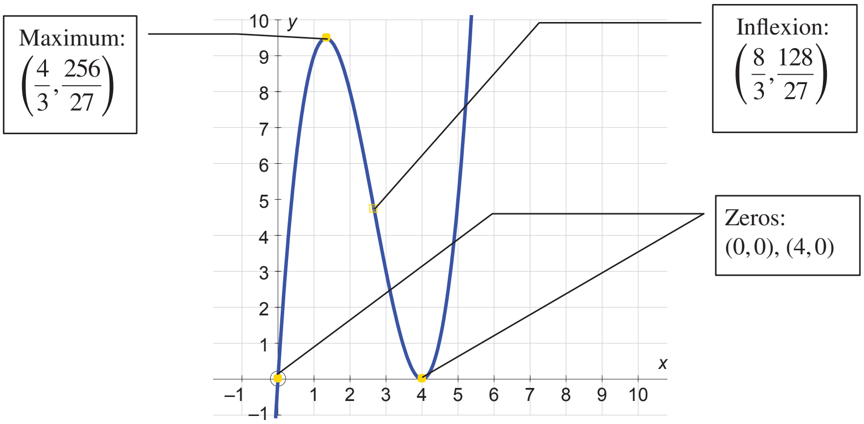
$y'' = 1\times(3x-4) + (x-4)\times 3 = 6x - 16 \Rightarrow y''(4) = 24 - 16 = 8 > 0,\ y''\left(\dfrac{4}{3}\right) = 8 - 16 = -8 < 0;$

therefore, the maximum is at: $x = \dfrac{4}{3}, y = \dfrac{4}{3}\left(\dfrac{4}{3} - 4\right)^2 = \dfrac{256}{27} \Rightarrow \left(\dfrac{4}{3}, \dfrac{256}{27}\right)$

iii) The second derivative test comes in really handy in this part, since the point of inflexion is a zero of the second derivative.

$y'' = 0 \Rightarrow 6x - 16 = 0 \Rightarrow x = \dfrac{8}{3}, y = \dfrac{8}{3}\left(\dfrac{8}{3} - 4\right)^2 = \dfrac{128}{27} \Rightarrow \left(\dfrac{8}{3}, \dfrac{128}{27}\right)$

b)



9 $f(x) = \dfrac{x^2 - 3x + 4}{(x+1)^2}$

Whenever the power of the denominator is greater than or equal to 2, it is advisable to change the quotient into product form for ease of differentiation.

a) $f(x) = (x^2 - 3x + 4)(x+1)^{-2} \Rightarrow f'(x) = (2x-3)(x+1)^{-2} + (x^2 - 3x + 4)(-2)(x+1)^{-3}$

$= (x+1)^{-3}\left((2x-3)(x+1) - 2(x^2 - 3x + 4)\right) = (x+1)^{-3}\left(\cancel{2x^2} - x - 3 - \cancel{2x^2} + 6x - 8\right) = \dfrac{5x - 11}{(x+1)^3}$

b) $f'(x) = (5x - 11)(x+1)^{-3} \Rightarrow f''(x) = 5(x+1)^{-3} + (5x-11)(-3)(x+1)^{-4}$

$= (x+1)^{-4}(5(x+1) - 3(5x - 11)) = (x+1)^{-4}(5x + 5 - 15x + 33) = \dfrac{-10x + 38}{(x+1)^4}$

c) The point of inflexion is a point where the second derivative is equal to zero, but note that the second derivative changes its sign at that point.

$f''(x) = 0 \Rightarrow \dfrac{-10x + 38}{(x+1)^4} = 0 \Rightarrow -10x + 38 = 0 \Rightarrow 38 = 10x \Rightarrow x = \dfrac{38}{10} = 3.8$ and we notice that the denominator is always positive on the domain, so the sign depends only on the numerator. The numerator is linear; therefore, it changes sign at 3.8. To verify, we calculate: $-10 \times 3.7 + 38 = 1 > 0$ and $-10 \times 3.9 + 38 = -1 < 0$.

10 $f(x) = \dfrac{x - a}{x + a} \Rightarrow f'(x) = \dfrac{1 \times (x+a) - (x-a) \times 1}{(x+a)^2} = \dfrac{2a}{(x+a)^2} = 2a(x+a)^{-2}$

$f''(x) = 2a \times (-2)(x+a)^{-3} = \dfrac{-4a}{(x+a)^3}$

Again, we notice that it is easier to use the product rule rather than the quotient rule.

11 $y = \dfrac{1}{1-x} = (1-x)^{-1} \Rightarrow \dfrac{dy}{dx} = -(1-x)^{-2} \times (-1) = (1-x)^{-2}$

$\dfrac{d^2y}{dx^2} = -2 \times (1-x)^{-3} \times (-1) = 2(1-x)^{-3}, \ \dfrac{d^3y}{dx^3} = 2 \times (-3) \times (1-x)^{-4} \times (-1) = 6(1-x)^{-4}$

$\dfrac{d^4y}{dx^4} = 6 \times (-4) \times (1-x)^{-5} \times (-1) = 24(1-x)^{-5}, \ \dfrac{d^5y}{dx^5} = 24 \times (-5) \times (1-x)^{-6} \times (-1) = 120(1-x)^{-6}$

We notice that the power of $(1-x)$ is always negative and equal to 1 less than the negative derivative index, whilst the constant factor is always the factorial of the derivative index.

$\dfrac{d^ny}{dx^n} = n!(1-x)^{-(n+1)} = \dfrac{n!}{(1-x)^{n+1}}$

12 $g(x) = \dfrac{8}{4+x^2}$

a) $g(x) = 8\left(4+x^2\right)^{-1} \Rightarrow g'(x) = 8\times(-1)\times\left(4+x^2\right)^{-2}\times 2x = -16x\left(4+x^2\right)^{-2} \Rightarrow$

$g'(x) = 0 \Rightarrow -16x\left(4+x^2\right)^{-2} = 0 \Rightarrow x = 0$; so, there is only one stationary point: $(0, 2)$.

$g''(x) = -16\times\left(4+x^2\right)^{-2} - 16x\times(-2)\times\left(4+x^2\right)^{-3}\times 2x = 16\left(4+x^2\right)^{-3}\left(-4-x^2+4x^2\right)$

$= 16\left(4+x^2\right)^{-3}\left(3x^2-4\right) \Rightarrow g''(x) = 0 \Rightarrow 16\left(4+x^2\right)^{-3}\left(3x^2-4\right) = 0 \Rightarrow x_{1,2} = \pm\sqrt{\dfrac{4}{3}} = \pm\dfrac{2\sqrt{3}}{3}$; so there are two points of inflexion: $\left(-\dfrac{2\sqrt{3}}{3}, \dfrac{3}{2}\right)$ and $\left(\dfrac{2\sqrt{3}}{3}, \dfrac{3}{2}\right)$.

b) **i)** There are no negative values of the function since all the values in the formula are positive or non-negative; therefore, their sum and the quotient cannot be negative.

ii) The function cannot have a value of zero because, in that case, the numerator would be equal to zero, but that is not possible.

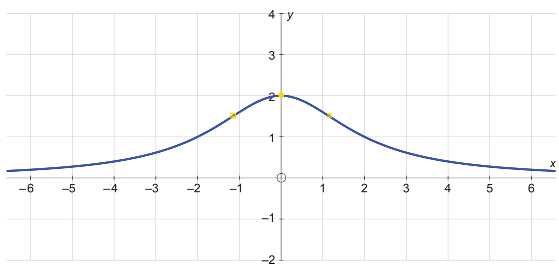
iii) We can now conclude that $g(x) > 0$ for all $x \in \mathbb{R}$.

c) Since the function is even, $g(-x) = \dfrac{8}{4+(-x)^2} = \dfrac{8}{4+x^2} = g(x)$, we can conclude that:

i) $\lim\limits_{x\to-\infty} g(x) = \lim\limits_{x\to+\infty} g(x)$, so it is sufficient to find just one limit.

ii) $\lim\limits_{x\to\infty} g(x) = \lim\limits_{x\to\infty}\dfrac{8}{4+x^2} = \lim\limits_{x\to\infty}\dfrac{\frac{8}{x^2}}{\frac{4}{x^2}+1} = \dfrac{0}{0+1} = 0$; therefore, the graph will have a horizontal asymptote: $y = 0$, the x-axis itself.

d)



13 $\dfrac{d}{dx}(c\cdot f(x)) = \dfrac{d}{dx}c\cdot f(x) + c\cdot\dfrac{d}{dx}f(x) = 0\cdot f(x) + c\cdot\dfrac{d}{dx}f(x) = c\cdot\dfrac{d}{dx}f(x)$

14 $y = x^4 - 6x^2 = x^2\left(x^2-6\right) \Rightarrow \dfrac{dy}{dx} = 4x^3 - 12x = 4x\left(x^2-3\right) \Rightarrow \dfrac{d^2y}{dx^2} = 12x^2 - 12 = 12\left(x^2-1\right)$

$\Rightarrow \dfrac{d^3y}{dx^3} = 24x$

When $0 < x < 1$ then $x^2 > 0$ and $x^2 - 6 < 0 \Rightarrow y < 0$.

When $0 < x < 1$ then $4x > 0$ and $x^2 - 3 < 0 \Rightarrow \dfrac{dy}{dx} < 0$.

When $0 < x < 1$ then $12 > 0$ and $x^2 - 1 < 0 \Rightarrow \dfrac{d^2y}{dx^2} < 0$.

When $0 < x < 1$ then $24x > 0 \Rightarrow \dfrac{d^3y}{dx^3} > 0$.

Exercise 15.2

1 **a)** $y = x^2 e^x \Rightarrow \frac{dy}{dx} = 2xe^x + x^2 e^x = x(2+x)e^x$

b) $y = 8^x \Rightarrow \frac{dy}{dx} = 8^x \times \ln 8 = \ln 8 \times 8^x$

c) $y = \tan e^x \Rightarrow \frac{dy}{dx} = \sec^2 e^x \times e^x$

d) $y = \frac{x}{1+\cos x} \Rightarrow \frac{dy}{dx} = \frac{1\times(1+\cos x) - x(-\sin x)}{(1+\cos x)^2} = \frac{1+\cos x + x\sin x}{(1+\cos x)^2}$

e) $y = \frac{e^x}{x} \Rightarrow \frac{dy}{dx} = \frac{e^x x - e^x}{x^2} = \frac{e^x(x-1)}{x^2}$

f) $y = \frac{1}{3}\sec^3 2x - \sec 2x \Rightarrow \frac{dy}{dx} = \frac{1}{\cancel{3}} \times \cancel{3}\sec^2 2x \times \left(-\sin 2x \sec^2 2x\right) \times 2 - \left(-\sin 2x \sec^2 2x\right) \times 2$

$= -2\sin 2x \times \sec^4 2x + 2\sin 2x \times \sec^2 2x = 2\sin 2x \times \sec^2 2x\left(1 - \sec^2 2x\right)$

g) $y = 4^{-x} \Rightarrow \frac{dy}{dx} = 4^{-x} \times \ln 4 \times (-1) = -\ln 4 \times 4^{-x}$

h) Sometimes, the easiest way to solve trigonometric functions is to find the simplest form of the function and then differentiate it.

$$y = \cos x \tan x = \cancel{\cos x}\,\frac{\sin x}{\cancel{\cos x}} = \sin x \Rightarrow \frac{dy}{dx} = \cos x$$

If we were differentiating this function without simplifying, the calculations are more complicated, but the final answer is the same.

$$y = \cos x \tan x \Rightarrow \frac{dy}{dx} = -\sin x \times \tan x + \cancel{\cos x} \times \frac{1}{\cos^{\cancel{2}} x} = \frac{-\sin^2 x + 1}{\cos x} = \frac{\cos^{\cancel{2}} x}{\cancel{\cos x}} = \cos x$$

i) $y = \frac{x}{e^x - 1} \Rightarrow \frac{dy}{dx} = \frac{1\times\left(e^x - 1\right) - x \times e^x}{\left(e^x-1\right)^2} = \frac{e^x - 1 - xe^x}{\left(e^x - 1\right)^2}$

j) $y = 4\cos(\sin 3x) \Rightarrow \frac{dy}{dx} = 4 \times (-\sin(\sin 3x)) \times \cos 3x \times 3 = -12\sin(\sin 3x)\cos 3x$

k) $y = 2^{x+1} \Rightarrow \frac{dy}{dx} = 2^{x+1} \times \ln 2 \times 1 = \ln 2 \times 2^{x+1}$

l) $y = \frac{1}{\csc x - \sec x} = \frac{\cos x \sin x}{\cos x - \sin x} \Rightarrow$

$$\frac{dy}{dx} = \frac{((-\sin x)\sin x + \cos x \cos x)(\cos x - \sin x) - \cos x \sin x(-\sin x - \cos x)}{(\cos x - \sin x)^2}$$

$$= \frac{\cancel{-\sin^2 x \cos x} + \cos^3 x - \sin^3 x - \cancel{\cos^2 x \sin x} + \cancel{\sin^2 x \cos x} + \cancel{\cos^2 x \sin x}}{(\cos x - \sin x)^2}$$

$$= \frac{\cos^3 x - \sin^3 x}{(\cos x - \sin x)^2}$$

$$= \frac{(\cos x - \sin x)(\cos^2 x + \cos x \sin x + \sin^2 x)}{(\cos x - \sin x)^2}$$

$$= \frac{1 + \cos x \sin x}{\cos x - \sin x}$$

2 a) $x = \frac{\pi}{3} \Rightarrow y = \sin\frac{\pi}{3} = \frac{\sqrt{3}}{2} \Rightarrow P\left(\frac{\pi}{3}, \frac{\sqrt{3}}{2}\right)$

$y = \sin x \Rightarrow y' = \cos x \Rightarrow m = y'\left(\frac{\pi}{3}\right) = \cos\left(\frac{\pi}{3}\right) = \frac{1}{2}$

Equation of tangent: $y = \frac{1}{2}\left(x - \frac{\pi}{3}\right) + \frac{\sqrt{3}}{2} \Rightarrow y = \frac{1}{2}x + \frac{3\sqrt{3} - \pi}{6}$

b) $x = 0 \Rightarrow y = 0 + e^0 = 1 \Rightarrow P(0, 1)$

$y = x + e^x \Rightarrow y' = 1 + e^x \Rightarrow m_T = y'(0) = 1 + e^0 = 2$

Equation of tangent: $y = 2(x - 0) + 1 \Rightarrow y = 2x + 1$

c) $x = \frac{\pi}{8} \Rightarrow y = 4 \times \tan\left(\not{2} \times \frac{\pi}{\not{8}\,4}\right) = 4 \Rightarrow P\left(\frac{\pi}{8}, 4\right)$

$y' = 4 \times \sec^2 2x \times 2 = 8\sec^2 2x \Rightarrow m_T = y'\left(\frac{\pi}{8}\right) = 8\sec^2\left(\frac{\pi}{4}\right) = 8 \times 2 = 16$

Equation of tangent: $y = 16\left(x - \frac{\pi}{8}\right) + 4 \Rightarrow y = 16x + 4 - 2\pi$

3 a) $y' = 1 + 2(-\sin x) = 1 - 2\sin x \Rightarrow y' = 0 \Rightarrow 1 - 2\sin x = 0 \Rightarrow \sin x = \frac{1}{2} \Rightarrow x = \frac{\pi}{6}$ or $x = \frac{5\pi}{6}$

b) $y'' = 0 - 2\cos x = -2\cos x \Rightarrow \begin{cases} y''\left(\frac{\pi}{6}\right) = -2\cos\left(\frac{\pi}{6}\right) = -2 \times \frac{\sqrt{3}}{2} = -\sqrt{3} < 0 \Rightarrow \text{maximum} \\ y''\left(\frac{5\pi}{6}\right) = -2\cos\left(\frac{5\pi}{6}\right) = -2 \times \left(-\frac{\sqrt{3}}{2}\right) = \sqrt{3} > 0 \Rightarrow \text{minimum} \end{cases}$

4 $y' = 1 - e^x \Rightarrow y' = 0 \Rightarrow 1 - e^x = 0 \Rightarrow e^x = 1 \Rightarrow x = 0 \Rightarrow y = 0 + e^0 = 1 \Rightarrow P(0, 1)$

$y'' = 0 - e^x = -e^x \Rightarrow y''(0) = -e^0 = -1 < 0 \Rightarrow P(0, 1)$ is a maximum point.

5 a) $f(x) = 4\sin x - \cos 2x \Rightarrow f'(x) = 4\cos x - (-\sin 2x) \times 2 = 4\cos x + 2\sin 2x$

$f'(x) = 0 \Rightarrow 4\cos x + 2\sin 2x = 0 \Rightarrow 4\cos x + 4\sin x\cos x = 0$

$\Rightarrow 4\cos x(1 + \sin x) = 0 \Rightarrow \cos x = 0$ or $1 + \sin x = 0 \Rightarrow x = \frac{\pi}{2}$ or $x = \frac{3\pi}{2}$

$x = \frac{\pi}{2} \Rightarrow y = 4\sin\frac{\pi}{2} - \cos\pi = 5$ or $x = \frac{3\pi}{2} \Rightarrow y = 4\sin\frac{3\pi}{2} - \cos 3\pi = -3$

$f''(x) = 4 \times (-\sin x) + 2 \times \cos 2x \times 2 = -4\sin x + 4\cos 2x$

$f''\left(\frac{\pi}{2}\right) = -4\sin\frac{\pi}{2} + 4\cos\pi = -8 < 0,\ f''\left(\frac{3\pi}{2}\right) = -4\sin\frac{3\pi}{2} + 4\cos 3\pi = 0$

$f^{(3)}(x) = -4\cos x - 8\sin 2x,\ f^{(3)}\left(\frac{3\pi}{2}\right) = -4\cos\frac{3\pi}{2} - 8\sin 3\pi = 0$

$f^{(4)}(x) = 4\sin x - 16\cos 2x,\ f^{(4)}\left(\frac{3\pi}{2}\right) = 4\sin\frac{3\pi}{2} - 16\cos 3\pi = 20 > 0$

So, we conclude that $\left(\frac{\pi}{2}, 5\right)$ is a maximum point and $\left(\frac{3\pi}{2}, -3\right)$ is a minimum point.

b) $g(x) = \tan x(\tan x + 2) = \tan^2 x + 2\tan x$

Note: It is much easier to differentiate a sum than a product.

$g'(x) = 2\tan x\sec^2 x + 2\sec^2 x = 2\sec^2 x(\tan x + 1)$

$$g'(x) = 0 \Rightarrow 2\sec^2 x(\tan x + 1) = 0 \Rightarrow \tan x = -1 \Rightarrow x = \frac{3\pi}{4} \text{ or } x = \frac{7\pi}{4}$$

$$x = \frac{3\pi}{4} \Rightarrow y = \tan\frac{3\pi}{4}\left(\tan\frac{3\pi}{4} + 2\right) = -1 \text{ or } x = \frac{7\pi}{4} \Rightarrow y = \tan\frac{7\pi}{4}\left(\tan\frac{7\pi}{4} + 2\right) = -1$$

$$g''(x) = 2 \times 2\sec x \times (-\sin x)\sec^2 x(\tan x + 1) + 2\sec^2 x \times \sec^2 x$$

$$= 2\sec^4 x(1 - 2\sin x\cos x(\tan x + 1))$$

In both cases: $x = \frac{3\pi}{4}, \frac{7\pi}{4} \Rightarrow \tan x + 1 = 0 \Rightarrow g'\left(\frac{3\pi}{4}\right) = g'\left(\frac{7\pi}{4}\right) = 2 > 0$

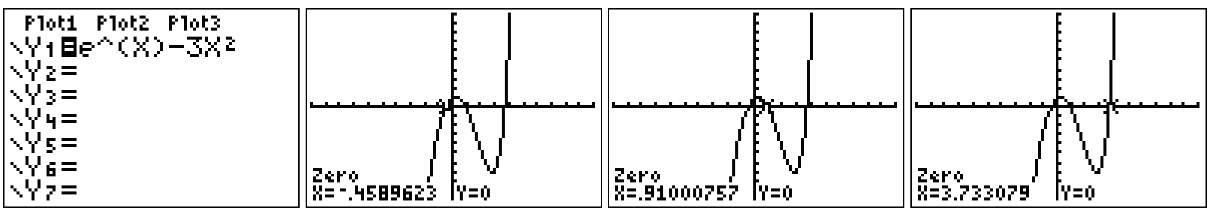
Therefore, there are two minimum points: $\left(\frac{3\pi}{4}, -1\right)$ and $\left(\frac{7\pi}{4}, -1\right)$.

6 $y = 3 + \sin\left(\frac{\pi}{2}\right) = 3 + 1 = 4 \Rightarrow P\left(\frac{\pi}{2}, 4\right)$

$y' = \cos x \Rightarrow m_T = y'\left(\frac{\pi}{2}\right) = \cos\left(\frac{\pi}{2}\right) = 0 \Rightarrow m_N$ is not defined. Therefore, the normal line is a vertical line through point P and its equation is $x = \frac{\pi}{2}$.

7 **a)** $f(x) = e^x - x^3 \Rightarrow f'(x) = e^x - 3x^2 \Rightarrow f''(x) = e^x - 6x$

b)



There are three possible answers: $x = -0.459$ or $x = 0.910$ or $x = 3.73$.

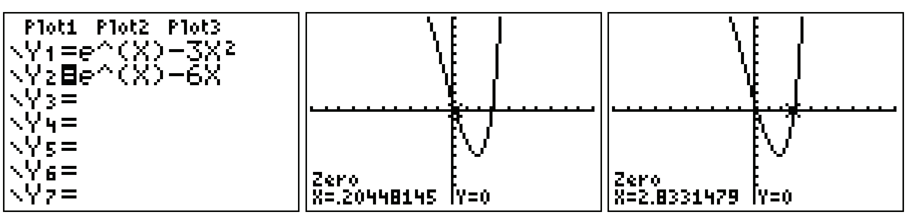
c) To answer this part, we are going to refer to the graph of the derivative function given in part **b** when the zeros of the derivative function were found.

The function f increases whenever: $f'(x) > 0 \Rightarrow x \in]-0.459, 0.910[\cup]3.73, \infty[$.

The function f decreases whenever: $f'(x) < 0 \Rightarrow x \in]-\infty, -0.459[\cup]0.910, 3.73[$.

d) At $x = -0.459$ and $x = 3.73$, we have minimum points since the derivative function changes its sign from negative to positive; therefore, the original function changes from decreasing to increasing, which is a point of minimum. At $x = 0.910$, we have a maximum point since the derivative changes its sign from positive to negative; therefore, the original function changes from increasing to decreasing, which is a point of maximum.

e)



There are two possible points of inflexion at: $x = 0.204$ or $x = 2.83$.

f) Concave up on $]-\infty, 0.204]$ and $[2.83, \infty[$; concave down on $[0.204, 2.83]$.

8 Let's find the points of intersection first.

$e^{-x} = e^{-x}\cos x \Rightarrow \underbrace{e^{-x}}_{>0}(1-\cos x) = 0 \Rightarrow 1-\cos x = 0 \Rightarrow x = 2k\pi, k \in \mathbb{Z}$. Given the interval $\left[-\frac{\pi}{2}, \frac{3\pi}{2}\right]$, the only solution is 0.

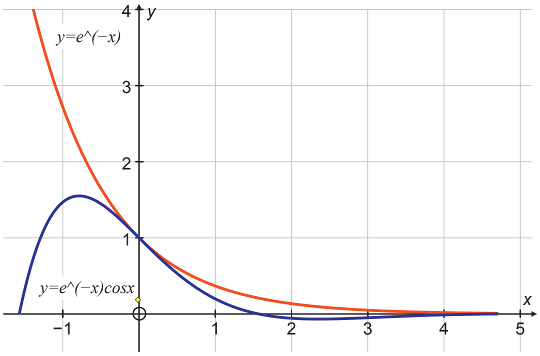
$y = e^{-x} \Rightarrow y' = -e^{-x}$, $y = e^{-x}\cos x \Rightarrow y' = -e^{-x}\cos x + e^{-x}(-\sin x) = -e^{-x}(\cos x + \sin x)$

We need to see that the curves have equal gradients at $x = 0$.

First curve: $y'(0) = -e^{0} = -1$

Second curve: $y'(0) = -e^{0}\left(\overbrace{\cos 0}^{1} + \overbrace{\sin 0}^{0}\right) = -1$

Since both curves pass through the same point, and have equal gradients at that point, we can conclude that the curves are tangent to each other.



9 $s(t) = 4\cos t - \cos 2t$

Given that the particle comes to rest after T seconds, $T > 0$, we can conclude that its velocity at T is equal to zero.

$v(t) = s'(t) = -4\sin t + \sin 2t \times 2 = 2\sin 2t - 4\sin t \Rightarrow v(T) = 0 \Rightarrow 2\overbrace{\sin 2T}^{2\sin T\cos T} - 4\sin T = 0$

$4\sin T(\cos T - 1) = 0 \Rightarrow \sin T = 0$ or $\cos T = 1 \Rightarrow T = \pi$ is the first positive solution.

a) $a(t) = 2\cos 2t \times 2 - 4\cos t = 4\cos 2t - 4\cos t \Rightarrow a(\pi) = 4\overbrace{\cos 2\pi}^{1} - 4\overbrace{\cos \pi}^{-1} = 8$ m/s

b) The maximum speed is achieved when the acceleration is zero; therefore:

$a(t) = 0 \Rightarrow 4\cos 2t - 4\cos t = 0 / \div 4 \Rightarrow 2\cos^2 t - 1 - \cos t = 0 \Rightarrow (2\cos t + 1)(\cos t - 1) = 0$

$\Rightarrow \cos t = -\frac{1}{2}$ or $\cos t = 1$

Since $0 < t < \pi$, we can discard the second case.

$\cos t = -\frac{1}{2} \Rightarrow t = \frac{2\pi}{3} \Rightarrow v\left(\frac{2\pi}{3}\right) = 2\overbrace{\sin\left(\frac{4\pi}{3}\right)}^{-\frac{\sqrt{3}}{2}} - 4\overbrace{\sin\left(\frac{2\pi}{3}\right)}^{\frac{\sqrt{3}}{2}} = -3\sqrt{3}$

Since the velocity is negative, the particle is moving in the opposite direction and the speed is $3\sqrt{3} \approx 5.20$ m/s.

10 $y = e^{x} \Rightarrow y' = e^{x}$

If the tangent line passes through the origin, we need just one more point, let's call it $(x_0, y_0) \Rightarrow y_0 = e^{x_0}$, to find the equation of the line T. We can use both facts to find the slope.

$e^{x_0} = \frac{e^{x_0}}{x_0} \Rightarrow e^{x_0} \times x_0 = e^{x_0} / \div e^{x_0}, e^{x_0} \neq 0 \Rightarrow x_0 = 1 \Rightarrow y_0 = e \Rightarrow$ Equation of tangent: $y = ex$

11 a) $f(x) = 2^x \Rightarrow f'(x) = 2^x \times \ln 2$

b) $m_T = f'(0) = 2^0 \times \ln 2 = \ln 2 \Rightarrow$ Equation of tangent: $y = \ln 2(x - 0) + 1 \Rightarrow y = \ln 2 \times x + 1$

c) Stationary point must satisfy the equation $f'(x) = 0 \Rightarrow 2^x \times \ln 2 = 0 \Rightarrow 2^x = 0 \Rightarrow x \notin \mathbb{R}$

12 a) $y = \frac{x^2 - 3}{e^x} \Rightarrow y' = \frac{2x\cancel{e^x} - (x^2 - 3)\cancel{e^x}}{e^{\cancel{2}x}} = \frac{2x - x^2 + 3}{e^x}$. Since the fraction is equal to zero, the numerator must be equal to zero and so we need to solve the following:

$2x - x^2 + 3 = 0 \Rightarrow x^2 - 2x - 3 = 0 \Rightarrow (x-3)(x+1) = 0 \Rightarrow x_1 = 3$ or $x_2 = -1$

Now, we need to find the corresponding y-coordinates: $y_1 = \frac{3^2 - 3}{e^3} = \frac{6}{e^3}$ and $y_2 = \frac{(-1)^2 - 3}{e^{-1}} = -2e$

So, the stationary points are: $P_1 = (-1, -2e)$ and $P_2 = \left(3, \frac{6}{e^3}\right)$.

b) $y' = \frac{2x - x^2 + 3}{e^x}$ and since the denominator is always positive we need to investigate the sign of the numerator. The numerator is a quadratic expression and the parabola opens downwards; hence, the expression is positive between the zeros, and negative otherwise.

x	$x < -1$	-1	$-1 < x < 3$	3	$x > 3$
$f'(x)$	negative	0	positive	0	negative
$f(x)$	decreases	minimum	increases	maximum	decreases

$P_1 = (-1, -2e)$ is a minimum point, whilst $P_2 = \left(3, \frac{6}{e^3}\right)$ is a maximum point.

c) The function is a quotient of a quadratic function and an exponential, and at infinity it is going to behave in the same way as the exponential, since it is steeper than the quadratic.

i) $\lim_{x\to\infty} e^x = +\infty \Rightarrow \lim_{x\to\infty} \frac{x^2 - 3}{e^x} = 0$

ii) $\lim_{x\to-\infty} e^x = 0 \Rightarrow \lim_{x\to-\infty} \frac{x^2 - 3}{e^x} = +\infty$

Notice that both numerator and denominator are positive.

d) Looking back at the previous part, we notice that the x-axis ($y = 0$) is the horizontal asymptote since the graph of function f approaches the positive part of the x-axis as x becomes very large.

e) To find the points of intersection with the axes, we need to set the values of x and y to zero in turn:

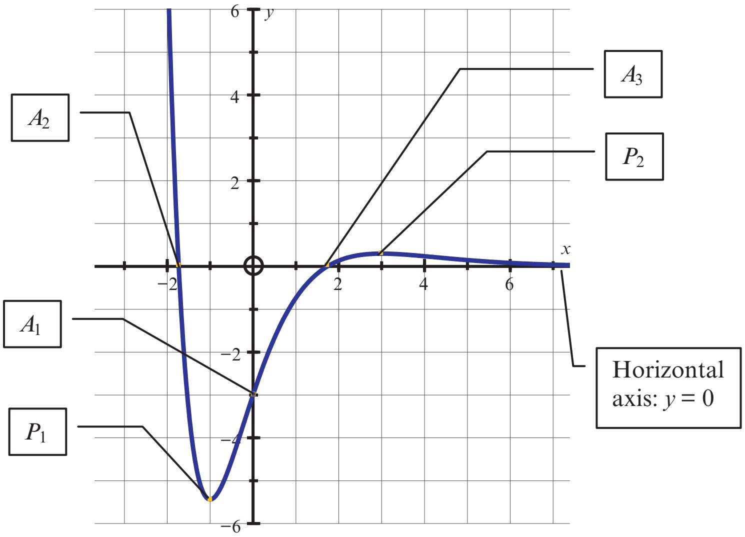
$x = 0 \Rightarrow y = \frac{0^2 - 3}{e^0} = -3 \Rightarrow A_1(0, -3)$

$y = 0 \Rightarrow 0 = \frac{x^2 - 3}{e^x} \Rightarrow x^2 - 3 = 0 \Rightarrow$

$x = -\sqrt{3}$ or $x = \sqrt{3}$

So, there are two points of intersection with the x-axis: $A_2\left(-\sqrt{3}, 0\right)$ and $A_3\left(\sqrt{3}, 0\right)$.

Using all of the above information, we sketch the graph:



13 a) We know that the derivatives of the sine function follow the pattern:
$y = \sin x \Rightarrow y' = \cos x \Rightarrow y'' = -\sin x \Rightarrow y^{(3)} = -\cos x$, and now we need to rewrite all of these expressions as a translated sine wave.

$$\cos x = \sin\left(x + \frac{\pi}{2}\right) \Rightarrow a = \frac{\pi}{2}$$

$$-\sin(x) = \sin(x + \pi) \Rightarrow b = \pi$$

$$-\cos x = \sin\left(x + \frac{3\pi}{2}\right) \Rightarrow c = \frac{3\pi}{2}$$

b) $\frac{d^{(n)}}{dx^{(n)}}(\sin x) = \sin\left(x + n \cdot \frac{\pi}{2}\right), n \in \mathbb{Z}^+$

14 a) $y = xe^x \Rightarrow \frac{dy}{dx} = 1 \times e^x + xe^x = (1 + x)e^x \Rightarrow \frac{d^2y}{dx^2} = 1 \times e^x + (1 + x)e^x = (2 + x)e^x$

$$\Rightarrow \frac{d^3y}{dx^3} = 1 \times e^x + (2 + x)e^x = (3 + x)e^x$$

b) $\frac{d^ny}{dx^n} = (n + x)e^x, n = 0, 1, 2, \ldots$, where the zero derivative is the original function.

c) We notice that this induction starts with zero and therefore we check the base for zero.

$n = 0 \Rightarrow \frac{d^0y}{dx^0} = (0 + x)e^x \Rightarrow y = xe^x$, which is true.

Now, we need to assume that the formula works for $n = k \Rightarrow \frac{d^ky}{dx^k} = (k + x)e^x$.

The next step is to see whether the formula works for the next value.

$n = k + 1 \Rightarrow \frac{d^{k+1}y}{dx^{k+1}} = \frac{d\left(\frac{d^ky}{dx^k}\right)}{dx} = \frac{d\left((k + x)e^x\right)}{dx} = 1 \times e^x + (k + x)e^x = (1 + k + x)e^x$, which shows that the formula works for $k + 1$.

Since the formula works for $n = 0$, and, from the assumption that the formula works for $n = k$, we confirm that it works for $n = k + 1$, and, by the principle of mathematical induction, we can conclude that the formula works for all $n = 0, 1, 2, \ldots$.

Exercise 15.3

1 $x^2+y^2=16 \Big/ \dfrac{d}{dx} \Rightarrow 2x+2yy'=0 \Rightarrow y'=\dfrac{-\cancel{2}x}{\cancel{2}y}=-\dfrac{x}{y}$

2 $x^2y+xy^2=6 \Big/ \dfrac{d}{dx} \Rightarrow 2xy+x^2y'+y^2+x\times 2yy'=0 \Rightarrow y'\left(x^2+2xy\right)=-2xy-y^2 \Rightarrow y'=\dfrac{-y^2-2xy}{x^2+2xy}$

3 $x=\tan y \Big/ \dfrac{d}{dx} \Rightarrow 1=\sec^2 y\times y' \Rightarrow y'=\cos^2 y$

4 $x^2-3xy^2+y^3x-y^2=2 \Big/ \dfrac{d}{dx} \Rightarrow 2x-3y^2-3x\times 2yy'+3y^2y'x+y^3-2yy'=0$

$\Rightarrow 2x-3y^2+y^3=y'\left(6xy-3xy^2+2y\right) \Rightarrow y'=\dfrac{2x-3y^2+y^3}{6xy-3xy^2+2y}$

5 $\dfrac{x}{y}-\dfrac{y}{x}=1 \Big/ \dfrac{d}{dx} \Rightarrow \dfrac{y-xy'}{y^2}-\dfrac{y'x-y}{x^2}=0 \Rightarrow \dfrac{x^2y-x^3y'-y^2y'x+y^3}{x^2y^2}=0$

$\Rightarrow x^2y-x^3y'-y^2y'x+y^3=0 \Rightarrow x^2y+y^3=\left(x^3+y^2x\right)y' \Rightarrow y'=\dfrac{x^2y+y^3}{x^3+xy^2}$

6 $xy\sqrt{x+y}=1 \Big/ \dfrac{d}{dx} \Rightarrow y\sqrt{x+y}+xy'\sqrt{x+y}+\dfrac{xy}{2\sqrt{x+y}}(1+y')=0 \Big/ \times 2\sqrt{x+y}$

$\Rightarrow 2y(x+y)+2xy'(x+y)+xy(1+y')=0 \Rightarrow 2x^2y'+2xyy'+xyy'=-2xy-2y^2-xy$

$\Rightarrow y'\left(2x^2+3xy\right)=-\left(2y^2+3xy\right) \Rightarrow y'=\dfrac{-2y^2-3xy}{2x^2+3xy}$

7 $x+\sin y=xy \Big/ \dfrac{d}{dx} \Rightarrow 1+\cos y\times y'=y+xy' \Rightarrow 1-y=y'(x-\cos y) \Rightarrow y'=\dfrac{1-y}{x-\cos y}$

8 $x^2y^3=x^4-y^4 \Big/ \dfrac{d}{dx} \Rightarrow 2xy^3-3x^2y^2y'=4x^3+4y^3y' \Rightarrow 2xy^3-4x^3=y'\left(3x^2y^2+4y^3\right)$

$\Rightarrow y'=\dfrac{2xy^3-4x^3}{3x^2y^2+4y^3}$

9 $xy+e^y=0 \Big/ \dfrac{d}{dx} \Rightarrow y+xy'+e^y\times y'=0 \Rightarrow y'\left(x+e^y\right)=-y \Rightarrow y'=\dfrac{-y}{x+e^y}$

10 $(x+2)^2+(y+3)^2=25 \Big/ \dfrac{d}{dx} \Rightarrow 2(x+2)+2(y+3)y'=0 \Rightarrow y'=\dfrac{-\cancel{2}(x+2)}{\cancel{2}(y+3)}=-\dfrac{x+2}{y+3}$

11 $x=\tan(x+y) \Big/ \dfrac{d}{dx} \Rightarrow 1=\sec^2(x+y)(1+y') \Rightarrow \cos^2(x+y)=1+y' \Rightarrow \cos^2(x+y)-1=y'$

$\Rightarrow y'=-\sin^2(x+y)$

12 $y+\sqrt{xy}=3x^3 \Big/ \dfrac{d}{dx} \Rightarrow y'+\dfrac{1}{2\sqrt{xy}}\times(y+xy')=9x^2 \Big/ 2\sqrt{xy} \Rightarrow 2\sqrt{xy}\,y'+y+xy'=18x^2\sqrt{xy}$

$\Rightarrow y'\left(x+2\sqrt{xy}\right)=18x^2\sqrt{xy}-y \Rightarrow y'=\dfrac{18x^2\sqrt{xy}-y}{x+2\sqrt{xy}}$

13 $x^3-xy-3y^2=0 \Big/ \dfrac{d}{dx} \Rightarrow 3x^2-(y+xy')-6yy'=0 \Rightarrow 3x^2-y=y'(x+6y) \Rightarrow y'=\dfrac{3x^2-y}{x+6y}$

a) $m_T = y'(2,-2) = \frac{3\times 2^2 - (-2)}{2+6\times(-2)} = \frac{14}{-10} = -\frac{7}{5} \Rightarrow$

Equation of tangent: $y = -\frac{7}{5}(x-2) - 2 \Rightarrow y = -\frac{7}{5}x + \frac{4}{5}$

b) $m_N = -\frac{1}{m_T} = \frac{5}{7} \Rightarrow$ Equation of normal: $y = \frac{5}{7}(x-2) - 2 \Rightarrow y = \frac{5}{7}x - \frac{24}{7}$

14 $16x^4 + y^4 = 32 \Big/ \frac{d}{dx} \Rightarrow 64x^3 + 4y^3y' = 0 \Rightarrow y' = -\frac{\cancel{64}\,16x^3}{\cancel{4}\,y^3} = -\frac{16x^3}{y^3}$

a) $m_T = y'(1,2) = -\frac{16\times 1^3}{2^3} = -2 \Rightarrow$ Equation of tangent: $y = -2(x-1) + 2 \Rightarrow y = -2x + 4$

b) $m_N = -\frac{1}{m_T} = \frac{1}{2} \Rightarrow$ Equation of normal: $y = \frac{1}{2}(x-1) + 2 \Rightarrow y = \frac{1}{2}x + \frac{3}{2}$

15 $2xy + \pi \sin y = 2\pi \Big/ \frac{d}{dx} \Rightarrow 2y + 2xy' + \pi \cos y \times y' = 0 \Rightarrow y' = -\frac{2y}{2x + \pi\cos y}$

a) $m_T = y'\left(1, \frac{\pi}{2}\right) = -\frac{\cancel{2}\times\frac{\pi}{\cancel{2}}}{2\times 1 + \pi\cos\left(\frac{\pi}{2}\right)} = -\frac{\pi}{2} \Rightarrow$

Equation of tangent: $y = -\frac{\pi}{2}(x-1) + \frac{\pi}{2} \Rightarrow y = -\frac{\pi}{2}x + \pi$

b) $m_N = -\frac{1}{m_T} = \frac{2}{\pi} \Rightarrow$ Equation of normal: $y = \frac{2}{\pi}(x-1) + \frac{\pi}{2} \Rightarrow y = \frac{2}{\pi}x + \frac{\pi^2 - 4}{2\pi}$

16 $\sqrt[3]{xy} = 14x + y \Big/ \frac{d}{dx} \Rightarrow \frac{1}{3\sqrt[3]{x^2y^2}}(y + xy') = 14 + y' \Big/ \times\left(3\sqrt[3]{x^2y^2}\right) \Rightarrow$

$y + xy' = 42\sqrt[3]{x^2y^2} + 3\sqrt[3]{x^2y^2} \times y' \Rightarrow y'\left(x - 3\sqrt[3]{x^2y^2}\right) = 42\sqrt[3]{x^2y^2} - y \Rightarrow y' = \frac{42\sqrt[3]{x^2y^2} - y}{x - 3\sqrt[3]{x^2y^2}}$

a) $m_T = y'(2,-32) = \frac{42\sqrt[3]{2^2(-32)^2} + 32}{2 - 3\sqrt[3]{2^2(-32)^2}} = \frac{42\times 16 + 32}{2 - 3\times 16} = -\frac{352}{23} \Rightarrow$

Equation of tangent: $y = -\frac{352}{23}(x-2) - 32 \Rightarrow y = -\frac{352}{23}x - \frac{32}{23}$

b) $m_N = -\frac{1}{m_T} = \frac{23}{352} \Rightarrow$ Equation of normal: $y = \frac{23}{352}(x-2) - 32 \Rightarrow y = \frac{23}{352}x - \frac{5655}{176}$

17 $x^2 + y^2 = r^2 \Big/ \frac{d}{dx} \Rightarrow 2x + 2yy' = 0 \Rightarrow y' = -\frac{x}{y}$

$m_T = y'(x_1, y_1) = -\frac{x_1}{y_1}, m_L = \frac{y_1 - 0}{x_1 - 0} = \frac{y_1}{x_1} \Rightarrow m_T \times m_L = -\frac{x_1}{y_1} \times \frac{y_1}{x_1} = -1$

Since the product of the gradients of the two lines is -1, we can deduce that the lines are perpendicular.

18 The function $x^2 + xy + y^2 = 7$ is given.

a) Points of intersection with the x-axis: $y = 0 \Rightarrow x^2 = 7 \Rightarrow x_{1,2} = \pm\sqrt{7}$. Now, we need to find the gradients to the curve at those points.

$x^2 + xy + y^2 = 7 \Big/ \frac{d}{dx} \Rightarrow 2x + y + xy' + 2yy' = 0 \Rightarrow y' = -\frac{2x + y}{x + 2y}$

$$y'\left(-\sqrt{7},0\right)=-\frac{2\times\left(-\sqrt{7}\right)+0}{-\sqrt{7}+2\times 0}=-2,\ y'\left(\sqrt{7},0\right)=-\frac{2\times\sqrt{7}+0}{\sqrt{7}+2\times 0}=-2$$

Since the gradients are equal, the tangents at those two points are parallel.

b) To find a point where the tangent line is parallel to the x-axis, we need to make the slope equal to zero.

$$y'=-\frac{2x+y}{x+2y}=0\Rightarrow 2x+y=0\Rightarrow y=-2x$$

Also, the point must be on the curve:

$$x^2+x\times(-2x)+(-2x)^2=7\Rightarrow 3x^2=7\Rightarrow x=\pm\sqrt{\frac{7}{3}}=\pm\frac{\sqrt{21}}{3}$$

So, there are two points: $\left(\frac{\sqrt{21}}{3},-\frac{2\sqrt{21}}{3}\right)$ and $\left(-\frac{\sqrt{21}}{3},\frac{2\sqrt{21}}{3}\right)$.

c) To find a point where the tangent line is parallel to the y-axis, we say that the slope is undefined; therefore, the denominator of the derivative must be equal to zero.

$$y'=-\frac{2x+y}{x+2y}\Rightarrow x+2y=0\Rightarrow x=-2y$$

Also, the point must be on the curve:

$$(-2y)^2+(-2y)\times y+y^2=7\Rightarrow 3y^2=7\Rightarrow y=\pm\sqrt{\frac{7}{3}}=\pm\frac{\sqrt{21}}{3}$$

So, there are two points: $\left(\frac{2\sqrt{21}}{3},-\frac{2\sqrt{21}}{3}\right)$ and $\left(-\frac{2\sqrt{21}}{3},\frac{\sqrt{21}}{3}\right)$.

19 $x^2+2xy-3y^2=0\Big/\frac{d}{dx}\Rightarrow 2x+2\left(y+xy'\right)-6yy'=0\Rightarrow 2x+2y=6yy'-2xy'\Rightarrow$

$$y'=\frac{\cancel{2}(x+y)}{\cancel{2}(3y-x)}=\frac{x+y}{3y-x}$$

$m_N=-\frac{1}{y'(1,1)}=-\frac{3-1}{1+1}=-2\Rightarrow$ Equation of normal: $y=-2(x-1)+1\Rightarrow y=-2x+3$

Now, we need to find the point of intersection with the curve.

$$\begin{cases} y=-2x+3 \\ x^2+2xy-3y^2=0 \end{cases}\Rightarrow\begin{cases} y=-2x+3 \\ x^2+2x\times(-2x+3)-3(-2x+3)^2=0 \end{cases}\Rightarrow\begin{cases} y=-2x+3 \\ -15x^2+42x-27=0/\div(-3) \end{cases}$$

$$\Rightarrow\begin{cases} y=-2x+3 \\ 5x^2-14x+9=0 \end{cases}\Rightarrow\begin{cases} y=-2x+3 \\ (x-1)(5x-9)=0 \end{cases}\Rightarrow\begin{cases} y=1 \text{ or } y=-\frac{3}{5} \\ x=1 \text{ or } x=\frac{9}{5} \end{cases}$$

This system has two solutions. One solution is expected, $(1,1)$, whilst the other solution is the point that we are looking for: $\left(\frac{9}{5},-\frac{3}{5}\right)$.

20 $4x^2+9y^2=36 \Big/ \frac{d}{dx} \Rightarrow 8x+18yy'=0 \Rightarrow y'=-\frac{4x}{9y}$

This is the first derivative and we now need to differentiate it again to obtain the second derivative.

$$y'=-\frac{4x}{9y}\Big/\frac{d}{dx} \Rightarrow y''=-\frac{4\times 9y-4x\times 9y'}{(9y)^2}=-\frac{36y-4x\times \cancel{9}\left(-\frac{4x}{\cancel{9}y}\right)}{(9y)^2}=-\frac{36y^2+16x^2}{81y^3}$$

An alternative method simply involves proceeding with the differentiation.

$$8x+18yy'=0\Big/\frac{d}{dx} \Rightarrow 8+18\left(y'\times y'+y\times y''\right)=0 \Rightarrow$$

$$y''=-\frac{8+18(y')^2}{18y}=-\frac{\cancel{8}4+\cancel{18}9\left(-\frac{4x}{9y}\right)^2}{\cancel{18}9y}=-\frac{4+\cancel{9}\times\frac{16x^2}{\cancel{81}9y^2}}{9y}=-\frac{36y^2+16x^2}{81y^3}$$

21 $xy=2x-3y\Big/\frac{d}{dx} \Rightarrow y+xy'=2-3y' \Rightarrow y'=\frac{2-y}{x+3}$

$$y'=\frac{2-y}{x+3}\Big/\frac{d}{dx} \Rightarrow y''=\frac{-y'(x+3)-(2-y)}{(x+3)^2}=\frac{-\frac{2-y}{\cancel{x+3}}\cancel{(x+3)}-(2-y)}{(x+3)^2}=\frac{2y-4}{(x+3)^2}$$

22 **a)** $xy^3=1 \Rightarrow y=\sqrt[3]{\frac{1}{x}} \Rightarrow y=x^{-\frac{1}{3}}\Big/\frac{d}{dx} \Rightarrow y'=-\frac{1}{3}\times x^{-\frac{4}{3}}\Big/\frac{d}{dx} \Rightarrow y''=\frac{4}{9}x^{-\frac{7}{3}}=\frac{4}{9x^{\frac{7}{3}}}$

b) $xy^3=1\Big/\frac{d}{dx} \Rightarrow y^3+x\times 3y^2y'=0 \Rightarrow y'=-\frac{y^{\cancel{3}}}{3x\cancel{y^2}}=-\frac{y}{3x}$

$$y'=-\frac{y}{3x}\Big/\frac{d}{dx} \Rightarrow y''=-\frac{y'\times\cancel{3}x-y\times\cancel{3}}{\cancel{9}3x^2}=\frac{-xy'+y}{3x^2}=\frac{-\cancel{x}\times\left(-\frac{y}{3\cancel{x}}\right)+y}{3x^2}=\frac{\frac{4}{3}y}{3x^2}=\frac{4y}{9x^2}$$

$$=\frac{4x^{-\frac{1}{3}}}{9x^2}=\frac{4}{9x^{\frac{7}{3}}}$$

23 $x^2+y^2=\left(x^2+y^2-x\right)^2\Big/\frac{d}{dx} \Rightarrow \cancel{2}x+\cancel{2}yy'=\cancel{2}\left(x^2+y^2-x\right)\times(2x+2yy'-1)$

$$\Rightarrow x-(2x-1)\left(x^2+y^2-x\right)=yy'\left[2\left(x^2+y^2-x\right)-1\right] \Rightarrow y'=\frac{x-(2x-1)\left(x^2+y^2-x\right)}{y\left[2\left(x^2+y^2-x\right)-1\right]}$$

To find the slope, we simply input the coordinates of the point.

$$m=y'\left(0,\frac{1}{2}\right)=\frac{0-(2\times 0-1)\left(0^2+\left(\frac{1}{2}\right)^2-0\right)}{\frac{1}{2}\left[2\left(0^2+\left(\frac{1}{2}\right)^2-0\right)-1\right]}=\frac{\frac{1}{4}}{\frac{1}{4}}=1$$

Equation of tangent: $y=1\times(x-0)+\frac{1}{2} \Rightarrow y=x+\frac{1}{2}$

24 $y=\ln\left(x^3+1\right) \Rightarrow y'=\frac{3x^2}{x^3+1}$

25 $y = \ln(\sin x) \Rightarrow y' = \dfrac{\cos x}{\sin x} = \cot x$

For questions 26–33, in order to simplify the calculation of the derivative of logarithmic functions, we need to rewrite the original functions in sum or product form, as these are much easier to differentiate than the products and powers respectively.

26 $y = \log_5 \sqrt{x^2 - 1} \Rightarrow y = \dfrac{1}{2}(\log_5(x+1) + \log_5(x-1)) \Rightarrow y' = \dfrac{1}{2}\left(\dfrac{1}{(x+1)\ln 5} + \dfrac{1}{(x-1)\ln 5}\right)$

$= \dfrac{1}{2\ln 5} \times \dfrac{x \cancel{-1} + x \cancel{+1}}{(x+1)(x-1)} = \dfrac{1}{\cancel{2}\ln 5} \times \dfrac{\cancel{2}x}{(x+1)(x-1)} = \dfrac{x}{\ln 5\,(x^2-1)}$

27 $y = \ln\sqrt{\dfrac{1+x}{1-x}} = \dfrac{1}{2}(\ln(1+x) - \ln(1-x)) \Rightarrow y' = \dfrac{1}{2}\left(\dfrac{1}{1+x} - \dfrac{-1}{1-x}\right) = \dfrac{1}{2} \times \dfrac{1 \cancel{-x} + 1 \cancel{+x}}{(1+x)(1-x)}$

$= \dfrac{\cancel{2}1}{\cancel{2}(1+x)(1-x)} = \dfrac{1}{1-x^2}$

28 $y = \sqrt{\log_{10} x} \Rightarrow y' = \dfrac{1}{2\sqrt{\log_{10} x}} \times \dfrac{1}{x\ln 10} = \dfrac{1}{2x\ln 10\sqrt{\log_{10} x}}$

Notice that we did not rewrite the function since it is a power of a logarithm, not a logarithm of a power.

29 $y = \ln\left(\dfrac{a-x}{a+x}\right) = \ln(a-x) - \ln(a+x) \Rightarrow y' = \dfrac{-1}{a-x} - \dfrac{1}{a+x} = \dfrac{-a \cancel{-x} - a \cancel{+x}}{(a-x)(a+x)} = \dfrac{-2a}{a^2 - x^2}\left(= \dfrac{2a}{x^2 - a^2}\right)$

30 $y = \ln\left(e^{\cos x}\right) = \cos x \Rightarrow y' = -\sin x$

If we don't notice the simpler form of the function, we will use more time and it is more likely that during the process we will make a mistake.

$y = \ln\left(e^{\cos x}\right) \Rightarrow y' = \dfrac{1}{\cancel{e^{\cos x}}} \times \cancel{e^{\cos x}} \times (-\sin x) = -\sin x$

31 $y = \dfrac{1}{\log_3 x} \Rightarrow y' = -\dfrac{1}{(\log_3 x)^2} \times \dfrac{1}{x\ln 3} = -\dfrac{1}{x\ln 3\,(\log_3 x)^2}$

32 $y = x\ln x - x \Rightarrow y' = \ln x + \cancel{x} \times \dfrac{1}{\cancel{x}} - 1 = \ln x \cancel{+1} \cancel{-1} = \ln x$

33 $y = \ln(ax) - \ln b \log_b x \Rightarrow y' = \dfrac{\cancel{a}1}{\cancel{a}x} - \cancel{\ln b} \times \dfrac{1}{x\cancel{\ln b}} = \dfrac{1}{x} - \dfrac{1}{x} = 0$

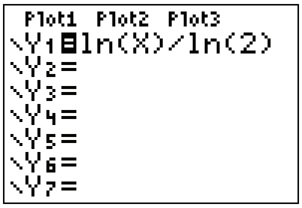
Again, if we spend just a bit more time rewriting the function, the differentiation is much simpler to perform.

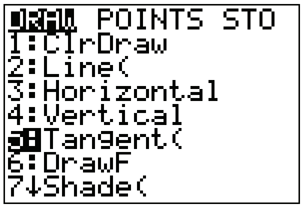
$y = \ln(ax) - \ln b \log_b x = \ln(ax) - \cancel{\ln b} \times \dfrac{\ln x}{\cancel{\ln b}} = \ln\left(\dfrac{a\cancel{x}}{\cancel{x}}\right) = \ln a$, which is a constant; therefore, $y' = 0$.

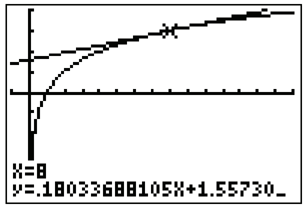
34 $x = 8 \Rightarrow y = \log_2 8 = 3 \Rightarrow P(8, 3)$, $y = \log_2 x \Rightarrow y' = \dfrac{1}{x\ln 2}$, $m = y'(8) = \dfrac{1}{8\ln 2}$

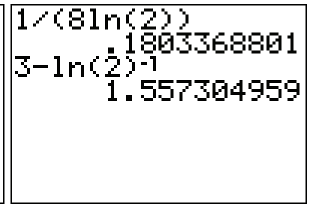
Equation of tangent: $y = \dfrac{1}{8\ln 2}(x-8) + 3 \Rightarrow y = \dfrac{1}{8\ln 2}x - \dfrac{\cancel{8}1}{\cancel{8}\ln 2} + 3 \Rightarrow y = \dfrac{1}{8\ln 2}x - \dfrac{1}{\ln 2} + 3$

When using a calculator to graph this function, we need to use the change of base formula since there are only two types of logarithmic functions given on the calculator, base 10 and base *e*.









The final screen gives the numerical values for the slope and y-intercept of our exact solutions.

35 $y = \sqrt{\frac{x^2-1}{x^2+1}} \Rightarrow \ln y = \ln\sqrt{\frac{x^2-1}{x^2+1}} = \frac{1}{2}\left(\ln\left(x^2-1\right)-\ln\left(x^2+1\right)\right) \Rightarrow$

$\ln y = \frac{1}{2}\left(\ln\left(x^2-1\right)-\ln\left(x^2+1\right)\right) \Big/ \frac{d}{dx} \Rightarrow \frac{y'}{y} = \frac{1}{2}\left(\frac{2x}{x^2-1}-\frac{2x}{x^2+1}\right) = \frac{\cancel{2}x}{\cancel{2}} \times \frac{\cancel{x^2}+1-\cancel{x^2}+1}{\left(x^2-1\right)\left(x^2+1\right)} \Rightarrow$

$y' = y \times \frac{2x}{\left(x^2-1\right)\left(x^2+1\right)} = \sqrt{\frac{x^2-1}{x^2+1}} \times \frac{2x}{\left(x^2-1\right)\left(x^2+1\right)} = \frac{2x}{\left(x^2-1\right)^{\frac{1}{2}}\left(x^2+1\right)^{\frac{3}{2}}}$

36 $y = x^2\ln\left(x^2\right) = 2x^2\ln x \Rightarrow y' = 4x\ln x + 2x^{\cancel{2}} \times \frac{1}{\cancel{x}} = 2x\,(2\ln x + 1)$

$y'' = 2\,(2\ln x + 1) + 2\cancel{x} \cdot \frac{2}{\cancel{x}} = 4\ln x + 2 + 4 = 4\ln x + 6$

$y'' = 0 \Rightarrow 4\ln x + 6 = 0 \Rightarrow \ln x = -\frac{\cancel{6}\,3}{\cancel{4}\,2} \Rightarrow x = e^{-\frac{3}{2}}$ or $\frac{1}{e^{\frac{3}{2}}}$

37 $g(x) = \frac{\ln x}{x} \Rightarrow g'(x) = \frac{\frac{1}{\cancel{x}} \times \cancel{x} - \ln x \times 1}{x^2} = \frac{1-\ln x}{x^2} = (1-\ln x)\,x^{-2}$

$g''(x) = -\frac{1}{x} \times x^{-2} + (1-\ln x)(-2)\,x^{-3} = \frac{-1-2+2\ln x}{x^3} = \frac{2\ln x - 3}{x^3}$

$g'(x) = \frac{1-\ln x}{x^2} = 0 \Rightarrow 1-\ln x = 0 \Rightarrow \ln x = 1 \Rightarrow x = e;\ y = \frac{\ln e}{e} = \frac{1}{e} \Rightarrow P\left(e, \frac{1}{e}\right)$

To show that point P is a maximum, we are going to use the second derivative test.

$g''(e) = \frac{2\ln e - 3}{e^3} = \frac{-1}{e^3} < 0$; therefore, it is a maximum. To show that it is an absolute maximum, we are going to show that the function on its domain is always increasing before point P and always decreasing after point P, by using a table of signs of the first derivative.

Since the denominator of the first derivative is always positive, we are simply going to investigate the sign of the numerator.

$0 < x < e \Rightarrow \ln x < 1 \Rightarrow 1-\ln x > 0 \Rightarrow g'(x) > 0,\ x > e \Rightarrow \ln x > 1 \Rightarrow 1-\ln x < 0 \Rightarrow g'(x) < 0$

x	$0 < x < e$	e	$x > e$
$f'(x)$	positive	0	negative
$f(x)$	increases	maximun	decreases

Therefore, point P is an absolute maximum.

38 $y = \arctan(x+1) \Rightarrow y' = \frac{1}{1+(x+1)^2} = \frac{1}{1+x^2+2x+1} = \frac{1}{x^2+2x+2}$

39 $y = \arcsin\left(\frac{x}{\sqrt{1+x^2}}\right) \Rightarrow y' = \frac{1}{\sqrt{1-\left(\frac{x}{\sqrt{1+x^2}}\right)^2}} \times \frac{\sqrt{1+x^2} - x\frac{2x}{2\sqrt{1+x^2}}}{1+x^2}$

$$= \frac{1}{\sqrt{\frac{1+x^2-x^2}{1+x^2}}} \times \frac{\frac{1+x^2-x^2}{\sqrt{1+x^2}}}{1+x^2} = \sqrt{1+x^2} \times \frac{1}{(1+x^2)\sqrt{1+x^2}} = \frac{1}{1+x^2}$$

40 $y = \arccos\left(\frac{3}{x^2}\right) \Rightarrow y' = -\frac{1}{\sqrt{1-\left(\frac{3}{x^2}\right)^2}} \times 3(-2)x^{-3} = \frac{6x^2}{x^3\sqrt{x^4-9}} = \frac{6}{x\sqrt{x^4-9}}$

41 $\ln y = x \arctan x \Big/ \frac{d}{dx} \Rightarrow \frac{1}{y}y' = \arctan x + \frac{x}{1+x^2} \Rightarrow y' = y\left(\arctan x + \frac{x}{1+x^2}\right)$

Now, if we find the function in explicit form, we obtain the following:

$\ln y = x \arctan x \Rightarrow y = e^{x \arctan x} \Rightarrow y' = e^{x \arctan x}\left(\arctan x + \frac{x}{1+x^2}\right)$

42 $f(x) = \arcsin x + \arccos x \Rightarrow f'(x) = \frac{1}{\sqrt{1-x^2}} + \frac{-1}{\sqrt{1-x^2}} = 0$, so we can conclude that the function is a constant. We can find the value of that constant by using trigonometric identities.

$\sin(\arcsin x + \arccos x) = \sin(\arcsin x)\cos(\arccos x) + \cos(\arcsin x)\sin(\arccos x)$

$$= x \times x + \sqrt{1-\sin^2(\arcsin x)} \times \sqrt{1-\cos^2(\arccos x)}$$

$$= x^2 + \sqrt{1-x^2} \times \sqrt{1-x^2}$$

$$= x^2 + 1 - x^2 = 1$$

$$\Rightarrow \arcsin x + \arccos x = \frac{\pi}{2} + 2k\pi, k \in \mathbb{Z}$$

43 **a)** $y = \arctan\left(\frac{x}{a}\right) \Rightarrow \frac{dy}{dx} = \frac{1}{1+\left(\frac{x}{a}\right)^2} \times \frac{1}{a} = \frac{a^2}{a^2+x^2} \times \frac{1}{a} = \frac{a}{a^2+x^2}$

b) $y = \arcsin\left(\frac{x}{a}\right) \Rightarrow \frac{dy}{dx} = \frac{1}{\sqrt{1-\left(\frac{x}{a}\right)^2}} \times \frac{1}{a} = \frac{a}{\sqrt{a^2-x^2}} \times \frac{1}{a} = \frac{1}{\sqrt{a^2-x^2}}$

44 $y = 4x \arctan 2x \Rightarrow y' = 4\arctan 2x + 4\,2x\frac{\frac{1}{2}}{\frac{1}{4}+x^2} = 4\arctan 2x + \frac{8x}{1+4x^2}$

$x = \frac{1}{2} \Rightarrow y = 4\,2 \times \frac{1}{2} \times \arctan 1 = 2 \times \frac{\pi}{4\,2} = \frac{\pi}{2}$

$m = y'\left(\frac{1}{2}\right) = 4\arctan 1 + \frac{4}{1+1} = 4 \times \frac{\pi}{4} + 2 = \pi + 2$

Equation of tangent: $y = (\pi+2)\left(x - \frac{1}{2}\right) + \frac{\pi}{2} = (\pi+2)x - \frac{\pi}{2} - 1 + \frac{\pi}{2} = (\pi+2)x - 1$

45 **a)** A function is linear if its derivative is a constant.

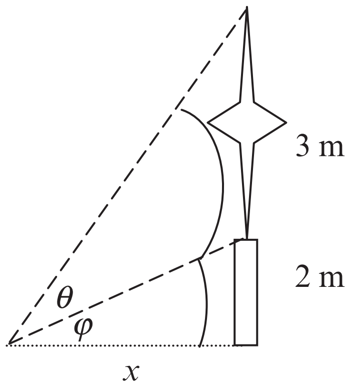
$$f(x) = \arcsin(\cos x) \Rightarrow f'(x) = \frac{1}{\sqrt{1-\cos^2 x}} \times (-\sin x) = \frac{1}{\sin x} \times (-\sin x) = -1$$

b) As we have seen in question 42:

$$\arcsin\theta + \arccos\theta = \frac{\pi}{2} \Rightarrow \arcsin(\cos x) + \overbrace{\arccos(\cos x)}^{x} = \frac{\pi}{2} \Rightarrow \arcsin(\cos x) = \frac{\pi}{2} - x$$

Therefore, $a = -1$ and $b = \frac{\pi}{2}$.

46



$$\tan\varphi = \frac{2}{x},\ \tan(\theta+\varphi) = \frac{5}{x} \Rightarrow \frac{\tan\theta + \tan\varphi}{1 - \tan\theta \times \tan\varphi} = \frac{5}{x}$$

$$\Rightarrow \frac{\tan\theta + \frac{2}{x}}{1 - \tan\theta \times \frac{2}{x}} = \frac{5}{x} \Rightarrow \frac{\frac{x\tan\theta + 2}{\cancel{x}}}{\frac{x - 2\tan\theta}{\cancel{x}}} = \frac{5}{x}$$

$$\Rightarrow x^2\tan\theta + 2x = 5x - 10\tan\theta \Rightarrow \tan\theta\left(x^2+10\right) = 3x$$

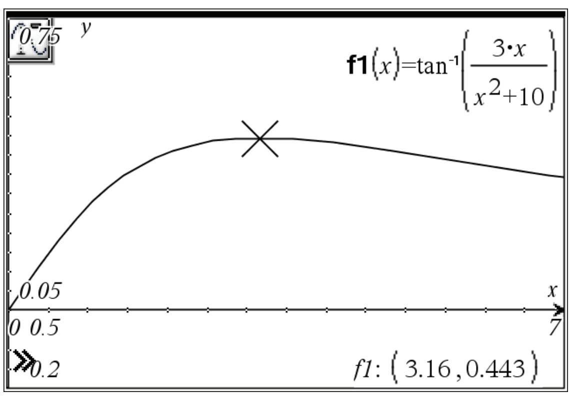
$\Rightarrow \tan\theta = \frac{3x}{x^2+10} \Rightarrow \theta = \arctan\left(\frac{3x}{x^2+10}\right)$ and we need to find the value of x so that θ is maximum.

$$\frac{d\theta}{dx} = \frac{1}{1+\left(\frac{3x}{x^2+10}\right)^2} \times \frac{3\left(x^2+10\right) - 3x \times 2x}{\left(x^2+10\right)^2} = \frac{\cancel{\left(x^2+10\right)^2}}{\left(x^2+10\right)^2 + 9x^2} \times \frac{-3x^2+30}{\cancel{\left(x^2+10\right)^2}} = \frac{-3\left(x^2-10\right)}{\left(x^2+10\right)^2 + 9x^2}$$

$$\frac{d\theta}{dx} = 0 \Rightarrow x^2 - 10 = 0 \Rightarrow \cancel{x = -\sqrt{10}} \text{ and } x = \sqrt{10}.$$

We have discarded one solution since the distance cannot be negative.

Note: Once we find the function, we can input it into a GDC and find the point at which the angle is a maximum.



Notice that in this case we can also see the value of the angle (in radians).

47 $s(t) = \arctan\sqrt{t} \Rightarrow v(t) = s'(t) = \frac{1}{1+t} \times \frac{1}{2\sqrt{t}} = \frac{1}{2\sqrt{t}\,(1+t)}$

a) **i)** $v(1) = \frac{1}{2\sqrt{1}\,(1+1)} = \frac{1}{4}$ m/s

ii) $v(4) = \frac{1}{2\sqrt{4}\,(1+4)} = \frac{1}{20}$ m/s

b) $s'(t) = \dfrac{1}{2\sqrt{t}\,(1+t)} = \dfrac{1}{2}t^{-\frac{1}{2}}(1+t)^{-1} \Rightarrow a(t) = s''(t) = \dfrac{1}{2}\left(-\dfrac{1}{2}t^{-\frac{3}{2}}(1+t)^{-1} + t^{-\frac{1}{2}} \times (-1)(1+t)^{-2}\right)$

$= -\dfrac{1}{4}t^{-\frac{3}{2}}(1+t)^{-2}((1+t)+2t) = -\dfrac{1+3t}{4t^{\frac{3}{2}}(1+t)^2}$

i) $a(1) = -\dfrac{1+3}{4 \times 1^{\frac{3}{2}} \times (1+1)^2} = -\dfrac{1}{4}\ \text{m/s}^2$

ii) $a(4) = -\dfrac{1+3\times 4}{4 \times 4^{\frac{3}{2}} \times (1+4)^2} = -\dfrac{13}{800}\ \text{m/s}^2$

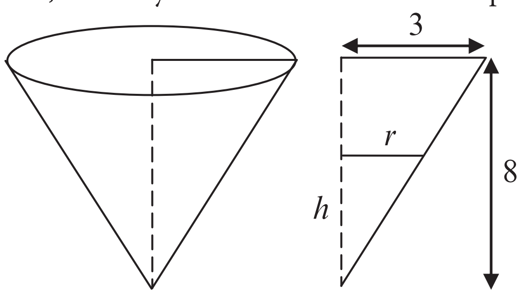
c) The particle is moving to the right and slowing down as the velocity decreases and the acceleration is negative.

d) $\lim\limits_{t\to\infty} s(t) = \lim\limits_{t\to\infty} \arctan\sqrt{t} = \dfrac{\pi}{2}$, as the curve $y = \arctan x$ has a horizontal asymptote: $y = \dfrac{\pi}{2}$.

Exercise 15.4

1 $\dfrac{dV}{dt} = -2\ \text{m}^3/\text{min}$

We can express the volume in terms of the height h only or in terms of the radius r only. In order to do this, we firstly need to find the relationship between h and r. Consider the cross-section of the cone:



By similar triangles:

$\dfrac{r}{h} = \dfrac{3}{8} \Rightarrow r = \dfrac{3}{8}h$ or $h = \dfrac{8}{3}r$

a) $V = \dfrac{1}{3}r^2h\pi \Rightarrow V(h) = \dfrac{1}{3}\left(\dfrac{3}{8}h\right)^2 \times h\pi = \dfrac{3}{64}h^3\pi \Rightarrow \dfrac{dV}{dh} = \dfrac{9}{64}h^2\pi$

$\dfrac{dh}{dt} = \dfrac{dh}{dV} \times \dfrac{dV}{dt} = \dfrac{1}{\frac{dV}{dh}} \times \dfrac{dV}{dt} = \dfrac{64}{9\pi h^2} \times (-2) \Rightarrow \dfrac{dh}{dt}(5) = \dfrac{64}{9\pi \times 25} \times (-2) = -\dfrac{128}{225\pi} \approx -0.181\ \text{m/min}$

b) $V = \dfrac{1}{3}r^2h\pi \Rightarrow V(r) = \dfrac{1}{3}r^2 \times \dfrac{8}{3}r\pi = \dfrac{8}{9}r^3\pi \Rightarrow \dfrac{dV}{dr} = \dfrac{8}{3}r^2\pi;\ h = 5 \Rightarrow r = \dfrac{15}{8}$

$\dfrac{dr}{dt} = \dfrac{dr}{dV} \times \dfrac{dV}{dt} = \dfrac{1}{\frac{dV}{dr}} \times \dfrac{dV}{dt} \Rightarrow \dfrac{dr}{dt}\left(\dfrac{15}{8}\right) = \dfrac{3}{8\left(\frac{15}{8}\right)^2\pi} \times (-2) = -\dfrac{4}{75\pi} \approx -0.0679\ \text{m/min}$

2 $\dfrac{dV}{dt} = 240\ \text{cm}^3/\text{sec},\ V = \dfrac{4}{3}r^3\pi \Rightarrow \dfrac{dV}{dr} = \dfrac{4}{\cancel{3}} \times \cancel{3}r^2\pi = 4r^2\pi$

a) $\dfrac{dr}{dt} = \dfrac{dr}{dV} \times \dfrac{dV}{dt} = \dfrac{1}{\frac{dV}{dr}} \times \dfrac{dV}{dt} \Rightarrow \dfrac{dr}{dt}(r=8) = \dfrac{1}{4 \times 8^2\pi} \times 240 = \dfrac{15}{16\pi} \approx 0.298\ \text{cm/sec}$

b) When $t = 5 \Rightarrow V = 240 \times 5 = 1200; 1200 = \frac{4}{3}\pi r^3 \Rightarrow r = \sqrt[3]{\frac{900}{\pi}}$

$$\frac{dr}{dt}\left(r = \sqrt[3]{\frac{900}{\pi}}\right) = \frac{1}{4 \times \left(\sqrt[3]{\frac{900}{\pi}}\right)^2 \pi} \times 240 \approx 0.439 \text{ cm/sec}$$

3 $\frac{dr}{dt} = 1\text{ cm/hr}, C = 2r\pi \Rightarrow \frac{dC}{dr} = 2\pi, A = r^2\pi \Rightarrow \frac{dA}{dr} = 2r\pi$

a) $\frac{dC}{dt} = \frac{dC}{dr} \times \frac{dr}{dt} \Rightarrow \frac{dr}{dt}(4) = 2\pi \times 1 = 2\pi \approx 6.28 \text{ cm/hr}$

b) $\frac{dA}{dt} = \frac{dA}{dr} \times \frac{dr}{dt} \Rightarrow \frac{dA}{dt}(4) = 2 \times 4 \times \pi \times 1 = 8\pi \approx 25.1 \text{ cm}^2/\text{hr}$

4 $\frac{dh}{dt} = 50 \text{ m/min}, \tan\theta = \frac{h}{150} \Rightarrow \theta = \arctan\left(\frac{h}{150}\right) \Rightarrow \frac{d\theta}{dh} = \frac{150}{150^2 + h^2} = \frac{150}{22\,500 + h^2}$

$$\frac{d\theta}{dt} = \frac{d\theta}{dh} \times \frac{dh}{dt} \Rightarrow \frac{d\theta}{dt}(h = 250) = \frac{150}{22\,500 + 250^2} \times 50 = \frac{7500}{22\,500 + 62\,500} = \frac{3}{34} \approx 0.0882 \text{ radians/min}$$

5 $h = 72 \text{ m}, \frac{dx}{dt} = 6 \text{ m/s}, s = \sqrt{72^2 + x^2} \Rightarrow \frac{ds}{dx} = \frac{\cancel{2}x}{\cancel{2}\sqrt{72^2 + x^2}} = \frac{x}{\sqrt{72^2 + x^2}}$, where s is the length of the string and x is the horizontal distance over which the kite is carried by the wind. When the length of the string is 120 m, we need to calculate the horizontal distance: $x = \sqrt{120^2 - 72^2} = 96$ m.

$$\frac{ds}{dt} = \frac{ds}{dx} \times \frac{dx}{dt} \Rightarrow \frac{ds}{dt}(s = 120, x = 96) = \frac{96}{120} \times 6 = 4.8 \text{ m/sec}$$

6 Let x be the distance of the boy from the lamp post, and s the length of the shadow.

$\frac{dx}{dt} = 6 \text{ ft/sec}, \frac{x}{20 - 5} = \frac{s}{5} \Rightarrow s = \frac{1}{3}x \Rightarrow \frac{ds}{dx} = \frac{1}{3}$

$$\frac{ds}{dt} = \frac{ds}{dx} \times \frac{dx}{dt} \Rightarrow \frac{ds}{dt} = \frac{1}{3} \times 6 = 2 \text{ ft/sec}$$

7 Let x be the distance between the two cars after t hours.

$x = \sqrt{w^2 + n^2} = \sqrt{(60t)^2 + (35t)^2} = 5\sqrt{193}\,t \Rightarrow \frac{dx}{dt} = 5\sqrt{193} \approx 69.5 \text{ km/hr}$

Notice that the rate of increase doesn't depend on time.

8 $\frac{dx}{dt} = 4, y = \sqrt{x^2 + 1} \Rightarrow \frac{dy}{dx} = \frac{\cancel{2}x}{\cancel{2}\sqrt{x^2 + 1}} = \frac{x}{\sqrt{x^2 + 1}}$

$$\frac{dy}{dt} = \frac{dy}{dx} \times \frac{dx}{dt} \Rightarrow \frac{dy}{dt}(x = 3) = \frac{3}{\sqrt{3^2 + 1}} \times 4 = \frac{12}{\sqrt{10}} = \frac{6\sqrt{10}}{5} \approx 3.79$$

9 Let x be the length of the base of the triangle, and h the height of the water level. From the similarity of the shape formed by the water and the trough dimensions, we calculate the relationship between x and h.

$$\frac{dV}{dt} = 0.03 \text{ m}^3/\text{sec}, \frac{1.5}{x} = \frac{1}{h} \Rightarrow x = \frac{3h}{2}, V = \frac{xh}{\cancel{2}} \times \cancel{4}2 = 2xh = \cancel{2} \times \frac{3h}{\cancel{2}} \times h = 3h^2$$

If the time is 25 seconds, we can calculate the volume of water to find the height of the water level.

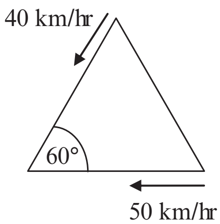
$$t = 25 \Rightarrow V = 0.03 \times 25 = 0.75; 0.75 = 3h^2 \Rightarrow h = 0.5$$

$$\frac{dh}{dt} = \frac{dh}{dV} \times \frac{dV}{dt} = \frac{1}{\frac{dV}{dh}} \times \frac{dV}{dt} \Rightarrow \frac{dh}{dt} = \frac{1}{6h} \times \frac{dV}{dt} = \frac{1}{6 \times 0.5} \times 0.03 = 0.01 \text{ m/sec}$$

10 $\frac{dr}{dt} = 3 \text{ mm/sec}, V = \frac{4}{3} r^3 \pi \Rightarrow \frac{dV}{dr} = \frac{4}{\cancel{3}} \times \cancel{3} r^2 \pi = 4r^2\pi, A = 4r^2\pi = 10 \text{ mm}^2$

$$\frac{dV}{dt} = \frac{dV}{dr} \times \frac{dr}{dt} = 4r^2\pi \times \frac{dr}{dt} \Rightarrow \frac{dV}{dt}(A = 10) = 10 \times 3 = 30 \text{ mm}^3/\text{sec}$$

11



$$s(t) = \sqrt{(2-40t)^2 + (2-50t)^2 - 2(2-40t)(2-50t)\cos 60^\circ} = \sqrt{4 - 180t + 2100t^2}$$

$$\frac{ds}{dt} = \frac{-180 + 4200t}{2\sqrt{4 - 180t + 2100t^2}} = \frac{-90 + 2100t}{\sqrt{4 - 180t + 2100t^2}} \Rightarrow \frac{ds}{dt}(t = 0) = \frac{-90}{\sqrt{4}} = -45$$

The distance is changing at the rate of 45 km/hr.

12 Let x be the length of a diagonal, and a the length of a side.

$$\frac{dx}{dt} = 8 \text{ cm/sec}, x = a\sqrt{3} \Rightarrow a = \frac{1}{\sqrt{3}} x \Rightarrow \frac{da}{dx} = \frac{1}{\sqrt{3}}, \frac{da}{dt} = \frac{da}{dx} \times \frac{dx}{dt} = \frac{1}{\sqrt{3}} \times 8 = \frac{8}{\sqrt{3}} \approx 4.62 \text{ cm/sec}$$

13 $l = r\theta \Rightarrow \theta = \frac{l}{10} \Rightarrow \frac{d\theta}{dt} = \frac{1}{10}\frac{dl}{dt} = 0.3 \text{ units/sec}$

If the vertical distance to the x-axis is 5, we can calculate:

$$\sin\theta = \frac{\cancel{5}1}{\cancel{10}2} \Rightarrow \theta = \frac{\pi}{6} \text{ or } \theta = \frac{5\pi}{6}, x = 10\cos\theta \Rightarrow \frac{dx}{d\theta} = -10\sin\theta$$

$$\frac{dx}{dt} = \frac{dx}{d\theta} \times \frac{d\theta}{dt} \Rightarrow \left|\frac{dx}{dt}\left(\theta = \frac{\pi}{6}\right)\right| = \left|-\sin\frac{\pi}{6}\right| \times 10 \times 0.3 = 1.5 \text{ units/sec}$$

14 Let x be the horizontal distance travelled by the jet, and θ the angle of elevation to the plane.

$$\frac{d\theta}{dt} = \frac{1}{60} \text{ rad/sec}, \tan\theta = \frac{10\,000}{x} \Rightarrow x = 10\,000\cot\theta \Rightarrow \frac{dx}{d\theta} = -\frac{10\,000}{\sin^2\theta}$$

$$\frac{dx}{dt} = \frac{dx}{d\theta} \times \frac{d\theta}{dt} \Rightarrow \frac{dx}{dt}\left(\theta = \frac{\pi}{3}\right) = -\frac{10\,000}{\sin^2\left(\frac{\pi}{3}\right)} \times \frac{1}{60} = -\frac{2000}{9} \approx -222.22...$$

Therefore, we can say that the speed is 222.22 m/sec or 800 km/hr.

15 Firstly, we need to take care of the units. Since the speed of the car is given in km/hr and the speed of the camera should be given in degrees per second, we have to convert km/hr into m/sec:

$$288 \text{ km/hr} = \frac{288}{3.6} = 80 \text{ m/sec}$$

To find the distance travelled by the car, we have the formula $x = 80t$, where t is time given in seconds. There is also a relationship between the angle of the camera and the distance on the track that the car has covered: $\tan\theta = \frac{x}{40} \Rightarrow \theta = \arctan\left(\frac{x}{40}\right)$.

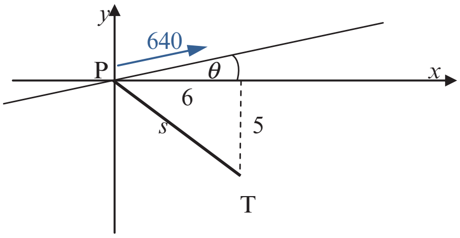
a) $\frac{d\theta}{dt} = \frac{d\theta}{dx} \times \frac{dx}{dt} = \frac{40}{40^2 + x^2} \times 80 = \frac{3200}{40^2 + x^2} \Rightarrow$

$\frac{d\theta}{dt}(x = 0) = \frac{3200}{40^2} = 2 \text{ rad/s} = 2 \times \frac{180^\circ}{\pi} = 115^\circ$ deg/sec, correct to three significant figures.

b) Half a second later, the horizontal distance travelled by the car is: $x = 80 \cdot \frac{1}{2} = 40$ m.

$\frac{d\theta}{dt}(x = 40) = \frac{3200}{40^2 + 40^2} = 1 \text{ rad/sec} = \frac{180^\circ}{\pi} = 57.3^\circ$ deg/sec

16 We need to set up the coordinate system in such a way that the plane is at the origin, the trajectory of the plane is a straight line, $y = x \tan \theta$, and the tower is at the point $(6, -5)$.



$\frac{dy}{dt} = 180 \text{ m/min} = 10.8 \text{ km/h} \Rightarrow \sin \theta = \frac{10.8}{640} \Rightarrow \theta = \arcsin\left(\frac{10.8}{640}\right)$

$x = 640 \cos \theta \times t \Rightarrow \frac{dx}{dt} = 640 \cos \theta$

$s = TP = \sqrt{(x-6)^2 + (x \tan \theta + 5)^2} = \sqrt{x^2 - 12x + 36 + x^2 \tan^2 \theta + 10x \tan \theta + 25}$

$= \sqrt{\left(1 + \tan^2 \theta\right) x^2 + (10 \tan \theta - 12)\, x + 61}$

$\frac{ds}{dx} = \frac{2\left(1 + \tan^2 \theta\right) x + (10 \tan \theta - 12)}{2\sqrt{\left(1 + \tan^2 \theta\right) x^2 + (10 \tan \theta - 12)\, x + 61}} = \frac{\left(1 + \tan^2 \theta\right) x + 5 \tan \theta - 6}{\sqrt{\left(1 + \tan^2 \theta\right) x^2 + (10 \tan \theta - 12)\, x + 61}}$

$\frac{ds}{dt} = \frac{ds}{dx} \times \frac{dx}{dt} \Rightarrow \frac{ds}{dt}(t = 0) = \frac{5 \tan \theta - 6}{\sqrt{61}} \times 640 \cos \theta \approx -485$ km/hr

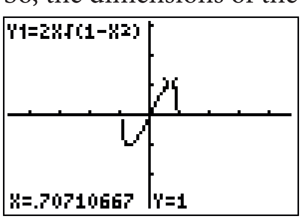
Exercise 15.5

1 The point in the first quadrant has coordinates $\left(x, \sqrt{1 - x^2}\right)$, so the length and width of the rectangle are $l = 2x$ and $w = \sqrt{1 - x^2}$ respectively. The area of the rectangle can then be written as $A(x) = 2x\sqrt{1 - x^2}$. Now, we need to find the maximum area:

$A'(x) = 2 \times \sqrt{1 - x^2} + \cancel{2}x \times \frac{1}{\cancel{2}\sqrt{1 - x^2}} \times (-2x) = \frac{2\left(1 - x^2\right) - 2x^2}{\sqrt{1 - x^2}} = \frac{2\left(1 - 2x^2\right)}{\sqrt{1 - x^2}}$

$A'(x) = 0 \Rightarrow \frac{2\left(1 - 2x^2\right)}{\sqrt{1 - x^2}} = 0 \Rightarrow 1 - 2x^2 = 0 \Rightarrow x = \frac{1}{\sqrt{2}}$, since the point is in the first quadrant.

So, the dimensions of the rectangle are $l = 2 \times \frac{1}{\sqrt{2}} = \sqrt{2}$ and $w = \sqrt{1 - \frac{1}{2}} = \frac{1}{\sqrt{2}} = \frac{\sqrt{2}}{2}$.



Note: We can also find the area of the rectangle: $A\left(\frac{1}{\sqrt{2}}\right) = 2\frac{1}{\sqrt{2}}\sqrt{1-\frac{1}{2}} = 2\frac{1}{\sqrt{2}} \times \frac{1}{\sqrt{2}} = 1.$

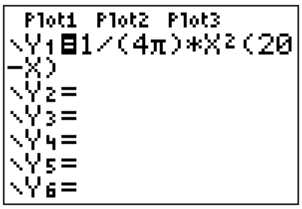
2 Let the rectangle have dimensions $x \times y$, where x is the fold into the base and y is the height of the cylinder. The radius of the base will be: $2r\pi = x \Rightarrow r = \frac{x}{2\pi}$. Therefore, the volume of the cylinder is: $V = r^2\pi h \Rightarrow V = \left(\frac{x}{2\pi}\right)^2 \pi y = \frac{x^2}{4\pi^{\not 2}} \not\pi y = \frac{1}{4\pi}x^2 y$. Since the perimeter is equal to 40 cm, we can express the volume in terms of x only: $2x + 2y = 40 \Rightarrow x + y = 20 \Rightarrow y = 20 - x$, and so $V(x) = \frac{1}{4\pi}x^2(20 - x)$. To find the maximum volume, we need to differentiate the volume with respect to x and find the zero of the derivative:

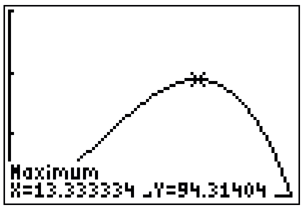
$$V'(x) = \frac{1}{4\pi}\left(2x(20-x) + x^2 \times (-1)\right) = \frac{1}{4\pi}x(40-3x) \Rightarrow V'(x) = 0 \Rightarrow \frac{1}{4\pi}x(40-3x) = 0$$

$$x = 0 \text{ or } 40 - 3x = 0 \Rightarrow x = \frac{40}{3}$$

The first solution is not possible, so we take the second and calculate y: $y = 20 - x \Rightarrow y = \frac{20}{3}$

So, the dimensions of the rectangle are $\frac{40}{3}$ cm and $\frac{20}{3}$ cm.





Notice that the value of y on the graph represents the maximum volume.

3 Any point on the graph has coordinates $(x, \sqrt{x})$; therefore, to find the distance to the given point, we will use the distance formula. To make the calculation simpler, we will use the square of the distance and then, at the end, we will simply take the square root of the value we obtain.

$$g(x) = \left(x - \frac{3}{2}\right)^2 + \left(\sqrt{x} - 0\right)^2 = x^2 - 3x + \frac{9}{4} + x = x^2 - 2x + \frac{9}{4} \Rightarrow g'(x) = 2x - 2$$

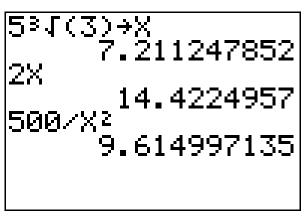
$$g'(x) = 0 \Rightarrow 2x - 2 = 0 \Rightarrow x = 1 \Rightarrow g(1) = 1^2 - 2 \times 1 + \frac{9}{4} = \frac{5}{4} \Rightarrow d = \sqrt{\frac{5}{4}} = \frac{\sqrt{5}}{2}$$

4 **a)** $V = l \times w \times h \Rightarrow 100 = 2x \times x \times h \Rightarrow h = \frac{\not{1000}\,500}{\not 2 x^2} = \frac{500}{x^2}$

b) $S = 2lw + 2lh + 2wh \Rightarrow S(x) = 4x^2 + (4x + 2x)\frac{500}{x^2} = 4x^2 + 6\not x \times \frac{500}{x^{\not 2}} = 4x^2 + \frac{3000}{x}$

c) $S'(x) = 8x - \frac{3000}{x^2} \Rightarrow S'(x) = 0 \Rightarrow 8x - \frac{3000}{x^2} = 0 \Rightarrow 8x = \frac{3000}{x^2} \Rightarrow x^3 = \frac{3000}{8} \Rightarrow$

$$x = \sqrt[3]{\frac{3000}{8}} = \frac{10\sqrt[3]{3}}{2} = 5\sqrt[3]{3} \approx 7.2112$$



So, the length, width and height are 7.21 cm, 14.4 cm and 9.61 cm respectively.

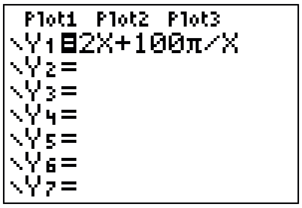
5 Let's denote the width of the rectangle by w, and then the radius of the semicircles is half of the width, $\frac{w}{2}$. Since the rectangle has an area of 100 cm^2, we can express w in terms of x only:

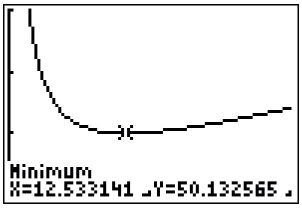
$100 = x \times w \Rightarrow w = \frac{100}{x}$. Now, the perimeter of the figure is equal to $P = 2x + 2r\pi = 2x + w\pi$, which can be expressed in terms of x only: $P(x) = 2x + \frac{100\pi}{x}$. To find the minimum value of the perimeter, we need to differentiate it with respect to x and find the zero of the derivative:

$$P'(x) = 2 - \frac{100\pi}{x^2} \Rightarrow P'(x) = 0 \Rightarrow 2 - \frac{100\pi}{x^2} = 0 \Rightarrow 2 = \frac{100\pi}{x^2} \Rightarrow x^2 = \frac{\cancel{100}\,50\pi}{\cancel{2}} \Rightarrow$$

$x = \sqrt{50\pi} = 5\sqrt{2\pi} \approx 12.533$. So, the answer is 12.5 cm, correct to three significant figures.

We are going to use a GDC to confirm our result. If there are problems with establishing an appropriate window, it is always advisable to look for the y-values in the table.



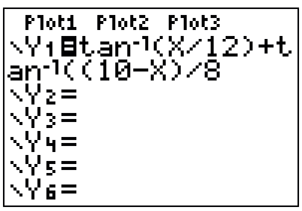


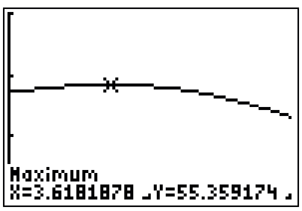
6 Let's denote the angles at the vertical posts by α and β. Using the property that angles along the same transversal have the same measure, we establish the relationship $\theta = \alpha + \beta$.

Now, using the right-angled triangle trigonometry formulae, both angles can be expressed in terms of x only:

$$\left.\begin{array}{l}\tan\alpha = \frac{x}{12} \\ \tan\beta = \frac{10-x}{8}\end{array}\right\} \Rightarrow \left.\begin{array}{l}\alpha = \arctan\left(\frac{x}{12}\right) \\ \beta = \arctan\left(\frac{10-x}{8}\right)\end{array}\right\} \Rightarrow \theta(x) = \arctan\left(\frac{x}{12}\right) + \arctan\left(\frac{10-x}{8}\right)$$

We cannot find the minimum of this function without using a calculator.





Here, not only can we see the value of x for which we obtain the maximum angle, $x = 3.62$, but we can also read the maximum angle, $\theta = 55.4^\circ$.

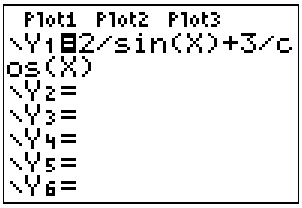
Note: The same problem could have been solved using angles in radians, but we need to remember to set the mode to radians.

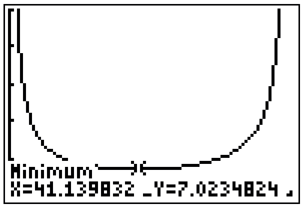
7 Let's denote the angle between the 2-metre wide hallway and the ladder by α. Then the angle between the 3-metre wide hallway and the ladder is: $180^\circ - (\alpha + 90^\circ) = 90^\circ - \alpha$. We split the total length of the ladder into two parts, a and b, to correspond to the lengths in the 2-metre wide and 3-metre wide hallways respectively. Now, from the corresponding right-angled triangles, we get:

$\sin\alpha = \frac{2}{a} \Rightarrow a = \frac{2}{\sin\alpha}$ and $\sin(90^\circ - \alpha) = \cos\alpha = \frac{3}{b} \Rightarrow b = \frac{3}{\cos\alpha}$.

Therefore, the total length of the ladder can be expressed in terms of α only:

$l = a + b \Rightarrow l(\alpha) = \frac{2}{\sin\alpha} + \frac{3}{\cos\alpha}$. Again, we will use a GDC to find the answer.



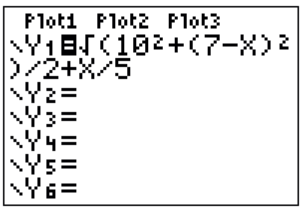


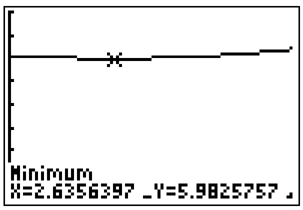
So, the longest ladder that can be carried around the corner is 7.02 m.

Note: The minimum of this function is the maximum length of the ladder that can be carried around the corner of the hallway.

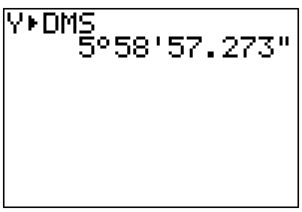
8 $AT = \sqrt{10^2 + (7-d)^2}$. The total time taken is the sum of the time that Charlie takes to walk through the sandy terrain and the time taken to walk on the road: $t = t_S + t_R \Rightarrow t(d) = \frac{\sqrt{10^2 + (7-d)^2}}{2} + \frac{d}{5}$.

This can be simplified, but we can immediately input this expression into the calculator.





So, point A is 2.64 km due west from the office. We can also read the minimum time it takes to walk from the office to the tower as 5.98 hours. On a GDC we can use an angle feature, DMS, to find the minutes and seconds of the minimum time.



So, we can conclude that the minimum time is 5 hours 58 minutes and 57 seconds.

9 We notice that such a rectangle has dimensions $2x \times y$; therefore, the area is calculated by the formula:

$$A = 2xy \Rightarrow A(x) = 2x \times \frac{8}{x^2+4} = \frac{16x}{x^2+4} \Rightarrow A'(x) = \frac{16\left(x^2+4\right) - 16x \times 2x}{\left(x^2+4\right)^2} = \frac{64 - 16x^2}{\left(x^2+4\right)^2}$$

$A'(x) = 0 \Rightarrow 64 - 16x^2 = 0 \Rightarrow x^2 = 4 \Rightarrow x = 2$, since we are taking just the positive x.

$$A(2) = \frac{16 \cdot 2}{2^2 + 4} = 4$$

10 Let x and y be the movements of the first and second ship respectively, and s the distance between the ships. For ease of calculation, we will use the square of the distance.

$$x = 12t,\ y = 10 + 16t \Rightarrow s^2 = (12t)^2 + (10 + 16t)^2 = 144t^2 + 100 + 320t + 256t^2 = 100 + 320t + 400t^2 \Rightarrow$$

$$(s^2)'(t) = 320 + 800t \Rightarrow (s^2)'(t) = 0 \Rightarrow 320 + 800t = 0 \Rightarrow t = -\frac{2}{5}$$

Notice that the minimum distance occurred before we started observing the ships.

$$s\left(-\frac{2}{5}\right) = \sqrt{100 + 320 \times \left(-\frac{2}{5}\right) + 400 \times \left(-\frac{2}{5}\right)^2} = \sqrt{100 - 128 + 64} = 6 \text{ nautical miles}$$

11 $V = r^2\pi h$ is the volume of the inscribed cylinder. If we look at the cross-section of the sphere, we can find the relationship between R, r and h.

$$R^2 = r^2 + \left(\frac{h}{2}\right)^2 \Rightarrow h = 2\sqrt{R^2 - r^2} \Rightarrow V(r) = 2r^2\pi\sqrt{R^2 - r^2} \Rightarrow$$

$$V'(r) = 4r\pi\sqrt{R^2 - r^2} + 2r^2\pi \frac{-\cancel{2}r}{\cancel{2}\sqrt{R^2 - r^2}} = \frac{2r\pi\left(2R^2 - 2r^2 - r^2\right)}{\sqrt{R^2 - r^2}} = \frac{2r\pi\left(2R^2 - 3r^2\right)}{\sqrt{R^2 - r^2}}$$

$$V'(r) = 0 \Rightarrow 2R^2 - 3r^2 = 0 \Rightarrow r_{1,2} = \pm\sqrt{\frac{2}{3}R^2} = \frac{\pm\sqrt{6}R}{3}$$

Since the radius r of the base of the cylinder cannot be negative, we have only one solution:

$$r = \frac{\sqrt{6}R}{3} \Rightarrow h = 2\sqrt{R^2 - \frac{2}{3}R^2} = \frac{2\sqrt{3}R}{3}$$

12 Let p be the distance between the points X and P, and t the time.

$$AP = \sqrt{a^2 + p^2},\ t(p) = \frac{\sqrt{a^2 + p^2}}{c} + \frac{b - p}{r} \Rightarrow t'(p) = \frac{\cancel{2}p}{\cancel{2}c\sqrt{a^2 + p^2}} - \frac{1}{r} = \frac{pr - c\sqrt{a^2 + p^2}}{cr\sqrt{a^2 + p^2}}$$

$$t'(p) = 0 \Rightarrow pr - c\sqrt{a^2 + p^2} = 0 \Rightarrow pr = c\sqrt{a^2 + p^2} \,/(\ )^2 \Rightarrow p^2r^2 = c^2a^2 + c^2p^2$$

$$p^2r^2 - c^2p^2 = a^2c^2 \Rightarrow p^2 = \frac{a^2c^2}{r^2 - c^2} \Rightarrow p = \sqrt{\frac{a^2c^2}{r^2 - c^2}} = \frac{ac}{\sqrt{r^2 - c^2}}$$, since p has to be positive.

13 Since the circumference of the base is: $2\pi \times 10 - x = 20\pi - x \Rightarrow r = \frac{20\pi - x}{2\pi} = 10 - \frac{x}{2\pi}$, we can also express the height h in terms of x.

$$h = \sqrt{100 - r^2} = \sqrt{100 - \left(10 - \frac{x}{2\pi}\right)^2} = \sqrt{\cancel{100} - \cancel{100} + 10\frac{x}{\pi} - \frac{x^2}{4\pi^2}} = \sqrt{10\frac{x}{\pi} - \frac{x^2}{4\pi^2}} = \frac{\sqrt{40\pi x - x^2}}{2\pi}$$

Now, the volume can be expressed in terms of x only.

$$V = \frac{1}{3}r^2\pi h = \frac{1}{3}\left(10 - \frac{x}{2\pi}\right)^2 \cancel{\pi}\frac{\sqrt{40\pi x - x^2}}{2\cancel{\pi}} = \frac{(20\pi - x)^2\sqrt{40\pi x - x^2}}{24\pi^2} \Rightarrow$$

$$V'(x) = \frac{1}{24\pi^2}\left[2(20\pi - x)\times(-1)\times\sqrt{40\pi x - x^2} + (20\pi - x)^2\frac{\cancel{2}(20\pi - x)}{\cancel{2}\sqrt{40\pi x - x^2}}\right]$$

$$= \frac{(20\pi - x)}{24\pi^2\sqrt{40\pi x - x^2}}\left[2\left(x^2 - 40\pi x\right) + (20\pi - x)^2\right]$$

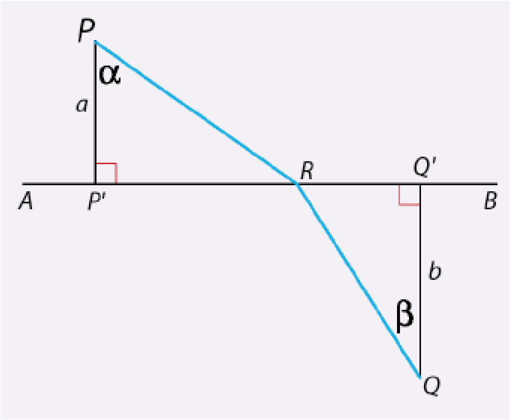
$$= \frac{(20\pi - x)}{24\pi^2\sqrt{40\pi x - x^2}}\left(2x^2 - 80\pi x + 400\pi^2 - 40\pi + x^2\right) = \frac{(20\pi - x)\left(3x^2 - 120\pi x + 400\pi^2\right)}{24\pi^2\sqrt{40\pi x - x^2}}$$

$$V'(x) = 0 \Rightarrow 3x^2 - 120\pi x + 400\pi^2 = 0 \Rightarrow x_{1,2} = \frac{120\pi \pm \sqrt{14\,400\pi^2 - 4800\pi^2}}{6} = 20\pi \pm \frac{20\pi\sqrt{6}}{3}$$

$x = 20\pi - \frac{20\pi\sqrt{6}}{3} \approx 11.5 \text{ cm} \Rightarrow V = \frac{2000\pi\sqrt{3}}{27} \approx 403 \text{ cm}^3$, given correct to three significant figures.

Notice that the other possible solution is discarded because the value of x exceeds the perimeter of the circle.

14



Let's set up the coordinate system in such a way that the origin is at the point R. The distance $P'Q'$ is a fixed positive number d and the distance $P'R$ is our variable x, $x > 0$. Then, the coordinates of the points are as follows: $P(-x, a)$, $R(0, 0)$, $Q(d - x, b)$.

$$t_{Total} = t_1 + t_2 = \frac{PR}{u} + \frac{RQ}{v} = \frac{\sqrt{x^2 + a^2}}{u} + \frac{\sqrt{(d-x)^2 + b^2}}{v}$$

Since the ray travels in such a way that the time is a minimum, we can deduce that $\frac{dt}{dx} = 0$.

$$\frac{dt}{dx} = \frac{\cancel{2}x}{u \times \cancel{2}\sqrt{x^2 + a^2}} + \frac{\cancel{2}(d-x)\times(-1)}{v \times \cancel{2}\sqrt{(d-x)^2 + b^2}} = 0 \Rightarrow \frac{1}{u} \times \frac{x}{\sqrt{x^2 + a^2}} = \frac{1}{u} \times \frac{(d-x)}{\sqrt{(d-x)^2 + b^2}}$$

By looking at triangles $PP'R$ and $RQ'Q$, we can establish the following relationships.

$\sin\alpha = \frac{P'R}{PR} = \frac{x}{\sqrt{x^2 + a^2}}$ and $\sin\beta = \frac{RQ'}{RQ} = \frac{d-x}{\sqrt{(d-x)^2 + b^2}}$; therefore, we obtain the formula:

$$\frac{1}{u} \times \sin\alpha = \frac{1}{u} \times \sin\beta \Rightarrow \frac{\sin\alpha}{\sin\beta} = \frac{u}{v}.$$

Chapter 16

Exercise 16.1

1 $\int (x+2)\,dx = \frac{1}{2}x^2 + 2x + c,\ c \in \mathbb{R}$

2 $\int \left(3t^2 - 2t + 1\right) dx = \cancel{3}\cdot\frac{1}{\cancel{3}}t^3 - \cancel{2}\cdot\frac{1}{\cancel{2}}t^2 + t + c = t^3 - t^2 + t + c,\ c \in \mathbb{R}$

3 $\int \left(\frac{1}{3} - \frac{2}{7}x^3\right) dx = \frac{1}{3}x - \frac{\cancel{2}1}{7}\cdot\frac{1}{\cancel{4}2}x^4 + c = \frac{1}{3}x - \frac{1}{14}x^4 + c,\ c \in \mathbb{R}$

4 $\int (t-1)(2t+3)\,dt = \int \left(2t^2 + t - 3\right) dt = 2\cdot\frac{1}{3}t^3 + \frac{1}{2}t^2 - 3t + c = \frac{2}{3}t^3 + \frac{1}{2}t^2 - 3t + c,\ c \in \mathbb{R}$

5 $\int \left(u^{\frac{2}{5}} - 4u^3\right) du = \frac{1}{\frac{7}{5}}u^{\frac{7}{5}} - \cancel{4}\cdot\frac{1}{\cancel{4}}u^4 + c = \frac{5}{7}u^{\frac{7}{5}} - u^4 + c,\ c \in \mathbb{R}$

6 $\int \left(2\sqrt{x} - \frac{3}{2\sqrt{x}}\right) dx = \int \left(2x^{\frac{1}{2}} - \frac{3}{2}x^{-\frac{1}{2}}\right) dx = 2\cdot\frac{1}{\frac{3}{2}}x^{\frac{3}{2}} - \frac{3}{\cancel{2}}\cdot\frac{1}{\frac{1}{\cancel{2}}}x^{\frac{1}{2}} + c = \frac{4}{3}x^{\frac{3}{2}} - 3x^{\frac{1}{2}} + c,\ c \in \mathbb{R}$

7 $\int (3\sin\theta + 4\cos\theta)\,d\theta = 3(-\cos\theta) + 4\sin\theta + c = -3\cos\theta + 4\sin\theta + c,\ c \in \mathbb{R}$

8 $\int \left(3t^2 - 2\sin t\right) dt = \cancel{3}\cdot\frac{1}{\cancel{3}}t^3 - 2\cdot(-\cos t) + c = t^3 + 2\cos t + c,\ c \in \mathbb{R}$

9 $\int \sqrt{x}\,(2x-5)\,dx = \int \left(2x^{\frac{3}{2}} - 5x^{\frac{1}{2}}\right) dx = 2\cdot\frac{1}{\frac{5}{2}}x^{\frac{5}{2}} - 5\cdot\frac{1}{\frac{3}{2}}x^{\frac{3}{2}} + c = \frac{4}{5}x^{\frac{5}{2}} - \frac{10}{3}x^{\frac{3}{2}} + c,\ c \in \mathbb{R}$

10 $\int \left(3\cos\theta - 2\sec^2\theta\right) d\theta = 3\sin\theta - 2\tan\theta + c,\ c \in \mathbb{R}$

11 $\int e^{3t-1}\,dt = \int \frac{1}{3}e^{3t-1}\,d(3t-1) = \frac{1}{3}e^{3t-1} + c,\ c \in \mathbb{R}$

12 $\int \frac{2}{t}\,dt = 2\int \frac{1}{t}\,dt = 2\ln|t| + c,\ c \in \mathbb{R}$

13 $\int \frac{t}{3t^2+5}\,dt = \int \frac{1}{6}\,\frac{6t\,dt}{3t^2+5} = \frac{1}{6}\int \frac{d\left(3t^2+5\right)}{3t^2+5} = \frac{1}{6}\ln\left(3t^2+5\right) + c,\ c \in \mathbb{R}$

14 $\int e^{\sin\theta}\cos\theta\,d\theta = \int e^{\sin\theta}\,d(\sin\theta) = e^{\sin\theta} + c,\ c \in \mathbb{R}$

15 $\int (3+2x)^2\,dx = \int \frac{1}{2}(3+2x)^2\,d(3+2x) = \frac{1}{2}\times\frac{1}{3}(3+2x)^3 + c = \frac{1}{6}(3+2x)^3 + c,\ c \in \mathbb{R}$

16 $f'(x) = \int \left(4x - 15x^2\right) dx = \cancel{4}2\times\frac{1}{\cancel{2}}x^2 - \cancel{15}5\times\frac{1}{\cancel{3}}x^3 + c = 2x^2 - 5x^3 + c,\ c \in \mathbb{R}$

$f(x) = \int \left(2x^2 - 5x^3 + c\right) dx = 2\times\frac{1}{3}x^3 - 5\times\frac{1}{4}x^4 + cx + k = \frac{2}{3}x^3 - \frac{5}{4}x^4 + cx + k;\ c, k \in \mathbb{R}$

17 $f'(x) = \int \left(1 + 3x^2 - 4x^3\right) dx = x + \cancel{3}\times\frac{1}{\cancel{3}}x^3 - \cancel{4}\times\frac{1}{\cancel{4}}x^4 + c = x + x^3 - x^4 + c,\ c \in \mathbb{R}$

In this problem we are told that $f'(0) = 2$, so we can calculate the constant c:
$f'(0) = 2 \Rightarrow 0 + c = 2 \Rightarrow c = 2 \Rightarrow f'(x) = 2 + x + x^3 - x^4$

Hence, $f(x) = \int (2 + x + x^3 - x^4)\, dx = 2x + \frac{x^2}{2} + \frac{x^4}{4} - \frac{x^5}{5} + k, k \in \mathbb{R}$. Also, we are told that $f(1) = 2$, so we can calculate the constant k:

$$f(1) = \cancel{2} + \frac{1}{2} + \frac{1}{4} - \frac{1}{5} + k = \cancel{2} \Rightarrow k = -\frac{11}{20} \Rightarrow f(x) = -\frac{x^5}{5} + \frac{x^4}{4} + \frac{x^2}{2} + 2x - \frac{11}{20}$$

18 $f'(t) = \int (8t - \sin t)\, dt = \cancel{8}4 \times \frac{1}{\cancel{2}} t^2 - (-\cos t) + c = 4t^2 + \cos t + c, c \in \mathbb{R}$

$$f(t) = \int (4t^2 + \cos t + c)\, dt = \frac{4}{3} t^3 + \sin t + ct + k; c, k \in \mathbb{R}$$

19 $f(x) = \int (12x^3 - 8x + 7)\, dx = \cancel{12}3 \times \frac{1}{\cancel{4}} x^4 - \cancel{8}4 \times \frac{1}{\cancel{2}} x^2 + 7x + c = 3x^4 - 4x^2 + 7x + c, c \in \mathbb{R}$

In this problem the initial condition is given, $f(0) = 3$, so we can calculate the constant c:

$$f(0) = 3 \Rightarrow 0 + c = 3 \Rightarrow c = 3 \Rightarrow f(x) = 3x^4 - 4x^2 + 7x + 3$$

20 $f(\theta) = \int (2\cos\theta - \sin(2\theta))\, d\theta = 2\int \cos\theta\, d\theta - \int \sin(2\theta) \times \frac{1}{2} d(2\theta) + c$

$$= 2\sin\theta - \frac{1}{2}(-\cos(2\theta)) + c = 2\sin\theta + \frac{1}{2}\cos(2\theta) + c, c \in \mathbb{R}$$

21 $\int x(3x^2+7)^5\, dx = \int (3x^2+7)^5 \times \frac{1}{6} d(3x^2+7) = \frac{1}{6} \times \frac{(3x^2+7)^6}{6} + c = \frac{(3x^2+7)^6}{36} + c, c \in \mathbb{R}$

22 $\int \frac{x}{(3x^2+5)^4}\, dx = \int (3x^2+5)^{-4} \times \frac{1}{6} d(3x^2+5) = \frac{1}{6} \times \frac{(3x^2+5)^{-3}}{-3} + c = -\frac{1}{18(3x^2+5)^3} + c, c \in \mathbb{R}$

23 $\int 2x^2 \sqrt[4]{5x^3+2}\, dx = \int (5x^3+2)^{\frac{1}{4}} \times \frac{2}{15} d(5x^3+2) = \frac{2}{15} \times \frac{(3x^2+7)^{\frac{5}{4}}}{\frac{5}{4}} + c = \frac{8\sqrt[4]{(5x^3+2)^5}}{75} + c, c \in \mathbb{R}$

24 $\int \frac{(3+2\sqrt{x})^5}{\sqrt{x}}\, dx = \int (3+2\sqrt{x})^5 \times d(3+2\sqrt{x}) = \frac{(3+2\sqrt{x})^6}{6} + c, c \in \mathbb{R}$

25 $\int t^2\sqrt{2t^3-7}\, dt = \int (2t^3-7)^{\frac{1}{2}} \times \frac{1}{6} d(2t^3-7) = \frac{1}{6} \times \frac{(2t^3-7)^{\frac{3}{2}}}{\frac{3}{2}} + c = \frac{\sqrt{(2t^3-7)^3}}{9} + c, c \in \mathbb{R}$

26 $\int \left(2+\frac{3}{x}\right)^5 \left(\frac{1}{x^2}\right) dx = \int \left(2+\frac{3}{x}\right)^5 \times (-3)\, d\left(2+\frac{3}{x}\right) = -\frac{\left(2+\frac{3}{x}\right)^6}{18} + c = -\frac{(2x+3)^6}{18x^6} + c, c \in \mathbb{R}$

27 $\int \sin(7x-3)\, dx = \int \sin(7x-3) \times \frac{1}{7} d(7x-3) = \frac{1}{7} \times (-\cos(7x-3)) + c = -\frac{\cos(7x-3)}{7} + c, c \in \mathbb{R}$

28 $\int \frac{\sin(2\theta-1)}{\cos(2\theta-1)+3}\, d\theta = \int \frac{1}{\cos(2\theta-1)+3} \times \left(-\frac{1}{2}\right) d(\cos(2\theta-1)+3) = -\frac{1}{2}\ln(\cos(2\theta-1)+3) + c, c \in \mathbb{R}$

29 $\int \sec^2(5\theta-2)\, d\theta = \int \sec^2(5\theta-2) \times \frac{1}{5} d(5\theta-2) = \frac{\tan(5\theta-2)}{5} + c, c \in \mathbb{R}$

30 $\int \cos(\pi x+3)\, dx = \int \cos(\pi x+3) \times \frac{1}{\pi} d(\pi x+3) = \frac{1}{\pi}\sin(\pi x+3) + c, c \in \mathbb{R}$

31 $\int \sec 2t \tan 2t\, dt = \int \frac{\sin 2t}{\cos^2 2t}\, dt = \int (\cos 2t)^{-2} \times -\frac{1}{2} d(\cos 2t) = \frac{1}{2} \times \frac{1}{(\cos 2t)} + c = \frac{1}{2}\sec 2t + c, c \in \mathbb{R}$

32 $\int xe^{x^2+1}\,dx = \int e^{x^2+1} \times \frac{1}{2}\,d\left(x^2+1\right) = \frac{1}{2}e^{x^2+1} + c, c \in \mathbb{R}$

33 $\int \sqrt{t}e^{2t\sqrt{t}}\,dt = \int t^{\frac{1}{2}}e^{2t^{\frac{3}{2}}}\,dt = \int e^{t^{\frac{3}{2}}} \times \frac{1}{3}\,d\left(2t^{\frac{3}{2}}\right) = \frac{1}{3}e^{2t^{\frac{3}{2}}} + c = \frac{1}{3}e^{2t\sqrt{t}} + c, c \in \mathbb{R}$

34 $\int \frac{2}{\theta}(\ln\theta)^2\,d\theta = \int (\ln\theta)^2 \times 2d\,(\ln\theta) = 2 \times \frac{(\ln\theta)^3}{3} + c = \frac{2\,(\ln\theta)^3}{3} + c, c \in \mathbb{R}$

35 $\int \frac{dz}{z\ln 2z} = \int \frac{1}{\ln 2z} \times d\,(\ln 2z) = \ln|\ln 2z| + c, c \in \mathbb{R}$

36 $\int t\sqrt{3-5t^2}\,dt = \int \left(3-5t^2\right)^{\frac{1}{2}} \times \left(-\frac{1}{10}\right)d\left(3-5t^2\right) = -\frac{1}{\not{10}\,5} \times \frac{\left(3-5t^2\right)^{\frac{3}{2}}}{\frac{3}{\not{2}}} + c = -\frac{\sqrt{\left(3-5t^2\right)^3}}{15} + c, c \in \mathbb{R}$

37 $\int \theta^2\sec^2\theta^3 d\theta = \int \sec^2\theta^3 \times \frac{1}{3}\,d\left(\theta^3\right) = \frac{1}{3}\tan\theta^3 + c, c \in \mathbb{R}$

38 $\int \frac{\sin\sqrt{t}}{2\sqrt{t}}\,dt = \int \sin\left(\sqrt{t}\right) \times d\left(\sqrt{t}\right) = -\cos\left(\sqrt{t}\right) + c, c \in \mathbb{R}$

39 $\int \tan^5 2t\sec^2 2t\,dt = \int (\tan 2t)^5 \times \frac{1}{2}\,d\,(\tan 2t) = \frac{1}{2}\frac{\tan^6 2t}{6} + c = \frac{\tan^6 2t}{12} + c, c \in \mathbb{R}$

40 $\int \frac{dx}{\sqrt{x}\left(\sqrt{x}+2\right)} = \int \frac{1}{\left(\sqrt{x}+2\right)} \times 2d\left(\sqrt{x}+2\right) = 2\ln\left(\sqrt{x}+2\right) + c, c \in \mathbb{R}$

41 $\int \sec^5 2t\tan 2t\,dt = \int (\sec 2t)^4\sec 2t\tan 2t\,dt = \int (\sec 2t)^4 \times \frac{1}{2}\,d\,(\sec 2t) = \frac{\sec^5 2t}{10} + c, c \in \mathbb{R}$

42 $\int \frac{x+3}{x^2+6x+7}\,dx = \int \frac{1}{x^2+6x+7} \times \frac{1}{2}\,d\left(x^2+6x+7\right) = \frac{1}{2}\ln\left|x^2+6x+7\right| + c, c \in \mathbb{R}$

43 $\int \frac{k^3x^3}{\sqrt{a^2-a^4x^4}}\,dx = \int \left(a^2-a^4x^4\right)^{-\frac{1}{2}} \times \left(-\frac{k^3}{4a^4}\right)d\left(a^2-a^4x^4\right) = -\frac{k^3}{\not{4}\,2a^4} \times \frac{\left(a^2-a^4x^4\right)^{\frac{1}{2}}}{\frac{1}{\not{2}}} + c$

$= -\frac{k^3\sqrt{a^2-a^4x^4}}{2a^4} + c = -\frac{k^3\,|a|\sqrt{1-a^2x^4}}{2a^4} + c = -\frac{k^3\sqrt{1-a^2x^4}}{2\,|a|^3} + c, c \in \mathbb{R}$

44 In this question we are going to use a proper method of substitution, not just a simple one.

$$\int 3x\sqrt{x-1}\,dx = \left[\begin{array}{c} x-1=t \Rightarrow x=t+1 \\ dx=dt \end{array}\right] = \int 3\,(t+1)\sqrt{t}dt = 3\int\left(t^{\frac{3}{2}}+t^{\frac{1}{2}}\right)dt = 3\left(\frac{t^{\frac{5}{2}}}{\frac{5}{2}}+\frac{t^{\frac{3}{2}}}{\frac{3}{2}}\right) + c$$

$$= \frac{2}{5}t^{\frac{3}{2}}\,(3t+5) + c = \frac{2}{5}t^{\frac{1}{2}}\left(3t^2+5t\right) + c = \frac{2}{5}\sqrt{x-1}\left(3\,(x-1)^2+5\,(x-1)\right) + c$$

$$= \frac{2}{5}\left(3x^2-x-2\right)\sqrt{x-1} + c$$

45 $\int \csc^2\pi t\,dt = \int \csc^2\pi t \times \frac{1}{\pi}\,d\,(\pi t) = -\frac{1}{\pi}\cot(\pi t) + c, c \in \mathbb{R}$

46 $\int \sqrt{1+\cos\theta}\sin\theta\,d\theta = \int (1+\cos\theta)^{\frac{1}{2}} \times (-d\,(1+\cos\theta)) = -\frac{(1+\cos\theta)^{\frac{3}{2}}}{\frac{3}{2}} + c$

$= -\frac{2}{3}\sqrt{(1+\cos\theta)^3} + c, c \in \mathbb{R}$

In questions 47–50, we are going to use a proper method of substitution.

47 $$\int t^2\sqrt{1-t}\,dt = \left[\begin{array}{c} 1-t=x \Rightarrow 1-x=t \\ -dt=dx \end{array}\right] = \int (1-x)^2\sqrt{x}\,(-dx) = -\int\left(x^{\frac{1}{2}}-2x^{\frac{3}{2}}+x^{\frac{5}{2}}\right)dx$$

$$= -\frac{x^{\frac{3}{2}}}{\frac{3}{2}} + 2\frac{x^{\frac{5}{2}}}{\frac{5}{2}} - \frac{x^{\frac{7}{2}}}{\frac{7}{2}} + c = \frac{2}{105}x^{\frac{1}{2}}\left(-35x+42x^2-15x^3\right)+c$$

$$= \frac{2\sqrt{1-t}}{105}\left(-35(1-t)+42(1-t)^2-15(1-t)^3\right)+c = \frac{2\sqrt{1-t}}{105}(t-1)\left(15t^2+12t+8\right)+c,\ c\in\mathbb{R}$$

48 $$\int \frac{r^2-1}{\sqrt{2r-1}}\,dr = \left[\begin{array}{c} 2r-1=x \Rightarrow r=\frac{x+1}{2} \\ 2dr=dx \Rightarrow dr=\frac{dx}{2} \end{array}\right] = \int \frac{\left(\frac{x+1}{2}\right)^2-1}{\sqrt{x}}\frac{dx}{2} = \frac{1}{8}\int\left(x^{\frac{3}{2}}+2x^{\frac{1}{2}}-3x^{-\frac{1}{2}}\right)dx$$

$$= \frac{1}{8}\left(\frac{x^{\frac{5}{2}}}{\frac{5}{2}}+2\frac{x^{\frac{3}{2}}}{\frac{3}{2}}-3\frac{x^{\frac{1}{2}}}{\frac{1}{2}}\right)+c = \frac{1}{60}x^{\frac{1}{2}}\left(3x^2+10x-45\right)+c$$

$$= \frac{1}{60}\sqrt{2r-1}\left(3(2r-1)^2+10(2r-1)-45\right)+c = \frac{\sqrt{2r-1}}{15}\left(3r^2+2r-13\right)+c,\ c\in\mathbb{R}$$

49 $$\int \frac{e^{x^2}-e^{-x^2}}{e^{x^2}+e^{-x^2}}\,x\,dx = \left[\begin{array}{c} e^{x^2}+e^{-x^2}=t \\ \left(e^{x^2}\times 2x+e^{-x^2}\times(-2x)\right)dx=dt \end{array}\right] = \int\frac{1}{t}\frac{dt}{2} = \frac{1}{2}\ln t + c$$

$$= \frac{1}{2}\ln\left(e^{x^2}+e^{-x^2}\right)+c,\ c\in\mathbb{R}$$

50 $$\int \frac{t^2+2}{\sqrt{t-5}}\,dt = \left[\begin{array}{c} t-5=x \Rightarrow t=x+5 \\ dt=dx \end{array}\right] = \int\frac{(x+5)^2+2}{\sqrt{x}}\,dx = \int\left(x^{\frac{3}{2}}+10x^{\frac{1}{2}}+27x^{-\frac{1}{2}}\right)dx$$

$$= \frac{x^{\frac{5}{2}}}{\frac{5}{2}}+10\frac{x^{\frac{3}{2}}}{\frac{3}{2}}+27\frac{x^{\frac{1}{2}}}{\frac{1}{2}}+c = \frac{2}{15}x^{\frac{1}{2}}\left(3x^2+50x+405\right)+c$$

$$= \frac{2}{15}\sqrt{t-5}\left(3(t-5)^2+50(t-5)+405\right)+c = \frac{2\sqrt{t-5}}{15}\left(3t^2+20t+230\right)+c,\ c\in\mathbb{R}$$

Exercise 16.2

As a rule, we use an expression for u that is not too difficult to differentiate and for dv an expression that is easier to integrate.

Note: The new integral should be simpler than the original.

1 To evaluate this integral, we are going to use a method of substitution.

$$\int x^2 e^{-x^3}\,dx = \left[\begin{array}{c} -x^3=t \\ -3x^2dx=dt \end{array}\right] = \int e^t\times\left(-\frac{1}{3}dt\right) = -\frac{1}{3}e^t+c = -\frac{1}{3}e^{-x^3}+c,\ c\in\mathbb{R}$$

In questions 2–4, we have to apply integration by parts twice.

2 $$\int x^2 e^{-x}\,dx = \begin{bmatrix} u = x^2 & du = 2x\,dx \\ dv = e^{-x}dx & v = -e^{-x} \end{bmatrix} = -x^2e^{-x} - \int -e^{-x}2x\,dx = -x^2e^{-x} + 2\int xe^{-x}\,dx$$

$$= \begin{bmatrix} u = x & du = dx \\ dv = e^{-x}dx & v = -e^{-x} \end{bmatrix} = -x^2e^{-x} + 2\left(-xe^{-x} + \int e^{-x}\,dx\right) = -x^2e^{-x} - 2xe^{-x} - 2e^{-x} + c$$

$$= -e^{-x}\left(x^2 + 2x + 2\right) + c,\ c \in \mathbb{R}$$

3 $$\int x^2 \cos 3x\,dx = \begin{bmatrix} u = x^2 & du = 2x\,dx \\ dv = \cos 3x\,dx & v = \frac{1}{3}\sin 3x \end{bmatrix} = \frac{1}{3}x^2 \sin 3x - \int \frac{1}{3}\sin 3x \cdot 2x\,dx$$

$$= \frac{1}{3}x^2 \sin 3x - \frac{2}{3}\int x \sin 3x\,dx = \begin{bmatrix} u = x & du = dx \\ dv = \sin 3x\,dx & v = -\frac{1}{3}\cos 3x \end{bmatrix}$$

$$= \frac{1}{3}x^2 \sin 3x - \frac{2}{3}\left(-\frac{1}{3}x\cos 3x - \int -\frac{1}{3}\cos 3x\,dx\right) = \frac{1}{3}x^2 \sin 3x + \frac{2}{9}x\cos 3x - \frac{2}{27}\sin 3x + c,\ c \in \mathbb{R}$$

4 $$\int x^2 \sin ax\,dx = \begin{bmatrix} u = x^2 & du = 2x\,dx \\ dv = \sin ax\,dx & v = -\frac{1}{a}\cos ax \end{bmatrix} = -\frac{1}{a}x^2\cos ax - \int -\frac{1}{a}\cos ax \cdot 2x\,dx$$

$$= -\frac{1}{a}x^2\cos ax + \frac{2}{a}\int x\cos ax\,dx = \begin{bmatrix} u = x & du = dx \\ dv = \cos ax\,dx & v = \frac{1}{a}\sin ax \end{bmatrix}$$

$$= -\frac{1}{a}x^2\cos ax + \frac{2}{a}\left(\frac{1}{a}x\sin ax - \int \frac{1}{a}\sin ax\,dx\right) = -\frac{1}{a}x^2\cos ax + \frac{2}{a^2}x\sin ax + \frac{2}{a^3}\cos ax + c$$

$$= \frac{1}{a^3}\left(-a^2x^2\cos ax + 2ax\sin ax + 2\cos ax\right) + c,\ c \in \mathbb{R}$$

5 $$\int \cos x \ln(\sin x)\,dx = \begin{bmatrix} u = \ln(\sin x) & du = \frac{1}{\sin x}\cos x\,dx \\ dv = \cos x\,dx & v = \sin x \end{bmatrix}$$

$$= \sin x \ln(\sin x) - \int \cancel{\sin x} \times \frac{1}{\cancel{\sin x}}\cos x\,dx = \sin x \ln(\sin x) - \sin x + c$$

$$= \sin x\left(\ln(\sin x) - 1\right) + c,\ c \in \mathbb{R}$$

6 $$\int x \ln x^2 dx = \begin{bmatrix} u = \ln x^2 = 2\ln x & du = \frac{2}{x}dx \\ dv = x\,dx & v = \frac{1}{2}x^2 \end{bmatrix} = x^2 \ln x - \int \frac{1}{\cancel{2}}x^{\cancel{2}} \times \frac{\cancel{2}}{\cancel{x}}dx = x^2\ln x - \int x\,dx$$

$$= x^2 \ln x - \frac{x^2}{2} + c = \frac{x^2}{2}(2\ln x - 1) + c,\ c \in \mathbb{R}$$

7 $\int x^2 \ln x\, dx = \begin{bmatrix} u = \ln x & du = \frac{1}{x} dx \\ dv = x^2 dx & v = \frac{1}{3} x^3 \end{bmatrix} = \frac{1}{3} x^3 \ln x - \int \frac{1}{3} x^{\cancel{3}2} \times \frac{1}{\cancel{x}} dx = \frac{1}{3} x^3 \ln x - \frac{1}{3} \int x^2\, dx$

$= \frac{1}{3} x^3 \ln x - \frac{1}{3} \times \frac{x^3}{3} + c = \frac{x^3}{9} (3 \ln x - 1) + c, c \in \mathbb{R}$

8 Here, we will firstly use the distribution property and split the integrals.

$\int x^2 (e^x - 1)\, dx = \int x^2 e^x\, dx - \int x^2\, dx = \begin{bmatrix} u = x^2 & du = 2x\, dx \\ dv = e^x dx & v = e^x \end{bmatrix} = x^2 e^x - 2\int x e^x dx - \frac{x^3}{3}$

$= \begin{bmatrix} u = x & du = dx \\ dv = e^x dx & v = e^x \end{bmatrix} = x^2 e^x - 2\left(x e^x - \int e^x\, dx\right) - \frac{x^3}{3}$

$= x^2 e^x - 2x e^x + 2e^x + c - \frac{x^3}{3} = e^x \left(x^2 - 2x + 2\right) - \frac{x^3}{3} + c, c \in \mathbb{R}$

9 $\int x \cos \pi x\, dx = \begin{bmatrix} u = x & du = dx \\ dv = \cos \pi x\, dx & v = \frac{1}{\pi} \sin \pi x \end{bmatrix} = \frac{1}{\pi} x \sin \pi x - \frac{1}{\pi} \int \sin \pi x\, dx$

$= \frac{1}{\pi} x \sin \pi x - \frac{1}{\pi} \times \left(-\frac{1}{\pi} \cos \pi x\right) + c = \frac{1}{\pi^2} (\pi x \sin \pi x + \cos \pi x) + c, c \in \mathbb{R}$

10 $\int e^{3t} \cos 2t\, dt = \begin{bmatrix} u = e^{3t} & du = 3e^{3t} dt \\ dv = \cos 2t\, dt & v = \frac{1}{2} \sin 2t \end{bmatrix} = \frac{1}{2} e^{3t} \sin 2t - \frac{3}{2} \int \sin 2t\, e^{3t} dt$

$= \begin{bmatrix} u = e^{3t} & du = 3e^{3t} dt \\ dv = \sin 2t\, dt & v = -\frac{1}{2} \cos 2t \end{bmatrix} = \frac{1}{2} e^{3t} \sin 2t - \frac{3}{2}\left(-\frac{1}{2} e^{3t} \cos 2t + \frac{3}{2} \int \cos 2t\, e^{3t} dt\right)$

$= \frac{1}{2} e^{3t} \sin 2t + \frac{3}{4} e^{3t} \cos 2t - \frac{9}{4} \int \cos 2t\, e^{3t} dt$

Notice that we have obtained the original integral, so we will solve this equation for the integral.

$\int e^{3t} \cos 2t\, dt = \frac{1}{2} e^{3t} \sin 2t + \frac{3}{4} e^{3t} \cos 2t - \frac{9}{4} \int \cos 2t\, e^{3t} dt \Rightarrow$

$\frac{13}{4} \int e^{3t} \cos 2t\, dt = \frac{1}{4} e^{3t} (2 \sin 2t + 3 \cos 2t) \Big/ \times \frac{4}{13} \Rightarrow$

$\int e^{3t} \cos 2t\, dt = \frac{1}{13} e^{3t} (2 \sin 2t + 3 \cos 2t) + c, c \in \mathbb{R}$

11 $\int \arcsin x\, dx = \begin{bmatrix} u = \arcsin x & du = \frac{1}{\sqrt{1 - x^2}} dx \\ dv = dx & v = x \end{bmatrix} = x \arcsin x - \int \frac{x}{\sqrt{1 - x^2}} dx$

$= x \arcsin x - \int \left(1 - x^2\right)^{-\frac{1}{2}} \times \left(-\frac{1}{2}\right) d\left(1 - x^2\right) = x \arcsin x + \frac{1}{2} \times \frac{\left(1 - x^2\right)^{\frac{1}{2}}}{\frac{1}{2}} + c$

$= x \arcsin x + \sqrt{1 - x^2} + c, c \in \mathbb{R}$

12 $\int x^3e^x\,dx = \begin{bmatrix} u = x^3 & du = 3x^2dx \\ dv = e^x dx & v = e^x \end{bmatrix} = x^3e^x - 3\int e^x x^2\,dx = \begin{bmatrix} u = x^2 & du = 2x\,dx \\ dv = e^x dx & v = e^x \end{bmatrix}$

$= x^3e^x - 3\left(x^2e^x - 2\int e^x x\,dx\right) = \begin{bmatrix} u = x & du = dx \\ dv = e^x dx & v = e^x \end{bmatrix} = x^3e^x - 3x^2e^x + 6\left(xe^x - \int e^x\,dx\right)$

$= x^3e^x - 3x^2e^x + 6xe^x - 6e^x + c = e^x\left(x^3 - 3x^2 + 6x - 6\right) + c,\ c \in \mathbb{R}$

13 $\int e^{-2x}\sin 2x\,dx = \begin{bmatrix} u = e^{-2x} & du = -2e^{-2x}dx \\ dv = \sin 2x\,dx & v = -\frac{1}{2}\cos 2x \end{bmatrix} = -\frac{1}{2}e^{-2x}\cos 2x - \int \sin 2x\,e^{-2x}dx$

$= \begin{bmatrix} u = e^{-2x} & du = -2e^{-2x}dx \\ dv = \cos 2x\,dx & v = \frac{1}{2}\sin 2x \end{bmatrix} = -\frac{1}{2}e^{-2x}\cos 2x - \left(\frac{1}{2}e^{-2x}\sin 2x + \int \sin 2t\,e^{-2x}dx\right)$

$= -\frac{1}{2}e^{-2x}\cos 2x - \frac{1}{2}e^{-2x}\sin 2x - \int \sin 2t\,e^{-2x}dx$

$\int e^{-2x}\sin 2x\,dx = -\frac{1}{2}e^{-2x}\cos 2x - \frac{1}{2}e^{-2x}\sin 2x - \int \sin 2t\,e^{-2x}dx \Rightarrow$

$2\int e^{-2x}\sin 2x\,dx = -\frac{1}{2}e^{-2x}(\cos 2x + \sin 2x) \Rightarrow$

$\int e^{-2x}\sin 2x\,dx = -\frac{1}{4}e^{-2x}(\cos 2x + \sin 2x) + c,\ c \in \mathbb{R}$

14 $\int \sin(\ln x)\,dx = \begin{bmatrix} u = \sin(\ln x) & du = \cos(\ln x)\frac{dx}{x} \\ dv = dx & v = x \end{bmatrix} = x\sin(\ln x) - \int \cancel{x}\cos(\ln x)\frac{dx}{\cancel{x}}$

$= \begin{bmatrix} u = \cos(\ln x) & du = -\sin(\ln x)\frac{dx}{x} \\ dv = dx & v = x \end{bmatrix} = x\sin(\ln x) - \left(x\cos(\ln x) + \int \cancel{x}\sin(\ln x)\frac{dx}{\cancel{x}}\right)$

$= x\sin(\ln x) - x\cos(\ln x) - \int \sin(\ln x)\,dx$

$2\int \sin(\ln x)\,dx = x\sin(\ln x) - x\cos(\ln x) \Rightarrow \int \sin(\ln x)\,dx = \frac{1}{2}x(\sin(\ln x) - \cos(\ln x)) + c,\ c \in \mathbb{R}$

15 $\int \cos(\ln x)\,dx = \begin{bmatrix} u = \cos(\ln x) & du = -\sin(\ln x)\frac{dx}{x} \\ dv = dx & v = x \end{bmatrix} = x\cos(\ln x) + \int \cancel{x}\sin(\ln x)\frac{dx}{\cancel{x}}$

$= \begin{bmatrix} u = \sin(\ln x) & du = \cos(\ln x)\frac{dx}{x} \\ dv = dx & v = x \end{bmatrix} = x\cos(\ln x) + \left(x\sin(\ln x) - \int \cancel{x}\cos(\ln x)\frac{dx}{\cancel{x}}\right)$

$= x\cos(\ln x) + x\sin(\ln x) - \int \cos(\ln x)\,dx$

$2\int \cos(\ln x)\,dx = x\cos(\ln x) + x\sin(\ln x) \Rightarrow \int \cos(\ln x)\,dx = \frac{1}{2}x(\cos(\ln x) + \sin(\ln x)) + c,\ c \in \mathbb{R}$

$$16\quad \int \ln\left(x+x^2\right)dx = \begin{bmatrix} u=\ln\left(x+x^2\right) & du=\dfrac{1+2x}{x+x^2}dx \\ dv=dx & v=x \end{bmatrix} = x\ln\left(x+x^2\right)-\int\frac{x+2x^2}{x+x^2}dx$$

$$= x\ln\left(x+x^2\right)-\int\frac{1+2x}{1+x}dx = x\ln\left(x+x^2\right)-\int\left(2-\frac{1}{1+x}\right)dx$$

$$= x\ln\left|x+x^2\right|-2x+\ln|1+x|+c,\ c\in\mathbb{R}$$

$$17\quad \int e^{kx}\sin x\,dx = \begin{bmatrix} u=e^{kx} & du=ke^{kx}dx \\ dv=\sin x\,dx & v=-\cos x \end{bmatrix} = -e^{kx}\cos x + k\int e^{kx}\cos x\,dx$$

$$= \begin{bmatrix} u=e^{kx} & du=ke^{kx}dx \\ dv=\cos x\,dx & v=\sin x \end{bmatrix} = -e^{kx}\cos x + k\left(e^{kx}\sin x - k\int e^{kx}\sin x\,dx\right)$$

$$= -e^{kx}\cos x + ke^{kx}\sin x - k^2\int e^{kx}\sin x\,dx$$

$$\left(1+k^2\right)\int e^{kx}\sin x\,dx = e^{kx}\left(k\sin x-\cos x\right) \Rightarrow \int e^{kx}\sin x\,dx = \frac{e^{kx}\left(k\sin x-\cos x\right)}{1+k^2}+c,\ c\in\mathbb{R}$$

$$18\quad \int x\sec^2 x\,dx = \begin{bmatrix} u=x & du=dx \\ dv=\sec^2 x\,dx & v=\tan x \end{bmatrix} = x\tan x-\int\tan x\,dx = x\tan x-\int\frac{\sin x}{\cos x}dx$$

$$= x\tan x-\int\frac{-d\left(\cos x\right)}{\cos x} = x\tan x+\ln|\cos x|+c,\ c\in\mathbb{R}$$

$$19\quad \int\sin x\sin 2x\,dx = \begin{bmatrix} u=\sin 2x & du=2\cos 2x\,dx \\ dv=\sin x\,dx & v=-\cos x \end{bmatrix} = -\sin 2x\cos x+2\int\cos x\cos 2x\,dx$$

$$= \begin{bmatrix} u=\cos 2x & du=-2\sin 2x\,dx \\ dv=\cos x\,dx & v=\sin x \end{bmatrix} = -\sin 2x\cos x+2\left(\cos 2x\sin x+2\int\sin x\sin 2x\,dx\right)$$

$$= -\sin 2x\cos x+2\cos 2x\sin x+4\int\sin x\sin 2x\,dx \Rightarrow$$

$$\sin 2x\cos x-2\cos 2x\sin x = 3\int\sin x\sin 2x\,dx \Rightarrow$$

$$\int\sin x\sin 2x\,dx = \frac{1}{3}\left(\sin 2x\cos x-2\cos 2x\sin x\right)+c,\ c\in\mathbb{R}$$

We obtained this solution by using integration by parts. A much simpler solution can be found by using the trigonometric identities and the method of simple integration.

$$\int\sin x\overbrace{\sin 2x}^{2\sin x\cos x}dx = 2\int\sin^2 x\cos x\,dx = 2\int\sin^2 x\,d\left(\sin x\right) = \frac{2}{3}\sin^3 x+c,\ c\in\mathbb{R}$$

Notice that the solutions of trigonometric integrals can be written in many different forms, but we can show that all of these forms are mutually equivalent or that they differ by a constant.

20 $\int x \arctan x \, dx = \begin{bmatrix} u = \arctan x & du = \dfrac{dx}{1+x^2} \\ dv = x\,dx & v = \dfrac{1}{2}x^2 \end{bmatrix} = \frac{1}{2}x^2 \arctan x - \frac{1}{2}\int \frac{x^2 dx}{1+x^2}$

$= \frac{1}{2}x^2 \arctan x - \frac{1}{2}\int\left(1 - \frac{1}{1+x^2}\right)dx = \frac{1}{2}x^2 \arctan x - \frac{1}{2}x + \frac{1}{2}\arctan x + c$

$= \frac{1}{2}\left((x^2+1)\arctan x - x\right) + c, c \in \mathbb{R}$

21 $\int \frac{\ln x}{\sqrt{x}}\,dx = \begin{bmatrix} u = \ln x & du = \dfrac{dx}{x} \\ dv = \dfrac{dx}{\sqrt{x}} & v = 2\sqrt{x} \end{bmatrix} = 2\sqrt{x}\ln x - 2\int x^{\frac{1}{2}}\frac{dx}{x} = 2\sqrt{x}\ln x - 2\int x^{-\frac{1}{2}}\,dx$

$= 2\sqrt{x}\ln x - 4\sqrt{x} + c = 2\sqrt{x}\,(\ln x - 2) + c, c \in \mathbb{R}$

22 Apart from the variable, t instead of x, this is the same question as question 18.

$\int t \sec^2 t \, dt = t \tan t + \ln|\cos t| + c, c \in \mathbb{R}$

23

u	dv	sign
	$\sin x\,dx$	
x^2	$-\cos x$	+
$2x$	$-\sin x$	−
2	$\cos x$	+

In the first column, we have a sequence of derivatives of u and in the second, the sequence of anti-derivatives of dv. The reason why the signs alternate is a direct consequence of the formula, where the second integral always takes a minus.

$\int x^2 \sin x \, dx = -x^2\cos x - \int -\cos x \, 2x\,dx = -x^2 \cos x - \left(2x(-\sin x) - \int(-\sin x)\,2\,dx\right)$

$= -x^2\cos x + 2x\sin x + 2\cos x + c, c \in \mathbb{R}$

24

	$\sin x$	
x^4	$-\cos x$	+
$4x^3$	$-\sin x$	−
$12x^2$	$\cos x$	+
$24x$	$\sin x$	−
24	$-\cos x$	+

$\int x^4 \sin x\,dx = -x^4\cos x + 4x^3\sin x + 12x^2\cos x - 24x\sin x - 24\cos x + c, c \in \mathbb{R}$

25

	$\cos x$	
x^5	$\sin x$	+
$5x^4$	$-\cos x$	−
$20x^3$	$-\sin x$	+
$60x^2$	$\cos x$	−
$120x$	$\sin x$	+
120	$-\cos x$	−

$\int x^5 \cos x\,dx = x^5\sin x + 5x^4\cos x - 20x^3\sin x - 60x^2\cos x + 120x\sin x + 120\cos x + c, c \in \mathbb{R}$

26

	e^x	
x^4	e^x	+
$4x^3$	e^x	−
$12x^2$	e^x	+
$24x$	e^x	−
24	e^x	+

$$\int x^4e^x\,dx = x^4e^x - 4x^3e^x + 12x^2e^x - 24xe^x + 24e^x + c = e^x\left(x^4 - 4x^3 + 12x^2 - 24x + 24\right) + c,\ c \in \mathbb{R}$$

27 The method used in question 23 cannot give the result because, in the second column, we have anti-derivatives, and the anti-derivative of the natural logarithm is a more complicated logarithmic expression.

$\int \ln x\,dx = x(\ln x - 1) + c$, and finding the anti-derivative would further complicate the integration.

Note: The logarithmic function is always taken as u since we need to differentiate it.

28 $$\int x^ne^x\,dx = \begin{bmatrix} u = x^n & du = nx^{n-1}dx \\ dv = e^xdx & v = e^x \end{bmatrix} = x^ne^x - \int e^xnx^{n-1}\,dx = x^ne^x - n\int x^{n-1}e^x\,dx$$

$$\int x^4e^x\,dx = x^4e^x - 4\int x^3e^x\,dx = x^4e^x - 4\left(x^3e^x - 3\int x^2e^x\,dx\right)$$

$$= x^4e^x - 4x^3e^x + 12\left(x^2e^x - 2\int xe^x\,dx\right) = x^4e^x - 4x^3e^x + 12x^2e^x - 24\left(xe^x - \int e^x\,dx\right)$$

$$= x^4e^x - 4x^3e^x + 12x^2e^x - 24xe^x + 24e^x + g,\ g \in \mathbb{R}$$

29 $$\int x^n\ln x\,dx = \begin{bmatrix} u = \ln x & du = \frac{1}{x}dx \\ dv = x^ndx & v = \frac{1}{n+1}x^{n+1} \end{bmatrix} = \frac{1}{n+1}x^{n+1}\ln x - \int \frac{1}{n+1}x^{\cancel{n+1}\,n} \times \frac{1}{\cancel{x}}\,dx$$

$$= \frac{x^{n+1}}{n+1}\ln x - \frac{1}{n+1}\int x^n\,dx = \frac{x^{n+1}}{n+1}\ln x - \frac{1}{n+1} \times \frac{x^{n+1}}{n+1} + c = \frac{x^{n+1}}{n+1}\ln x - \frac{x^{n+1}}{(n+1)^2} + c,\ c \in \mathbb{R}$$

30 $$\int e^{mx}\cos nx\,dx = \begin{bmatrix} u = e^{mx} & du = me^{mx}dx \\ dv = \cos nx\,dx & v = \frac{1}{n}\sin nx \end{bmatrix} = \frac{1}{n}e^{mx}\sin nx - \frac{m}{n}\int \sin nx\,e^{mx}dx$$

$$= \begin{bmatrix} u = e^{mx} & du = me^{mx}dx \\ dv = \sin nx\,dx & v = -\frac{1}{n}\cos nx \end{bmatrix} = \frac{1}{n}e^{mx}\sin nx - \frac{m}{n}\left(-\frac{1}{n}e^{mx}\cos nx + \frac{m}{n}\int \cos nx\,e^{mx}dx\right)$$

$$= \frac{1}{n}e^{mx}\sin nx + \frac{m}{n^2}e^{mx}\cos nx - \frac{m^2}{n^2}\int \cos nx\,e^{mx}dx \Rightarrow$$

$$\left(1 + \frac{m^2}{n^2}\right)\int \cos nx\,e^{mx}dx = \frac{e^{mx}(n\sin nx + m\cos nx)}{n^2} \Rightarrow$$

$$\int \cos nx\,e^{mx}dx = \frac{e^{mx}(n\sin nx + m\cos nx)}{\cancel{n^2}} \times \frac{\cancel{n^2}}{m^2 + n^2} + c = \frac{e^{mx}(n\sin nx + m\cos nx)}{m^2 + n^2} + c,\ c \in \mathbb{R}$$

31 $$\int e^{mx}\sin nx\,dx = \begin{bmatrix} u = e^{mx} & du = me^{mx}dx \\ dv = \sin nx\,dx & v = -\frac{1}{n}\cos nx \end{bmatrix} = -\frac{e^{kx}\cos nx}{n} + \frac{m}{n}\int e^{mx}\cos nx\,dx$$

$$= \begin{bmatrix} u = e^{mx} & du = me^{mx}dx \\ dv = \cos nx\,dx & v = \frac{1}{n}\sin nx \end{bmatrix} = -\frac{e^{kx}\cos nx}{n} + \frac{m}{n}\left(\frac{1}{n}e^{mx}\sin nx - \frac{m}{n}\int e^{mx}\sin nx\,dx\right)$$

$$= -\frac{e^{kx}\cos nx}{n} + \frac{me^{mx}\sin nx}{n^2} - \frac{m^2}{n^2}\int e^{mx}\sin nx\,dx \Rightarrow$$

$$\left(1 + \frac{m^2}{n^2}\right)\int e^{kx}\sin nx\,dx = \frac{-ne^{kx}\cos nx + me^{mx}\sin nx}{n^2} \Rightarrow$$

$$\int e^{kx}\sin nx\,dx = \left(\frac{-ne^{mx}\cos nx + me^{mx}\sin nx}{\cancel{n^2}}\right)\frac{\cancel{n^2}}{m^2+n^2} + c = \frac{e^{mx}(m\sin nx - n\cos nx)}{m^2+n^2} + c,\ c \in \mathbb{R}$$

Exercise 16.3

1 $$\int \sin^3 t\cos^2 t\,dt = \int \sin t\left(1-\cos^2 t\right)\cos^2 t\,dt = \int \sin t\cos^2 t\,dt - \int \sin t\cos^4 t\,dt$$

$$= \int \cos^2 t\,(-d(\cos t)) - \int \cos^4 t\,(-d(\cos t)) = \frac{\cos^5 t}{5} - \frac{\cos^3 t}{3} + c,\ c \in \mathbb{R}$$

2 $$\int \sin^3 t\cos^3 t\,dt = \int \sin t\left(1-\cos^2 t\right)\cos^3 t\,dt = \int \sin t\cos^3 t\,dt - \int \sin t\cos^5 t\,dt$$

$$= \int \cos^3 t\,(-d(\cos t)) - \int \cos^5 t\,(-d(\cos t)) = \frac{\cos^6 t}{6} - \frac{\cos^4 t}{4} + c,\ c \in \mathbb{R}$$

We could have performed a similar transformation with the cosine expression:

$$\int \sin^3 t\cos^3 t\,dt = \int \sin^3 t\cos t\left(1-\sin^2 t\right)dt = \int \sin^3 t\cos t\,dt - \int \sin^5 t\cos t\,dt$$

$$= \int \sin^3 t\,d(\sin t) - \int \sin^5 t\,d(\sin t) = \frac{\sin^4 t}{4} - \frac{\sin^6 t}{6} + c,\ c \in \mathbb{R}$$

3 $$\int \sin^3 3\theta\cos 3\theta\,d\theta = \int \sin^3 3\theta \times \frac{1}{3}d(\sin 3\theta) = \frac{1}{3}\times\frac{\cos^4 3\theta}{4} + c = \frac{\cos^4 3\theta}{12} + c,\ c \in \mathbb{R}$$

4 $$\int \frac{1}{t^2}\sin^5\left(\frac{1}{t}\right)\cos^2\left(\frac{1}{t}\right)dt = \begin{bmatrix} \frac{1}{t} = x \\ -\frac{1}{t^2}dt = dx \end{bmatrix} = \int -\sin^5 x\cos^2 x\,dx$$

$$= -\int \sin x\left(1-\cos^2 x\right)^2\cos^2 x\,dx = \int \cos^2 x\,d(\cos x) - 2\int \cos^4 x\,d(\cos x) + \int \cos^6 x\,d(\cos x)$$

$$= \frac{\cos^3 x}{3} - \frac{2\cos^5 x}{5} + \frac{\cos^7 x}{7} + c = \frac{\cos^3\left(\frac{1}{t}\right)}{3} - \frac{2\cos^5\left(\frac{1}{t}\right)}{5} + \frac{\cos^7\left(\frac{1}{t}\right)}{7} + c,\ c \in \mathbb{R}$$

5 $\int \frac{\sin^3 x}{\cos^2 x} dx = \int \frac{\sin^2 x}{\cos^2 x} \sin x\, dx = \int \frac{1-\cos^2 x}{\cos^2 x} \sin x\, dx = \int \left(\frac{1}{\cos^2 x} - 1 \right) \sin x\, dx$

$= \int \frac{1}{\cos^2 x} (-d \cos x) - \int \sin x\, dx = -\frac{\cos^{-1} x}{-1} + \cos x + c = \sec x + \cos x + c, c \in \mathbb{R}$

6 $\int \tan^5 3x \sec^2 3x\, dx = \left[\begin{array}{c} \tan 3x = t \\ \sec^2 3x \times 3\, dx = dt \end{array} \right] = \int t^5 \times \frac{1}{3} dt = \frac{1}{3} \times \frac{t^6}{6} + c = \frac{\tan^6 3x}{18} + c, c \in \mathbb{R}$

7 $\int \theta \tan^3 \theta^2 \sec^4 \theta^2 d\theta = \left[\begin{array}{c} \theta^2 = t \\ 2\theta\, d\theta = dt \end{array} \right] = \int \tan^3 t \sec^4 t \times \frac{1}{2} dt = \frac{1}{2} \int \tan^3 t \left(\tan^2 t + 1 \right) \sec^2 t\, dt$

$= \frac{1}{2} \int \left(\tan^5 t + \tan^3 t \right) d\left(\tan t \right) = \frac{1}{2} \left(\frac{\tan^6 t}{6} + \frac{\tan^4 t}{4} \right) + c = \frac{\tan^6 \theta^2}{12} + \frac{\tan^4 \theta^2}{8} + c, c \in \mathbb{R}$

8 $\int \frac{1}{\sqrt{t}} \tan^3 \sqrt{t} \sec^3 \sqrt{t}\, dt = \left[\begin{array}{c} \sqrt{t} = x \\ \frac{1}{2\sqrt{t}} dt = dx \end{array} \right] = \int \tan^3 x \sec^3 x \times 2\, dx = 2 \int \frac{\sin^3 x}{\cos^6 x} dx$

$= 2 \int \frac{\left(1-\cos^2 x\right) \sin x}{\cos^6 x} dx = 2 \int \left(\frac{1}{\cos^6 x} - \frac{1}{\cos^4 x} \right) \left(-d\left(\cos x\right)\right) = 2 \left(\frac{\cos^{-3} x}{-3} - \frac{\cos^{-5} x}{-5} \right) + c$

$= \frac{2 \sec^5 \sqrt{t}}{5} - \frac{2 \sec^3 \sqrt{t}}{3} + c, c \in \mathbb{R}$

9 $\int \tan^4 5t\, dt = \int \left(\left(\tan^4 5t - 1 \right) + 1 \right) dt = \int \left(\left(\tan^2 5t - 1 \right) \left(\tan^2 5t + 1 \right) + 1 \right) dt$

$\int \left(\left(\tan^2 5t - 1 \right) \sec^2 5t + 1 \right) dt = \int \tan^2 5t \times \frac{1}{5} d\left(\tan^2 5t \right) - \int \sec^2 5t\, dt + \int dt$

$= \frac{1}{5} \times \frac{\tan^3 5t}{3} - \frac{\tan 5t}{5} + t + c = \frac{\tan^3 5t}{15} - \frac{\tan 5t}{5} + t + c, c \in \mathbb{R}$

10 $\int \frac{dt}{1+\sin t} = \int \frac{1-\sin t}{(1+\sin t)(1-\sin t)} dt = \int \frac{1-\sin t}{1-\sin^2 t} dt = \int \frac{1-\sin t}{\cos^2 t} dt = \int \frac{dt}{\cos^2 t} - \int \frac{-d\left(\cos t\right)}{\cos^2 t}$

$= \tan t + \frac{\cos^{-1} t}{-1} + c = \tan t - \sec t + c, c \in \mathbb{R}$

11 $\int \frac{d\theta}{1+\cos\theta} = \int \frac{1-\cos\theta}{(1+\cos\theta)(1-\cos\theta)} d\theta = \int \frac{1-\cos\theta}{1-\cos^2\theta} d\theta = \int \frac{1-\cos\theta}{\sin^2\theta} d\theta$

$= \int \frac{d\theta}{\sin^2\theta} - \int \frac{d\left(\sin\theta\right)}{\sin^2\theta} = -\cot\theta - \frac{\sin^{-1}\theta}{-1} + c = -\cot\theta + \csc\theta + c, c \in \mathbb{R}$

12 $\int \frac{1+\sin t}{\cos t} dt = \int \frac{(1+\sin t)(1-\sin t)}{\cos t\,(1-\sin t)} dt = \int \frac{1-\sin^2 t}{\cos t\,(1-\sin t)} dt = \int \frac{\cos^{\not{2}} t}{\not{\cos t}\,(1-\sin t)} dt$

$= \int \frac{-d\left(1-\sin t\right)}{1-\sin t} = -\ln\left|1-\sin t\right| + c, c \in \mathbb{R}$

13 $\int \frac{\sin x - 5\cos x}{\sin x + \cos x} dx = \int \frac{-2\left(\sin x + \cos x\right) - 3\left(\cos x - \sin x\right)}{\sin x + \cos x} dx = -2\int dx - 3\int \frac{d\left(\sin x + \cos x\right)}{\sin x + \cos x}$

$= -2x - 3\ln\left|\sin x + \cos x\right| + c, c \in \mathbb{R}$

14 $\int \frac{\sec\theta\tan\theta}{1+\sec^2\theta}\,d\theta = \begin{bmatrix} \sec\theta = t \\ \sec\theta\tan\theta\,d\theta = dt \end{bmatrix} = \int \frac{dt}{1+t^2} = \arctan t + c = \arctan(\sec\theta) + c, c \in \mathbb{R}$

15 $\int \frac{\arctan t}{1+t^2}\,dt = \int \arctan t \; d(\arctan t) = \frac{\arctan^2 t}{2} + c, c \in \mathbb{R}$

16 $\int \frac{1}{(1+t^2)\arctan t}\,dt = \int \frac{1}{\arctan t}\,d(\arctan t) = \ln|\arctan t| + c, c \in \mathbb{R}$

17 $\int \frac{dx}{x\sqrt{1-\ln^2 x}} = \begin{bmatrix} \ln x = t \\ \frac{1}{x}\,dx = dt \end{bmatrix} = \int \frac{dt}{\sqrt{1-t^2}} = \arcsin t + c = \arcsin(\ln x) + c, c \in \mathbb{R}$

18 $\int \sin^3 x\,dx = \int \sin x\left(1-\cos^2 x\right)dx = \int \sin x\,dx - \int \cos^2 x\,(-d(\cos x))$

$= -\cos x + \frac{\cos^3 x}{3} + c, c \in \mathbb{R}$

19 $\int \frac{\sin^3 x}{\sqrt{\cos x}}\,dx = \int \frac{\sin x\left(1-\cos^2 x\right)}{\sqrt{\cos x}}\,dx = \int \frac{\sin x}{\sqrt{\cos x}}\,dx - \int \sin x \cos^{\frac{3}{2}} x\,dx$

$= \int \cos^{-\frac{1}{2}} x\,(-d(\cos x)) - \int \cos^{\frac{3}{2}} x\,(-d(\cos x)) = \frac{-\cos^{\frac{1}{2}} x}{\frac{1}{2}} + \frac{\cos^{\frac{5}{2}} x}{\frac{5}{2}} + c$

$= 2\sqrt{\cos x}\left(\frac{1}{5}\cos^2 x - 1\right) + c, c \in \mathbb{R}$

20 In this problem, after a simple substitution, we can use the result from question 18.

$\int \frac{\sin^3\sqrt{x}}{\sqrt{x}}\,dx = \begin{bmatrix} \sqrt{x} = t \\ \frac{1}{2\sqrt{x}}\,dx = dt \end{bmatrix} = \int \sin^3 t \times 2\,dt = 2\left(-\cos t + \frac{\cos^3 t}{3}\right) + c$

$= -2\cos\sqrt{x} + \frac{2\cos^3\sqrt{x}}{3} + c, c \in \mathbb{R}$

21 $\int \cos t \cos^3(\sin t)\,dt = \begin{bmatrix} \sin t = x \\ \cos t\,dt = dx \end{bmatrix} = \int \cos^3(x)\,dx = \int \cos x\left(1-\sin^2 x\right)dx$

$= \int \cos x\,dx - \int \sin^2 x\; d(\sin x) = \sin x - \frac{\sin^3 x}{3} + c = \sin(\sin t) - \frac{\sin^3(\sin t)}{3} + c, c \in \mathbb{R}$

22 $\int \frac{\cos\theta + \sin 2\theta}{\sin\theta}\,d\theta = \int \frac{\cos\theta + 2\sin\theta\cos\theta}{\sin\theta}\,d\theta = \int \frac{d(\sin\theta)}{\sin\theta} + 2\int \cos\theta\,d\theta$

$= \ln|\sin\theta| + 2\sin\theta + c, c \in \mathbb{R}$

23 $\int t\sec t\tan t\,dt = \begin{bmatrix} u = t & du = dt \\ dv = \sec t\tan t\,dt & v = \sec t \end{bmatrix} = t\sec t - \int \sec t\,dt$

To find the second integral, we need to apply some further transformations, so we will do it separately.

$\int \sec t\,dt = \int \sec t\,\frac{\sec t + \tan t}{\sec t + \tan t}\,dt = \int \frac{\sec^2 t + \sec t\tan t}{\sec t + \tan t}\,dt = \int \frac{d(\tan t + \sec t)}{\sec t + \tan t} = \ln|\sec t + \tan t| + c$

Therefore, the solution of our original integral is:

$\int t\sec t\tan t\,dt = t\sec t - \ln|\sec t + \tan t| + c, c \in \mathbb{R}$

24 $\int \frac{\cos x}{2-\sin x}\,dx = \begin{bmatrix} 2-\sin x = t \\ -\cos x\,dx = dt \end{bmatrix} = \int \frac{-dt}{t} = -\ln|t| + c = -\ln|2-\sin x| + c, c \in \mathbb{R}$

25 $\int e^{-2x}\tan\left(e^{-2x}\right)dx = \left[\begin{array}{c} e^{-2x} = t \\ e^{-2x}\times(-2)\,dx = dt \end{array}\right] = \int \tan t\times\left(-\frac{1}{2}\right)dt = -\frac{1}{2}\ln|\cos t| + c$

$= -\frac{1}{2}\ln\left|\cos\left(e^{-2x}\right)\right| + c,\ c\in\mathbb{R}$

26 $\int \frac{\sec\left(\sqrt{t}\right)}{\sqrt{t}}dt = \left[\begin{array}{c} \sqrt{t} = x \\ \frac{1}{2\sqrt{t}}dt = dx \end{array}\right] = \int 2\sec x\,dx = 2\ln|\sec x + \tan x| + c$

$= 2\ln\left|\sec\sqrt{t} + \tan\sqrt{t}\right| + c,\ c\in\mathbb{R}$

27 $\int \frac{dt}{1+\cos 2t} = \int \frac{dt}{\cancel{1} + 2\cos^2 t \cancel{-1}} = \frac{1}{2}\int \frac{dt}{\cos^2 t} = \frac{1}{2}\tan t + c,\ c\in\mathbb{R}$

28 $\int \sqrt{1-9x^2}\,dx = \left[\begin{array}{c} 3x = \sin\theta \\ 3\,dx = \cos\theta\,d\theta \end{array}\right] = \int \cos\theta\times\frac{1}{3}\cos\theta\,d\theta = \frac{1}{3}\int\frac{1+\cos 2\theta}{2}d\theta$

$= \frac{1}{6}\left(\theta + \frac{\sin 2\theta}{2}\right) + c = \frac{\arcsin(3x)}{6} + \frac{\sin(2\arcsin(3x))}{12} + c$

$= \frac{\arcsin(3x)}{6} + \frac{2\sin(\arcsin(3x))\cos(\arcsin(3x))}{12} + c = \frac{1}{6}\arcsin(3x) + \frac{1}{6}3x\times\sqrt{1-9x^2} + c$

$= \frac{1}{6}\arcsin(3x) + \frac{x\sqrt{1-9x^2}}{2} + c,\ c\in\mathbb{R}$

29 $\int \frac{dx}{\left(x^2+4\right)^{\frac{3}{2}}} = \left[\begin{array}{c} x = 2\tan\theta \\ dx = 2\sec^2\theta\,d\theta \end{array}\right] = \int \frac{2\sec^2\theta\,d\theta}{\left(4\left(\tan^2\theta+1\right)\right)^{\frac{3}{2}}} = \int\frac{2\sec^2\theta\,d\theta}{8\sec^3\theta} = \frac{1}{4}\int\cos\theta\,d\theta$

$= \frac{1}{4}\sin\theta + c = \frac{1}{4}\sin\left(\arctan\left(\frac{x}{2}\right)\right) + c = \frac{1}{4}\frac{\tan\left(\arctan\left(\frac{x}{2}\right)\right)}{\sqrt{1+\tan^2\left(\arctan\left(\frac{x}{2}\right)\right)}} + c = \frac{\frac{x}{\cancel{2}}}{4\times\frac{\sqrt{4+x^2}}{\cancel{2}}} + c$

$= \frac{x}{4\sqrt{4+x^2}} + c,\ c\in\mathbb{R}$

30 $\int \sqrt{4+t^2}\,dt = \left[\begin{array}{c} t = 2\tan\theta \\ dt = 2\sec^2\theta\,d\theta \end{array}\right] = \int 2\sqrt{1+\tan^2\theta}\times 2\sec^2\theta\,d\theta = 4\int\sec^3\theta\,d\theta$

The integral $\int\sec^3\theta\,d\theta$ is not a simple one, so we are going to solve it separately.

$\int\sec^3\theta\,d\theta = \int\sec\theta\left(\tan^2\theta+1\right)d\theta = \int\sec\theta\tan^2\theta\,d\theta + \int\sec\theta\,d\theta$

$= \left[\begin{array}{cc} u = \tan\theta & du = \sec^2\theta\,d\theta \\ dv = \sec\theta\tan\theta\,d\theta & v = \sec\theta \end{array}\right] = \sec\theta\tan\theta - \int\sec^3\theta\,d\theta + \int\sec\theta\,d\theta$

Following this step, we can see the desired integral again and so we can transfer it to the left side of the equation:

$2\int\sec^3\theta\,d\theta = \sec\theta\tan\theta + \int\sec\theta\,d\theta \Rightarrow \int\sec^3\theta\,d\theta = \frac{1}{2}\sec\theta\tan\theta + \frac{1}{2}\ln|\sec\theta + \tan\theta| + c$

So, we can now continue solving the original question.

$$\int \sqrt{4+t^2}\,dt = \ldots = 4\int \sec^3\theta\,d\theta = 2\sec\theta\tan\theta + 2\ln\left|\sec\theta+\tan\theta\right| + c$$

$$= 2\sec\left(\arctan\left(\frac{t}{2}\right)\right)\tan\left(\arctan\left(\frac{t}{2}\right)\right) + 2\ln\left|\sec\left(\arctan\left(\frac{t}{2}\right)\right) + \tan\left(\arctan\left(\frac{t}{2}\right)\right)\right| + c$$

$$= 2\sqrt{1+\left(\frac{t}{2}\right)^2}\times\frac{t}{2} + 2\ln\left|\sqrt{1+\left(\frac{t}{2}\right)^2}+\frac{t}{2}\right| + c = \frac{t\sqrt{4+t^2}}{2} + 2\ln\left|\frac{\sqrt{4+t^2}+t}{2}\right| + c,\ c\in\mathbb{R}$$

Note: If we rewrite the second logarithmic term, we obtain:

$$\ldots = \frac{t\sqrt{4+t^2}}{2} + 2\ln\left|\frac{\sqrt{4+t^2}+t}{2}\right| + c = \frac{t\sqrt{4+t^2}}{2} + 2\ln\left|t+\sqrt{4+t^2}\right| \underbrace{-2\ln 2 + c}_{c_2}$$

$$= \frac{t\sqrt{4+t^2}}{2} + 2\ln\left|t+\sqrt{4+t^2}\right| + c_2,\ c_2\in\mathbb{R}, \text{ as written in the textbook answers.}$$

31 $$\int \frac{3e^t\,dt}{4+e^{2t}} = \begin{bmatrix} e^t = x \\ e^t dt = dx \end{bmatrix} = \int \frac{3\,dx}{4+x^2} = \begin{bmatrix} x = 2\tan\theta \\ dx = 2\sec^2\theta\,d\theta \end{bmatrix} = 3\int \frac{2\sec^2\theta\,d\theta}{4\left(1+\tan^2\theta\right)} = \frac{3}{2}\int d\theta$$

$$= \frac{3}{2}\theta + c = \frac{3}{2}\arctan\left(\frac{x}{2}\right) + c = \frac{3}{2}\arctan\left(\frac{e^t}{2}\right) + c,\ c\in\mathbb{R}$$

32 $$\int \frac{1}{\sqrt{9-4x^2}}\,dx = \begin{bmatrix} x = \frac{3}{2}\cos\theta \\ dx = -\frac{3}{2}\sin\theta\,d\theta \end{bmatrix} = \int \frac{-\frac{3}{2}\sin\theta\,d\theta}{\sqrt{9-\not{4}\times\frac{9}{\not{4}}\cos^2\theta}} = \int \frac{-\frac{\not{3}}{2}\sin\theta\,d\theta}{\not{3}\sqrt{1-\cos^2\theta}} = -\frac{1}{2}\int d\theta$$

$$= -\frac{1}{2}\theta + c = -\frac{1}{2}\arccos\left(\frac{2x}{3}\right) + c,\ c\in\mathbb{R}$$

Note: If we had used the substitution $x = \frac{3}{2}\sin\theta$, then the result would have been $\frac{1}{2}\arcsin\left(\frac{2x}{3}\right) + c$ since the two results differ by a constant.

33 $$\int \frac{1}{\sqrt{4+9x^2}}\,dx = \begin{bmatrix} x = \frac{2}{3}\tan\theta \\ dx = \frac{2}{3}\sec^2\theta\,d\theta \end{bmatrix} = \int \frac{\frac{2}{3}\sec^2\theta\,d\theta}{\sqrt{4+\not{9}\times\frac{4}{\not{9}}\tan^2\theta}} = \int \frac{\frac{\not{2}}{3}\sec^2\theta\,d\theta}{\not{2}\sqrt{1+\tan^2\theta}} = \frac{1}{3}\int \sec\theta\,d\theta$$

$$= \frac{1}{3}\left(\ln\left|\sec\theta+\tan\theta\right|\right) + c = \frac{1}{3}\ln\left|\sec\left(\arctan\left(\frac{3x}{2}\right)\right) + \tan\left(\arctan\left(\frac{3x}{2}\right)\right)\right| + c$$

$$= \frac{1}{3}\ln\left|\sqrt{1+\left(\frac{3x}{2}\right)^2}+\frac{3x}{2}\right| + c = \frac{1}{3}\ln\left|\frac{\sqrt{4+9x^2}+3x}{2}\right| + c,\ c\in\mathbb{R}$$

34 $$\int \frac{\cos x}{\sqrt{1+\sin^2 x}}\,dx = \begin{bmatrix} \sin x = t \\ \cos x\,dx = dt \end{bmatrix} = \int \frac{dt}{\sqrt{1+t^2}} = \begin{bmatrix} t = \tan\theta \\ dt = \sec^2\theta\,d\theta \end{bmatrix} = \int \frac{\sec^2\theta\,d\theta}{\sqrt{1+\tan^2\theta}}$$

$$= \int \sec\theta\,d\theta = \ln\left|\sec\theta+\tan\theta\right| + c = \ln\left|\sec(\arctan t)+\tan(\arctan t)\right| + c = \ln\left|\sqrt{1+t^2}+t\right| + c$$

$$= \ln\left|\sqrt{1+\sin^2 x}+\sin x\right| + c,\ c\in\mathbb{R}$$

35 $$\int \frac{x}{\sqrt{4-x^2}}\,dx = \begin{bmatrix} 4-x^2 = t \\ -2x\,dx = dt \end{bmatrix} = \int \frac{-\frac{1}{2}dt}{\sqrt{t}} = -\frac{1}{2}\times\frac{t^{\frac{1}{2}}}{\frac{1}{2}} + c = -\sqrt{4-x^2} + c,\ c\in\mathbb{R}$$

36 $\int \frac{x}{x^2+16}dx = \left[\begin{array}{c} x^2+16=t \\ 2x\,dx = dt \end{array}\right] = \int \frac{\frac{1}{2}dt}{t} = \frac{1}{2}\ln t + c = \frac{1}{2}\ln\left(x^2+16\right)+c, c\in\mathbb{R}$

37 $\int \frac{\sqrt{4-x^2}}{x^2}dx = \left[\begin{array}{c} x = 2\cos\theta \\ dx = -2\sin\theta\, d\theta \end{array}\right] = \int \frac{\cancel{2}\sqrt{1-\cos^2\theta}\times\left(-\cancel{2}\sin\theta\right)d\theta}{\cancel{4}\cos^2\theta} = -\int \frac{\sin^2\theta\, d\theta}{\cos^2\theta}$

$= \int \frac{\cos^2\theta - 1}{\cos^2\theta}d\theta = \int d\theta - \int \frac{1}{\cos^2\theta}d\theta = \theta - \tan\theta + c = \arccos\left(\frac{x}{2}\right) - \tan\left(\arccos\left(\frac{x}{2}\right)\right)+c$

$= \arccos\left(\frac{x}{2}\right) - \frac{\sqrt{1-\left(\cos\left(\arccos\left(\frac{x}{2}\right)\right)\right)^2}}{\cos\left(\arccos\left(\frac{x}{2}\right)\right)} + c = \arccos\left(\frac{x}{2}\right) - \frac{\sqrt{4-x^2}}{x}+c, c\in\mathbb{R}$

Note: If we had used the substitution $x = \frac{3}{2}\sin\theta$, then the result would have been $-\arcsin\left(\frac{x}{2}\right) - \frac{\sqrt{4-x^2}}{x}+c$ since the two results differ by a constant.

38 $\int \frac{dx}{\left(9-x^2\right)^{\frac{3}{2}}} = \left[\begin{array}{c} x = 3\cos\theta \\ dx = -3\sin\theta\, d\theta \end{array}\right] = \int \frac{-3\sin\theta\, d\theta}{\left(9-9\cos^2\theta\right)^{\frac{3}{2}}} = \int \frac{-3\sin\theta\, d\theta}{27\left(1-\cos^2\theta\right)^{\frac{3}{2}}}$

$= -\frac{1}{9}\int \frac{\sin\theta\, d\theta}{\sin^3\theta} = -\frac{1}{9}\int \frac{d\theta}{\sin^2\theta} = -\frac{1}{9}(-\cot\theta)+c = \frac{1}{9}\cot\left(\arccos\left(\frac{x}{3}\right)\right)+c$

$= \frac{1}{9}\frac{\cos\left(\arccos\left(\frac{x}{3}\right)\right)}{\sqrt{1-\cos^2\left(\arccos\left(\frac{x}{3}\right)\right)}}+c = \frac{x}{9\sqrt{9-x^2}}+c, c\in\mathbb{R}$

39 $\int x\sqrt{1+x^2}\,dx = \left[\begin{array}{c} 1+x^2 = t \\ 2x\,dx = dt \end{array}\right] = \int \sqrt{t}\times\frac{1}{2}dt = \frac{1}{\cancel{2}}\times\frac{t^{\frac{3}{2}}}{\frac{3}{\cancel{2}}}+c = \frac{1}{3}\left(1+x^2\right)\sqrt{1+x^2}+c, c\in\mathbb{R}$

40 $\int e^{2x}\sqrt{1+e^{2x}}dx = \left[\begin{array}{c} 1+e^{2x} = t \\ e^{2x}\times 2dx = dt \end{array}\right] = \int \sqrt{t}\times\frac{1}{2}dt = \frac{1}{\cancel{2}}\times\frac{t^{\frac{3}{2}}}{\frac{3}{\cancel{2}}}+c = \frac{1}{3}\left(1+e^{2x}\right)\sqrt{1+e^{2x}}+c, c\in\mathbb{R}$

41 $\int e^x\sqrt{1-e^{2x}}dx = \left[\begin{array}{c} e^x = t \\ e^x dx = dt \end{array}\right] = \int \sqrt{1-t^2}\,dt = \left[\begin{array}{c} t = \sin\theta \\ dt = \cos\theta\, d\theta \end{array}\right] = \int \cos^2\theta\, d\theta$

$= \int \frac{1+\cos 2\theta}{2}d\theta = \frac{1}{2}\theta + \frac{1}{4}\sin 2\theta + c = \frac{1}{2}\arcsin t + \frac{1}{2}\sin(\arcsin t)\cos(\arcsin t)+c$

$= \frac{1}{2}\arcsin t + \frac{1}{2}t\sqrt{1-t^2}+c = \frac{1}{2}\left(\arcsin\left(e^x\right) + e^x\sqrt{1-e^{2x}}\right)+c, c\in\mathbb{R}$

42 $\int \frac{e^x}{\sqrt{e^{2x}+9}}dx = \left[\begin{array}{c} e^x = t \\ e^x dx = dt \end{array}\right] = \int \frac{dt}{\sqrt{t^2+9}} = \left[\begin{array}{c} t = 3\tan\theta \\ dt = 3\sec^2\theta\, d\theta \end{array}\right] = \int \frac{\cancel{3}\sec^2\theta\, d\theta}{\cancel{3}\sqrt{\tan^2\theta+1}} = \int \sec\theta\, d\theta$

$= \ln\left|\sec\theta + \tan\theta\right| + c = \ln\left|\sec\left(\arctan\frac{t}{3}\right) + \tan\left(\arctan\frac{t}{3}\right)\right| + c = \ln\left|\sqrt{1+\left(\frac{t}{3}\right)^2} + \frac{t}{3}\right| + c$

$= \ln\left|\frac{\sqrt{9+t^2}+t}{3}\right| + c = \ln\left(\frac{\sqrt{9+e^{2x}}+e^x}{3}\right)+c, c\in\mathbb{R}$

43 $\int \frac{\ln x}{\sqrt{x}}\,dx = \begin{bmatrix} u = \ln x & du = \frac{1}{x}\,dx \\ dv = \frac{dx}{\sqrt{x}} & v = 2\sqrt{x} \end{bmatrix} = 2\sqrt{x}\ln x - \int 2\sqrt{x} \times \frac{1}{x}\,dx = 2\sqrt{x}\ln x - 2\int \frac{1}{\sqrt{x}}\,dx$

$= 2\sqrt{x}\ln x - 4\sqrt{x} + c = 2\sqrt{x}(\ln x - 2) + c, c \in \mathbb{R}$

44 $\int \frac{x^3}{(x+2)^2}\,dx = \begin{bmatrix} x+2=t \Rightarrow x = t-2 \\ dx = dt \end{bmatrix} = \int \frac{(t-2)^3}{t^2}\,dt = \int \frac{t^3 - 6t^2 + 12t - 8}{t^2}\,dt$

$\int \left(t - 6 + 12t^{-1} - 8t^{-2}\right)dt = \frac{t^2}{2} - 6t + 12\ln|t| - 8\frac{t^{-1}}{-1} + c$

$= \frac{(x+2)^2}{2} - 6(x+2) + 12\ln|x+2| + \frac{8}{x+2} + c, c \in \mathbb{R}$

Since the constant could be any number, we can expand the first two expressions and eliminate the constants from them. Then we obtain the following result.

$= \frac{x^2+4x+4}{2} - (6x+12) + 12\ln|x+2| + \frac{8}{x+2} + c = \frac{x^2}{2} - 4x + 12\ln|x+2| + \frac{8}{x+2} + c, c \in \mathbb{R}$

45 $\int \frac{x}{x^2+9}\,dx = \begin{bmatrix} x^2+9=t \\ 2x\,dx = dt \end{bmatrix} = \int \frac{\frac{1}{2}dt}{t} = \frac{1}{2}\ln t + c_1 = \frac{1}{2}\ln\left(x^2+9\right) + c_1, c_1 \in \mathbb{R}$

$\int \frac{x}{x^2+9}\,dx = \begin{bmatrix} x = 3\tan\theta \\ dx = 3\sec^2\theta\,d\theta \end{bmatrix} = \int \frac{3\tan\theta}{9\tan^2\theta + 9} 3\sec^2\theta\,d\theta = \int \frac{\cancel{9}\tan\theta\sec^2\theta\,d\theta}{\cancel{9}\underbrace{\left(\tan^2\theta+1\right)}_{\sec^2\theta}} = \int \tan\theta\,d\theta$

$= -\ln|\cos\theta| + c_2 = -\ln\left|\cos\left(\arctan\left(\frac{x}{3}\right)\right)\right| + c_2 = -\ln\left|\frac{1}{\sqrt{1+\left(\frac{x}{3}\right)^2}}\right| + c = -\ln\left|\frac{3}{\sqrt{9+x^2}}\right| + c_2$

$= \ln\left(\sqrt{9+x^2}\right) - \ln 3 + c_2 = \frac{1}{2}\ln\left(9+x^2\right) - \ln 3 + c_2, c_2 \in \mathbb{R}$

Here, we can see that the two solutions differ by a constant.

46 $\int \frac{x^2}{x^2+9}\,dx = \int \frac{\left(x^2+9\right)-9}{x^2+9}\,dx = \int dx - \int \frac{9}{x^2+9}\,dx = x - 9 \times \frac{1}{3}\arctan\left(\frac{x}{3}\right) + c_1$

$= x - 3\arctan\left(\frac{x}{3}\right) + c_1, c_1 \in \mathbb{R}$

$\int \frac{x^2}{x^2+9}\,dx = \begin{bmatrix} x = 3\tan\theta \\ dx = 3\sec^2\theta\,d\theta \end{bmatrix} = \int \frac{9\tan^2\theta}{9\tan^2\theta+9} 3\sec^2\theta\,d\theta = \int \frac{\cancel{27}\,3\tan^2\theta\sec^2\theta\,d\theta}{\cancel{9}\underbrace{\left(\tan^2\theta+1\right)}_{\sec^2\theta}}$

$= 3\int \tan^2\theta\,d\theta = 3\int\left(\sec^2\theta - 1\right)d\theta = 3(\tan\theta - \theta) + c_2$

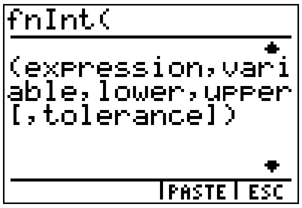
$= 3\left(\tan\left(\arctan\left(\frac{x}{3}\right)\right) - \arctan\left(\frac{x}{3}\right)\right) + c_2 = x - 3\arctan\left(\frac{x}{3}\right) + c_2, c_2 \in \mathbb{R}$

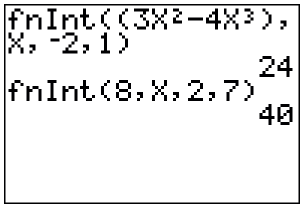
Exercise 16.4

1 $\int_{-2}^{1}\left(3x^2 - 4x^3\right)dx = \left[x^3 - x^4\right]_{-2}^{1} = \left(1^3 - 1^4\right) - \left((-2)^3 - (-2)^4\right) = 0 - (-24) = 24$

2 $\int_{2}^{7} 8\,dx = [8x]_2^7 = 56 - 16 = 40$

A GDC screen can show two answers at the same time. Using the finite integral feature in the Math menu, here are the solutions for questions 1 and 2.

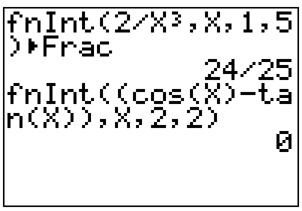




3 $\int_{1}^{5}\frac{2}{t^3}\,dt = \left[\cancel{2}\times\frac{t^{-2}}{-\cancel{2}}\right]_1^5 = \left(-\frac{1}{25}\right) - \left(-\frac{1}{1}\right) = \frac{24}{25}$

4 $\int_{2}^{2}(\cos t - \tan t)\,dt = 0$, since the upper border is equal to the lower border.

The GDC screen output for questions 3 and 4 is shown below.

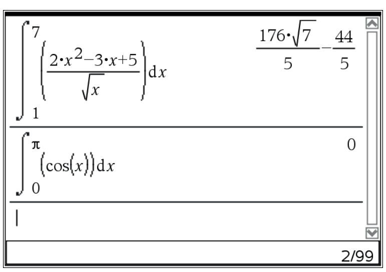


5 $\int_{1}^{7}\frac{2x^2 - 3x + 5}{\sqrt{x}}\,dx = \int_{1}^{7}\left(2x^{\frac{3}{2}} - 3x^{\frac{1}{2}} + 5x^{-\frac{1}{2}}\right)dx = \left[2\frac{x^{\frac{5}{2}}}{\frac{5}{2}} - 3\frac{x^{\frac{3}{2}}}{\frac{3}{2}} + 5\frac{x^{\frac{1}{2}}}{\frac{1}{2}}\right]_1^7$

$= \left(\frac{4}{5}\times 7^{\frac{5}{2}} - 2\times 7^{\frac{3}{2}} + 10\times 7^{\frac{1}{2}}\right) - \left(\frac{4}{5} - 2 + 10\right) = \frac{\sqrt{7}}{5}\left(4\times 7^2 - 2\times 7\times 5 + 10\times 5\right) - \frac{44}{5} = \frac{176\sqrt{7} - 44}{5}$

6 $\int_{0}^{\pi}\cos\theta\,d\theta = [\sin\theta]_0^{\pi} = \sin\pi - \sin 0 = 0$

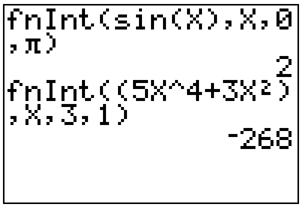
The GDC screen output for questions 5 and 6 is shown below.



7 $\int_{0}^{\pi}\sin\theta\,d\theta = [-\cos\theta]_0^{\pi} = -\cos\pi - (-\cos 0) = -(-1) - (-1) = 2$

8 $\int_{3}^{1}\left(5x^4 + 3x^2\right)dx = \left[\cancel{5}\frac{x^5}{\cancel{5}} + \cancel{3}\frac{x^3}{\cancel{3}}\right]_3^1 = \left(1^5 + 1^3\right) - \left(3^5 + 3^3\right) = 2 - (243 + 27) = -268$

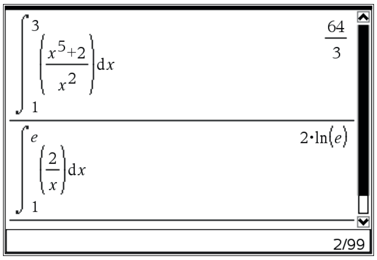
The GDC screen output for questions 7 and 8 is shown below.



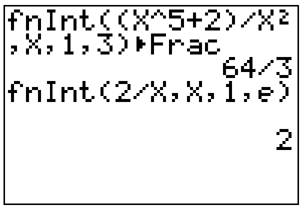
9 $\int_1^3 \frac{u^5+2}{u^2}\,du = \int_1^3 u^3 + 2u^{-2}\,du = \left[\frac{x^4}{4} + 2\frac{u^{-1}}{-1}\right]_1^3 = \left(\frac{3^4}{4} - \frac{2}{3}\right) - \left(\frac{1}{4} - 2\right) = 20 + \frac{4}{3} = \frac{64}{3}$

10 $\int_1^e \frac{2\,dx}{x} = [2\ln x]_1^e = 2\ln e - 2\ln(1) = 2$

Alternative GDC screen outputs for questions 9 and 10 are shown below.



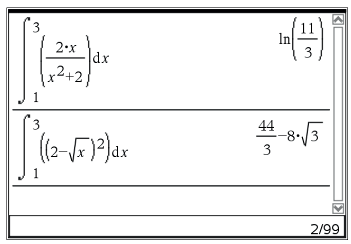
or



11 $\int_1^3 \frac{2x}{x^2+2}\,dx = \int_1^3 \frac{d\left(x^2+2\right)}{x^2+2} = \left[\ln\left(x^2+2\right)\right]_1^3 = \ln 11 - \ln 3 = \ln\left(\frac{11}{3}\right)$

12 $\int_1^3 \left(2-\sqrt{x}\right)^2\,dx = \int_1^3 \left(4 - 4\sqrt{x} + x\right)dx = \left[4x - 4\frac{x^{\frac{3}{2}}}{\frac{3}{2}} + \frac{x^2}{2}\right]_1^3$

$= \left(12 - \frac{8}{\cancel{3}}\cancel{3}\sqrt{3} + \frac{9}{2}\right) - \left(4 - \frac{8}{3} + \frac{1}{2}\right) = \frac{44}{3} - 8\sqrt{3}$

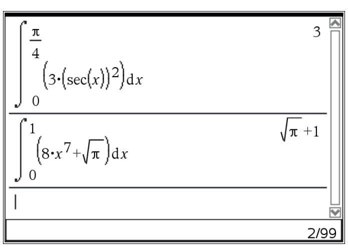


The GDC screen output for questions 11 and 12 is shown opposite.

13 $\int_0^{\frac{\pi}{4}} 3\sec^2\theta\,d\theta = [3\tan\theta]_0^{\frac{\pi}{4}} = 3\left(\tan\left(\frac{\pi}{4}\right) - \tan 0\right) = 3(1-0) = 3$

14 $\int_0^1 \left(8x^7 + \sqrt{\pi}\right)dx = \left[\cancel{8}\frac{x^8}{\cancel{8}} + \sqrt{\pi}\times x\right]_0^1 = 1 + \sqrt{\pi}$

The GDC screen output for questions 13 and 14 is shown below.

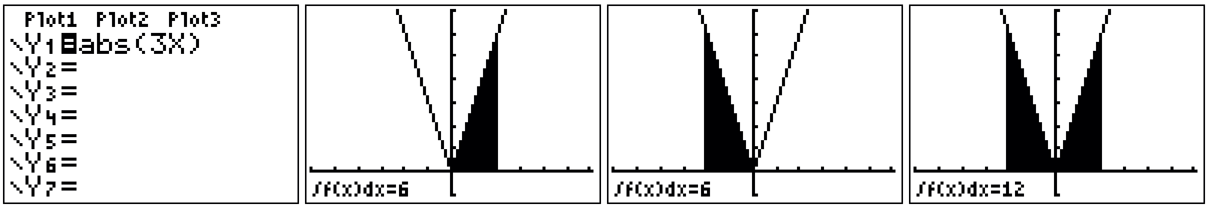


15 We will use the fact that $|3x| = \begin{cases} 3x, & x \geqslant 0 \\ -3x, & x < 0 \end{cases}$ in the following question:

a) $\int_0^2 |3x|\,dx = \int_0^2 3x\,dx = \left[3\dfrac{x^2}{2}\right]_0^2 = 6$

b) $\int_{-2}^0 |3x|\,dx = \int_{-2}^0 -3x\,dx = \left[-3\dfrac{x^2}{2}\right]_{-2}^0 = 0-(-6) = 6$

c) In this problem, we simply split the definite integral into the two previous parts: $\int_{-2}^2 |3x|\,dx = \int_{-2}^0 |3x|\,dx + \int_0^2 |3x|\,dx = 6+6 = 12$. The reason why we must split it like this is the definition of the absolute value function itself.



16 $\int_0^{\frac{\pi}{2}} \sin 2x\,dx = \left(-\dfrac{1}{2}\cos 2x\right)\Big]_0^{\frac{\pi}{2}} = -\dfrac{1}{2}\Big(\underbrace{\cos\pi}_{-1} - \underbrace{\cos 0}_{1}\Big) = -\dfrac{1}{2}\times(-2) = 1$

17 $\int_1^9 \dfrac{1}{\sqrt{x}}\,dx = \left(2\sqrt{x}\right)\Big]_1^9 = 2\left(\sqrt{9}-\sqrt{1}\right) = 4$

18 $\int_{-2}^2 \left(e^x - e^{-x}\right)dx = \left(e^x + e^{-x}\right)\Big]_{-2}^2 = \left(e^2+e^{-2}\right) - \left(e^{-2}+e^2\right) = 0$

19 $\int_{-1}^1 \dfrac{dx}{1+x^2} = \arctan x\Big]_{-1}^1 = \arctan(1) - \arctan(-1) = \dfrac{\pi}{4} - \left(-\dfrac{\pi}{4}\right) = \dfrac{\pi}{2}$

20 $\int_0^{\frac{1}{2}} \dfrac{dx}{\sqrt{1-x^2}} = \arcsin x\Big]_0^{\frac{1}{2}} = \arcsin\left(\dfrac{1}{2}\right) - \arcsin 0 = \dfrac{\pi}{6}$

21 $\int_{-1}^1 \dfrac{dx}{\sqrt{4-x^2}} = \arcsin\left(\dfrac{x}{2}\right)\Big]_{-1}^1 = \arcsin\left(\dfrac{1}{2}\right) - \arcsin\left(-\dfrac{1}{2}\right) = \dfrac{\pi}{6} - \left(-\dfrac{\pi}{6}\right) = \dfrac{\pi}{3}$

22 $\int_{-2}^0 \dfrac{dx}{4+x^2} = \dfrac{1}{2}\arctan\left(\dfrac{x}{2}\right)\Big]_{-2}^0 = \dfrac{1}{2}\arctan(0) - \dfrac{1}{2}\arctan(-1) = 0 - \left(-\dfrac{1}{2}\arctan(1)\right) = \dfrac{1}{2}\arctan(1) = \dfrac{\pi}{8}$

Note: In the integrals where we are using the method of substitution, we must not forget to change the boundaries of integration accordingly by using the same equation of substitution.

23 $\int_0^4 \dfrac{x^3dx}{\sqrt{x^2+1}} = \left[\begin{array}{c} x^2+1=t \\ 2x\,dx = dt \end{array}\right] = \int_1^{17} \dfrac{(t-1)\frac{1}{2}\,dt}{\sqrt{t}} = \dfrac{1}{2}\int_1^{17}\left(\sqrt{t} - \dfrac{1}{\sqrt{t}}\right)dt = \dfrac{1}{2}\left(\dfrac{2}{3}t^{\frac{3}{2}} - 2t^{\frac{1}{2}}\right)\Big]_1^{17}$

$= \left(\dfrac{1}{3}\times 17^{\frac{3}{2}} - \sqrt{17}\right) - \left(\dfrac{1}{3}\times 1^{\frac{3}{2}} - \sqrt{1}\right) = \dfrac{17\sqrt{17} - 3\sqrt{17} + 2}{3} = \dfrac{14\sqrt{17}+2}{3}$

24 $\int_1^{\sqrt{e}} \dfrac{\sin(\pi\ln x)\,dx}{x} = \left[\begin{array}{c} \pi\ln x = t \\ \dfrac{\pi}{x}\,dx = dt \end{array}\right] = \int_0^{\frac{\pi}{2}} \dfrac{\sin t\,dt}{\pi} = \dfrac{1}{\pi}(-\cos t)\Big]_0^{\frac{\pi}{2}}$

$= \dfrac{1}{\pi}\left(-\cos\left(\dfrac{\pi}{2}\right)\right) - \dfrac{1}{\pi}(-\cos 0) = \dfrac{1}{\pi} - \dfrac{\cos\left(\dfrac{\pi}{2}\right)}{\pi}$

25 $\int_e^{e^2} \frac{dt}{t \ln t} = \begin{bmatrix} \ln t = x \\ \frac{1}{t}\, dt = dx \end{bmatrix} = \int_1^2 \frac{dx}{x} = \ln |x| \Big]_1^2 = \ln(2) - \ln(1) = \ln 2$

26 $\int_{-1}^{2} 3x\sqrt{9 - x^2}\, dx = \begin{bmatrix} 9 - x^2 = t \\ -2x\, dx = dt \end{bmatrix} = \int_8^5 \frac{3\sqrt{t}\, dt}{-2} = \frac{3}{2}\int_5^8 \sqrt{t}\, dt = \frac{3}{2}\left(\frac{2}{3} t^{\frac{3}{2}}\right)\Big]_5^8 = 8^{\frac{3}{2}} - 5^{\frac{3}{2}} = 16\sqrt{2} - 5\sqrt{5}$

27 $\int_{-\frac{\pi}{3}}^{\frac{2\pi}{3}} \frac{\sin x}{\sqrt{3 + \cos x}}\, dx = \begin{bmatrix} 3 + \cos x = t \\ -\sin x\, dx = dt \end{bmatrix} = \int_{3+\frac{1}{2}}^{3-\frac{1}{2}} \frac{-dt}{\sqrt{t}} = \int_{\frac{5}{2}}^{\frac{7}{2}} \frac{dt}{\sqrt{t}} = \left(2\sqrt{t}\right)\Big]_{\frac{5}{2}}^{\frac{7}{2}} = 2\left(\sqrt{\frac{7}{2}} - \sqrt{\frac{5}{2}}\right) = \sqrt{14} - \sqrt{10}$

28 $\int_e^{e^2} \frac{\ln x}{x}\, dx = \begin{bmatrix} \ln x = t \\ \frac{1}{x}\, dx = dt \end{bmatrix} = \int_1^2 t\, dt = \frac{t^2}{2}\Big]_1^2 = 2 - \frac{1}{2} = \frac{3}{2}$

29 $\int_1^{\sqrt{3}} \frac{\sqrt{\arctan x}}{1 + x^2}\, dx = \begin{bmatrix} \arctan x = t \\ \frac{1}{1 + x^2}\, dx = dt \end{bmatrix} = \int_{\frac{\pi}{4}}^{\frac{\pi}{3}} \sqrt{t} dt = \left(\frac{2}{3} t\sqrt{t}\right)\Big]_{\frac{\pi}{4}}^{\frac{\pi}{3}} = \frac{2}{3}\left(\frac{\pi}{3}\sqrt{\frac{\pi}{3}} - \frac{\pi}{4}\sqrt{\frac{\pi}{4}}\right)$

$= \frac{2\pi}{216}\left(8\sqrt{3\pi} - 9\sqrt{\pi}\right) = \frac{\pi\sqrt{\pi}}{108}\left(8\sqrt{3} - 9\right)$

30 $\int_1^{\sqrt{e}} \frac{dx}{x\sqrt{1 - (\ln x)^2}} = \begin{bmatrix} \ln x = t \\ \frac{1}{x}\, dx = dt \end{bmatrix} = \int_0^{\frac{1}{2}} \frac{dt}{\sqrt{1 - t^2}} = \arcsin t\Big]_0^{\frac{1}{2}} = \arcsin\left(\frac{1}{2}\right) - \arcsin(0) = \frac{\pi}{6}$

31 $\int_{-\ln 2}^{\ln 2} \frac{e^{2x}}{e^{2x} + 9}\, dx = \begin{bmatrix} e^{2x} + 9 = t \\ e^{2x} \times 2\, dx = dt \end{bmatrix} = \int_{\frac{1}{4}+9}^{4+9} \frac{\frac{1}{2}\, dt}{t} = \left(\frac{1}{2}\ln|t|\right)\Big]_{\frac{37}{4}}^{13} = \frac{1}{2}\left(\ln 13 - \ln\left(\frac{37}{4}\right)\right) = \frac{1}{2}\ln\left(\frac{52}{37}\right)$

32 $\int_{\ln 2}^{\ln\left(\frac{2}{\sqrt{3}}\right)} \frac{e^{-2x}}{\sqrt{1 - e^{-4x}}}\, dx = \begin{bmatrix} e^{-2x} = t \\ e^{-2x} \times (-2)\, dx = dt \end{bmatrix} = \int_{\frac{1}{4}}^{\frac{3}{4}} \frac{-\frac{1}{2}\, dt}{\sqrt{1 - t^2}} = \left(-\frac{1}{2}\arcsin t\right)\Big]_{\frac{1}{4}}^{\frac{3}{4}}$

$= -\frac{1}{2}\left(\arcsin\left(\frac{3}{4}\right) - \arcsin\left(\frac{1}{4}\right)\right)$

33 $\int_0^{\frac{\pi}{4}} \sqrt{\tan x}\, \sec^2 x\, dx = \begin{bmatrix} \tan x = t \\ \sec^2 x\, dx = dt \end{bmatrix} = \int_0^1 \sqrt{t}\, dt = \left(\frac{2}{3} t\sqrt{t}\right)\Big]_0^1 = \frac{2}{3}$

34 $\int_0^{\sqrt{\pi}} 7x\cos\left(x^2\right) dx = \begin{bmatrix} x^2 = t \\ 2x\, dx = dt \end{bmatrix} = \int_0^{\pi} \frac{7}{2}\cos t\, dt = \left(\frac{7}{2}\sin t\right)\Big]_0^{\pi} = \frac{7}{2}\left(\underbrace{\sin \pi}_{0} - \underbrace{\sin 0}_{0}\right) = 0$

35 $\int_{\pi^2}^{4\pi^2} \frac{\sin\sqrt{x}}{\sqrt{x}}\, dx = \begin{bmatrix} \sqrt{x} = t \\ \frac{dx}{2\sqrt{x}} = dt \end{bmatrix} = \int_{\pi}^{2\pi} 2\sin t\, dt = (-2\cos t)\Big]_{\pi}^{2\pi} = -2\left(\underbrace{\cos(2\pi)}_{1} - \underbrace{\cos(\pi)}_{-1}\right) = -4$

36 $\int_0^1 \frac{\sqrt{3}x}{\sqrt{4 - 3x^4}}\, dx = \begin{bmatrix} \sqrt{3}x^2 = t \\ 2\sqrt{3}x\, dx = dt \end{bmatrix} = \int_0^{\sqrt{3}} \frac{\frac{1}{2}\, dt}{\sqrt{4 - t^2}} = \frac{1}{2}\left(\arcsin\left(\frac{t}{2}\right)\right)\Big]_0^{\sqrt{3}}$

$= \frac{1}{2}\left(\arcsin\left(\frac{\sqrt{3}}{2}\right) - \arcsin(0)\right) = \frac{\pi}{6}$

37 $\int_0^{\frac{2}{\sqrt{3}}} \frac{dx}{9+4x^2} = \frac{1}{9}\int_0^{\frac{2}{\sqrt{3}}} \frac{dx}{1+\left(\frac{2x}{3}\right)^2} = \left(\frac{1}{9}\times\frac{3}{2}\arctan\left(\frac{2x}{3}\right)\right)\Bigg]_0^{\frac{2}{\sqrt{3}}} = \frac{1}{6}\left(\arctan\left(\frac{4}{3\sqrt{3}}\right) - \arctan 0\right)$

$= \frac{1}{6}\arctan\left(\frac{4\sqrt{3}}{9}\right)$

38 $\int_1^{\sqrt{2}} \frac{x\,dx}{3+x^4} = \left[\begin{array}{c} x^2 = t \\ 2x\,dx = dt \end{array}\right] = \int_1^2 \frac{\frac{1}{2}dt}{3+t^2} = \left(\frac{1}{2}\times\frac{1}{\sqrt{3}}\arctan\left(\frac{t}{\sqrt{3}}\right)\right)\Bigg]_1^2$

$= \frac{1}{2\sqrt{3}}\left(\arctan\left(\frac{2}{\sqrt{3}}\right) - \arctan\left(\frac{1}{\sqrt{3}}\right)\right) = \frac{\arctan\left(\frac{2}{\sqrt{3}}\right)}{2\sqrt{3}} - \frac{\pi}{12\sqrt{3}}$

$= \frac{18\sqrt{3}\arctan\left(\frac{2}{\sqrt{3}}\right) - 3\sqrt{3}\pi}{36} \approx 0.0963$

39 $\int_0^{\frac{\pi}{6}} (1-\sin 3t)\cos 3t\,dt = \left[\begin{array}{c} 1-\sin 3t = x \\ -\cos 3t \times 3\,dt = dx \end{array}\right] = \int_1^0 x\times\left(-\frac{1}{3}\right)dx = \frac{1}{3}\int_0^1 x\,dx = \frac{1}{3}\times\frac{x^2}{2}\Bigg]_0^1 = \frac{1}{6}$

40 $\int_0^{\frac{\pi}{4}} e^{\sin 2\theta}\cos 2\theta\,d\theta = \left[\begin{array}{c} \sin 2\theta = t \\ \cos 2\theta\times 2\,d\theta = dt \end{array}\right] = \int_0^1 e^t\times\frac{1}{2}dt = \frac{1}{2}e^t\Bigg]_0^1 = \frac{e-1}{2}$

41 $\int_0^{\frac{\pi}{8}} \left(3+e^{\tan 2t}\right)\sec^2 2t\,dt = \left[\begin{array}{c} \tan 2t = x \\ \sec^2 2t\times 2\,dt = dx \end{array}\right] = \int_0^1 \left(3+e^x\right)\times\frac{1}{2}dx = \frac{1}{2}\left(3x+e^x\right)\Bigg]_0^1 = \frac{3+e-1}{2} = 1+\frac{e}{2}$

42 $\int_0^{\sqrt{\ln\pi}} 4t\,e^{t^2}\sin\left(e^{t^2}\right)dt = \left[\begin{array}{c} e^{t^2} = x \\ e^{t^2}\times 2t\,dt = dx \end{array}\right] = 2\int_1^{\pi}\sin x\,dx = 2\left(-\cos x\right)\Big]_1^{\pi}$

$= 2\left(-\underbrace{\cos\pi}_{-1} + \cos 1\right) = 2+2\cos 1$

43 $av(f) = \frac{1}{2-1}\int_1^2 x^4\,dx = \frac{x^5}{5}\Bigg]_1^2 = \frac{32}{5} - \frac{1}{5} = \frac{31}{5}$

44 $av(f) = \frac{1}{\frac{\pi}{2}-0}\int_0^{\frac{\pi}{2}}\cos x\,dx = \frac{2}{\pi}(\sin x)\Big]_0^{\frac{\pi}{2}} = \frac{2}{\pi}\left(\sin\left(\frac{\pi}{2}\right) - \sin 0\right) = \frac{2}{\pi}$

45 $av(f) = \frac{1}{\frac{\pi}{4}-\frac{\pi}{6}}\int_{\frac{\pi}{6}}^{\frac{\pi}{4}}\sec^2 x\,dx = \frac{12}{\pi}(\tan x)\Big]_{\frac{\pi}{6}}^{\frac{\pi}{4}} = \frac{12}{\pi}\left(\underbrace{\tan\left(\frac{\pi}{4}\right)}_{1} - \underbrace{\tan\left(\frac{\pi}{6}\right)}_{\frac{\sqrt{3}}{3}}\right) = \frac{12-4\sqrt{3}}{\pi}$

46 $av(f) = \frac{1}{4-0}\int_0^4 e^{-2x}\,dx = \frac{1}{4}\times\left(-\frac{1}{2}e^{-2x}\right)\Bigg]_0^4 = -\frac{1}{8}\left(e^{-8}-e^0\right) = \frac{1-e^{-8}}{8} = \frac{e^8-1}{8e^8}$

47 $av(f) = \frac{1}{0-\left(-\frac{\ln 3}{6}\right)}\int_{-\frac{\ln 3}{6}}^0 \frac{e^{3x}}{1+e^{6x}}dx = \left[\begin{array}{c} e^{3x} = t \\ e^{3x}\times 3\,dx = dt \end{array}\right] = \frac{6}{\ln 3}\int_{\frac{1}{\sqrt{3}}}^1 \frac{\frac{1}{3}dt}{1+t^2}$

$= \frac{2\cancel{6}}{\ln 3}\times\frac{1}{\cancel{3}}(\arctan t)\Bigg]_{\frac{1}{\sqrt{3}}}^1 = \frac{2}{\ln 3}\left(\arctan 1 - \arctan\left(\frac{1}{\sqrt{3}}\right)\right) = \frac{2}{\ln 3}\left(\frac{\pi}{4}-\frac{\pi}{6}\right) = \frac{\cancel{2}}{\ln 3}\times\frac{\pi}{\cancel{12}\,6} = \frac{\pi}{6\ln 3}$

48 $\dfrac{d}{dx}\displaystyle\int_2^x \frac{\sin t}{t}\,dt = \frac{\sin x}{x}$

49 $\dfrac{d}{dt}\displaystyle\int_t^3 \frac{\sin x}{x}\,dx = \frac{d}{dt}\int_3^t \left(-\frac{\sin x}{x}\right)dx = -\frac{\sin t}{t}$

50 $\dfrac{d}{dx}\displaystyle\int_{x^2}^0 \frac{\sin t}{t}\,dt = \frac{d}{dx}\int_0^{x^2}\left(-\frac{\sin t}{t}\right)dt = -2x\,\frac{\sin x^2}{x^2}$

51 $\dfrac{d}{dx}\displaystyle\int_0^{x^2} \frac{\sin u}{u}\,du = 2x\,\frac{\sin x^2}{x^2}$

52 $\dfrac{d}{dt}\displaystyle\int_{-\pi}^{t} \frac{\cos y}{1+y^2}\,dy = \frac{\cos t}{1+t^2}$

53 $\dfrac{d}{dx}\displaystyle\int_{ax}^{bx} \frac{dt}{5+t^4} = \frac{d}{dx}\left(\int_{ax}^{k} \frac{dt}{5+t^4} + \int_k^{bx}\frac{dt}{5+t^4}\right) = \frac{d}{dx}\left(\int_k^{ax} -\frac{dt}{5+t^4}\right) + \frac{d}{dx}\left(\int_k^{bx}\frac{dt}{5+t^4}\right)$

$$= \begin{bmatrix} u = ax \Rightarrow \dfrac{du}{dx} = a \\ v = bx \Rightarrow \dfrac{dv}{dx} = b \end{bmatrix} = \frac{d}{du}\left(\int_k^{u} -\frac{dt}{5+t^4}\right)\left(\frac{du}{dx}\right) + \frac{d}{dv}\left(\int_k^{v}\frac{dt}{5+t^4}\right)\left(\frac{dv}{dx}\right)$$

$$= -\frac{a}{5+u^4} + \frac{b}{5+v^4} = \frac{b}{5+(bx)^4} - \frac{a}{5+(ax)^4}$$

54 $\dfrac{d}{d\theta}\displaystyle\int_{\sin\theta}^{\cos\theta} \frac{1}{1-x^2}\,dx = \frac{d}{d\theta}\int_{\sin\theta}^{k} \frac{1}{1-x^2}\,dx + \frac{d}{d\theta}\int_k^{\cos\theta} \frac{1}{1-x^2}\,dx = \begin{bmatrix} u = \sin\theta \Rightarrow \dfrac{du}{d\theta} = \cos\theta \\ v = \cos\theta \Rightarrow \dfrac{dv}{d\theta} = -\sin\theta \end{bmatrix}$

$$= \left(\frac{d}{du}\int_k^u \frac{-1}{1-x^2}\,dx\right)\left(\frac{du}{d\theta}\right) + \left(\frac{d}{dv}\int_k^v \frac{1}{1-x^2}\,dx\right)\left(\frac{du}{d\theta}\right)$$

$$= \frac{-1}{\underbrace{1-\sin^2\theta}_{\cos^2\theta}} \times \cos\theta + \frac{1}{\underbrace{1-\cos^2\theta}_{\sin^2\theta}} \times (-\sin\theta) = -\sec\theta - \csc\theta$$

55 $\dfrac{d}{dx}\displaystyle\int_5^{x^{\frac{1}{4}}} e^{t^4+3t^2}\,dt = \begin{bmatrix} x^{\frac{1}{4}} = u \\ \dfrac{1}{4}x^{-\frac{3}{4}} = \dfrac{du}{dx} \end{bmatrix} = \left(\frac{d}{du}\int_5^u e^{t^4+3t^2}\,dt\right)\left(\frac{du}{dx}\right) = e^{x+3\sqrt{x}} \times \frac{1}{4x^{\frac{3}{4}}} = \frac{e^{x+3\sqrt{x}}}{4x^{\frac{3}{4}}}$

56 If $F(x) = \displaystyle\int_0^{2x-x^2} \cos\left(\frac{1}{1+t^2}\right)dt$ has an extreme value, then its derivative with respect to the variable x has to be zero.

$$\frac{d}{dx}F(x) = \frac{d}{dx}\int_0^{2x-x^2}\cos\left(\frac{1}{1+t^2}\right)dt = \begin{bmatrix} 2x - x^2 = u \\ 2-2x = \dfrac{du}{dx} \end{bmatrix} = \left(\frac{d}{dx}\int_0^u \cos\left(\frac{1}{1+t^2}\right)dt\right)\left(\frac{du}{dx}\right)$$

$$= \cos\left(\frac{1}{1+\left(2x-x^2\right)^2}\right) \times (2-2x)$$

We can see that there must be some extreme value since the expression can equal zero whenever

$$\cos\left(\frac{1}{1+\left(2x-x^2\right)^2}\right) = 0 \text{ or } x = 1.$$

57 a) $\displaystyle\int_0^k \frac{dx}{3x+2} = \frac{1}{3}\ln(3x+2)\Big]_0^k = \frac{1}{3}(\ln(3x+2) - \ln 2) = \frac{\ln\left(\frac{3}{2}k+1\right)}{3}$

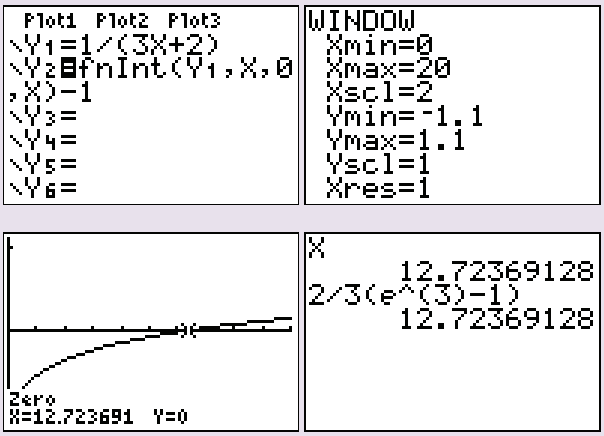
We did not take the absolute value of the natural logarithm since k must be positive. If k is negative,the interval $[k, 0]$ contains a vertical asymptote at $x = -\frac{2}{3}$; therefore, the function cannot be integrated.

Solution Paper 1 type

b $\displaystyle\frac{\ln\left(\frac{3}{2}k+1\right)}{3} = 1 \Rightarrow \ln\left(\frac{3}{2}k+1\right) = 3 \Rightarrow \frac{3}{2}k + 1 = e^3 \Rightarrow \frac{3}{2}k + 1 = e^3 \Rightarrow k = \frac{2(e^3-1)}{3}$

Solution Paper 2 type

b We can use Solver on a GDC, but by using the graphical mode we can see the tendency of the graph. There are multiple solutions possible.



58 $\displaystyle\int_0^1 x^p(1-x)^q dx = \begin{bmatrix} 1-x=t \\ -dx=dt \end{bmatrix} = \int_1^0 (1-t)^q\, t^p\,(-dt) = \int_0^1 t^q(1-t)^p dt;\ p, q \in \mathbb{N}$

Note: The integral doesn't change its value if we use a different variable.

59 a) $\displaystyle\int x(1-x)^k\, dx = \begin{bmatrix} 1-x=t \\ -dx=dt \end{bmatrix} = \int (1-t)\,t^k\,(-dt) = \int \left(t^{k+1} - t^k\right) dt = \frac{t^{k+2}}{k+2} - \frac{t^{k+1}}{k+1} + c$

$$= \frac{(1-x)^{k+2}}{k+2} - \frac{(1-x)^{k+1}}{k+1} + c;\ k \in \mathbb{N}, c \in \mathbb{R}$$

b) $\displaystyle\int_0^1 x(1-x)^k\, dx = \frac{(1-x)^{k+2}}{k+2} - \frac{(1-x)^{k+1}}{k+1}\Bigg]_0^1 = 0 - \left(\frac{1}{k+2} - \frac{1}{k+1}\right) = \frac{1}{(k+2)(k+1)}, k \in \mathbb{N}$

60 a) $\displaystyle F(3) = \int_3^3 \sqrt{5t^2+2}\, dt = 0$

b) $\displaystyle F'(x) = \frac{d}{dx}\int_3^x \sqrt{5t^2+2}\, dt = \sqrt{5x^2+2} \Rightarrow F'(3) = \sqrt{5\times 3^2+2} = \sqrt{47}$

c) $\displaystyle F''(x) = \frac{d}{dx}\left(\sqrt{5x^2+2}\right) = \frac{10x}{2\sqrt{5x^2+2}} \Rightarrow F''(3) = \frac{5\cancel{10}\times 3}{\cancel{2}\sqrt{5\times 3^2+2}} = \frac{15}{\sqrt{47}} = \frac{15\sqrt{47}}{47}$

61 If the function $f(x)$ is constant over the set of positive real numbers, then its derivative is equal to zero.

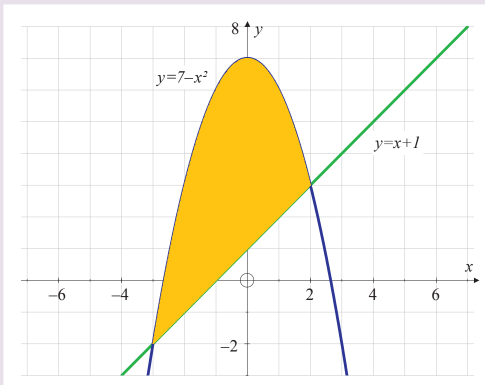
$$\frac{d}{dx}f(x)=\frac{d}{dx}\int_x^{3x}\frac{dt}{t}=\frac{d}{dx}\left(\int_x^{k}\frac{dt}{t}+\int_k^{3x}\frac{dt}{t}\right)=\frac{d}{dx}\left(\int_k^{3x}\frac{dt}{t}-\int_k^{x}\frac{dt}{t}\right)=\begin{bmatrix}3x=u\\ 3=\dfrac{du}{dx}\end{bmatrix}$$

$$=\left(\frac{d}{du}\int_k^{u}\frac{dt}{t}\right)\left(\frac{du}{dx}\right)-\left(\frac{d}{dx}\int_k^{x}\frac{dt}{t}\right)=\frac{1}{\cancel{3}x}\times\cancel{3}-\frac{1}{x}=0$$

Exercise 16.5

Solution Paper 1 type

1 Firstly, we sketch the line and the parabola, and then shade the enclosed area.



Now, we need to find the points of intersection by solving the system of equations.

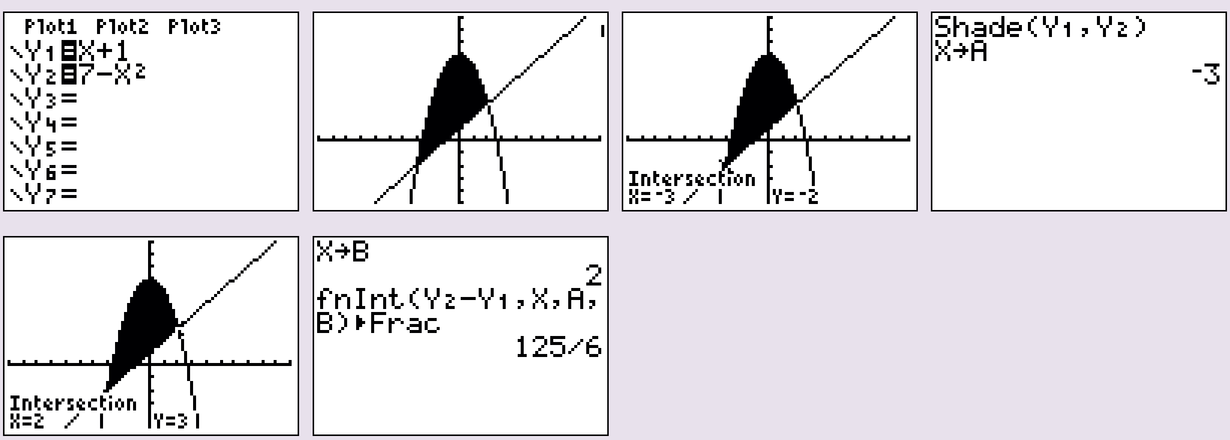
$$\left.\begin{matrix}y=x+1\\ y=7-x^2\end{matrix}\right\}\Rightarrow\left.\begin{matrix}y=x+1\\ x+1=7-x^2\end{matrix}\right\}\Rightarrow\left.\begin{matrix}y=x+1\\ x^2+x-6=0\end{matrix}\right\}\Rightarrow\left.\begin{matrix}y=x+1\\ (x-2)(x+3)=0\end{matrix}\right\}\Rightarrow$$

$\left.\begin{matrix}y_1=3,\ y_2=-2\\ x_1=2,\ x_2=-3\end{matrix}\right\}\Rightarrow(-3,-2)$ or $(2,3)$. So, the integral that we need to calculate is:

$$\int_{-3}^{2}\left[\left(7-x^2\right)-(x+1)\right]dx=\int_{-3}^{2}\left(6-x-x^2\right)dx=\left[6x-\frac{x^2}{2}-\frac{x^3}{3}\right]_{-3}^{2}=\left(12-2-\frac{8}{3}\right)-\left(-18-\frac{9}{2}+9\right)=\frac{125}{6}$$

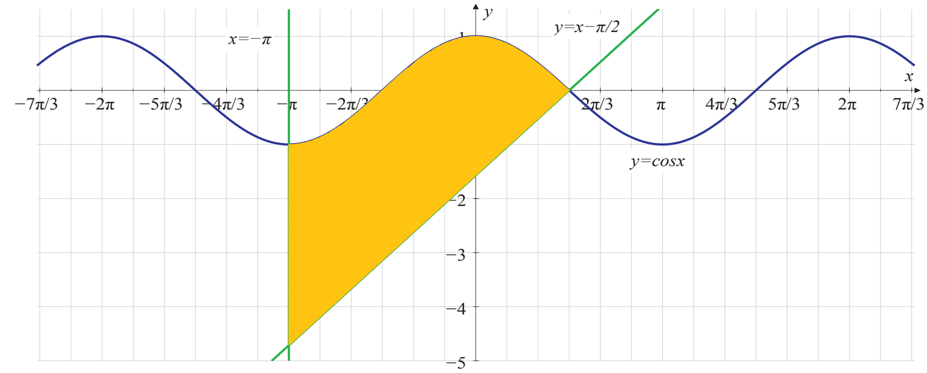
Solution Paper 2 type

1



Note: In questions 2–8, we are going to alternate between Paper 1 and Paper 2 type solutions.

2 Firstly, we sketch the cosine curve, the oblique line and the vertical line, and then shade the enclosed area.

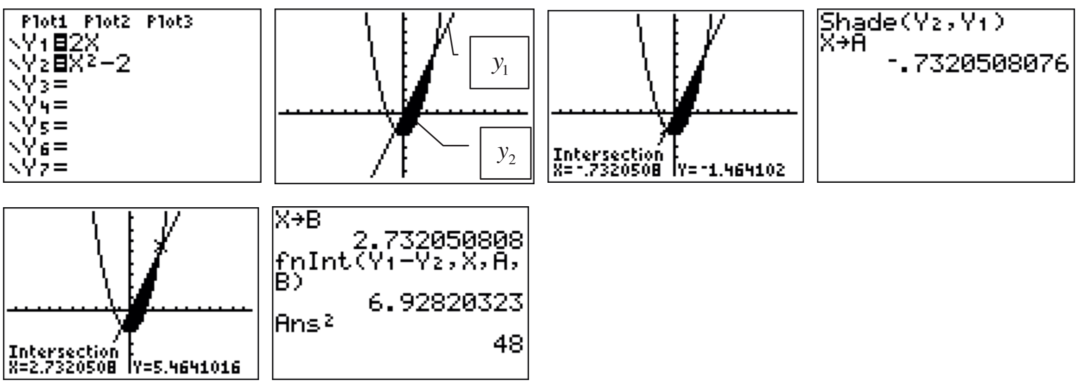


Now, we have to find the point of intersection of the curve and the oblique line. By inspection, we can see that the point is $\left(\frac{\pi}{2}, 0\right)$; therefore, to find the area, we need to solve the following integral:

$$\int_{-\pi}^{\frac{\pi}{2}} \left[\cos x - \left(x - \frac{\pi}{2}\right)\right] dx = \left[\sin x - \frac{x^2}{2} + \frac{\pi}{2} x\right]_{-\pi}^{\frac{\pi}{2}} = \left(1 - \frac{\pi^2}{8} + \frac{\pi^2}{4}\right) - \left(0 - \frac{\pi^2}{2} - \frac{\pi^2}{2}\right)$$

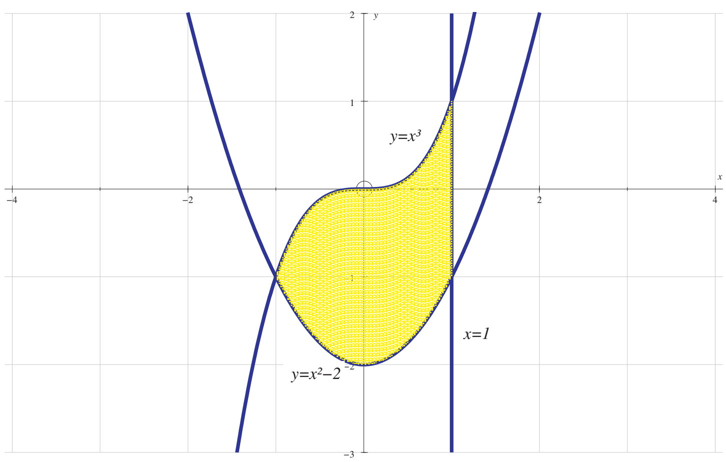
$$= 1 + \frac{\pi^2}{8} + \pi^2 = 1 + \frac{9\pi^2}{8}$$

3



Now, looking at the square of the answer, we can conclude that the exact form of the enclosed area is $\sqrt{48} = 4\sqrt{3}$.

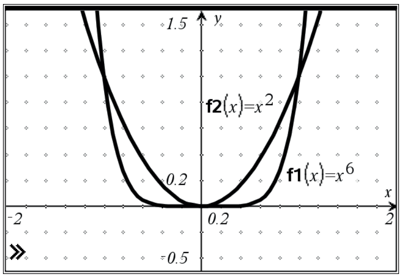
4 Sketching all the given curves and shading the enclosed area gives the following graph.

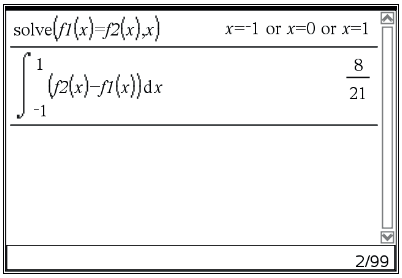


So, by inspection, we can see that the point of intersection of the curves is $(-1, -1)$; therefore, the integral that we need to calculate to find the area is:

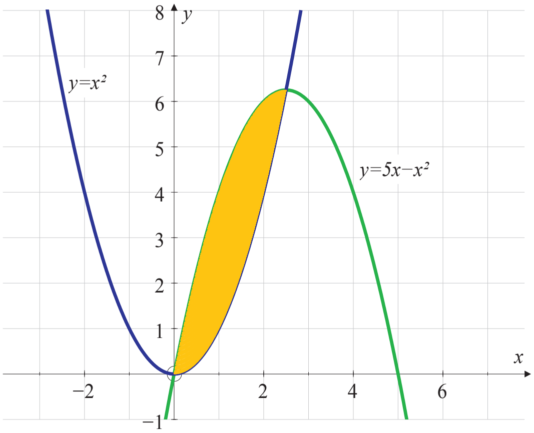
$$\int_{-1}^{1}\left[x^3-\left(x^2-2\right)\right]dx=\int_{-1}^{1}\left(x^3-x^2+2\right)dx=\left[\frac{x^4}{4}-\frac{x^3}{3}+2x\right]_{-1}^{1}=\left(\frac{1}{4}-\frac{1}{3}+2\right)-\left(\frac{1}{4}-\frac{-1}{3}-2\right)=\frac{10}{3}$$

5





6 Sketching the given curves and shading the enclosed area gives the following graph.



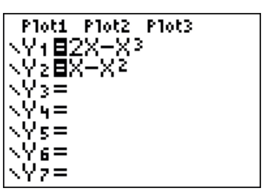
One point of intersection is obviously the origin, whilst the other looks like it has an x-coordinate of 2.5; but since we are not sure we will solve the simultaneous equations.

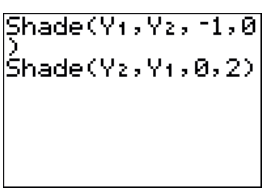
$$\left.\begin{array}{c}y=5x-x^2\\ y=x^2\end{array}\right\}\Rightarrow\left.\begin{array}{c}x^2=5x-x^2\\ y=x^2\end{array}\right\}\Rightarrow\left.\begin{array}{c}2x^2-5x=0\\ y=x^2\end{array}\right\}\Rightarrow\left.\begin{array}{c}x\left(2x-5\right)=0\\ y=x^2\end{array}\right\}\Rightarrow$$

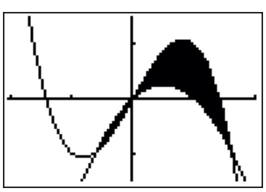
$$\left.\begin{array}{c}x_1=0,\ x_2=\frac{5}{2}\\ y_1=0,\ y_2=\frac{25}{4}\end{array}\right\}\Rightarrow(0,0)\text{ or }\left(\frac{5}{2},\frac{25}{4}\right)$$, so the integral we have to calculate is:

$$\int_0^{\frac{5}{2}}\left(\left(5x-x^2\right)-x^2\right)dx=\int_0^{\frac{5}{2}}\left(5x-2x^2\right)dx=\left[5\times\frac{x^2}{2}-2\times\frac{x^3}{3}\right]_0^{\frac{5}{2}}=\left(\frac{125}{8}-\frac{125}{12}\right)-0=\frac{125}{24}$$

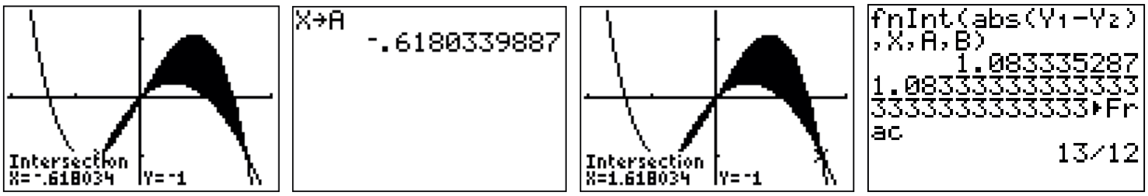
7



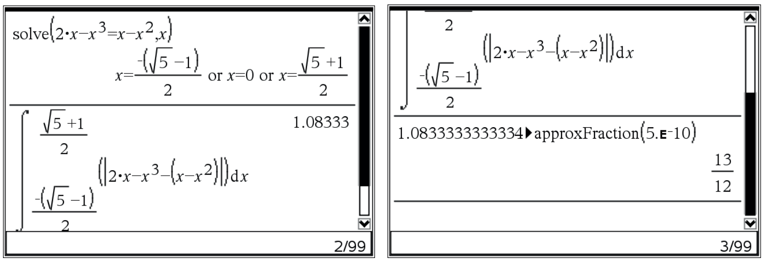




Notice that there are two areas that are defined by the two curves and the curves alternate between being upper and lower. In order to get both the areas enclosed by the curves shaded, we needed to apply the shade feature twice, with the given restriction.

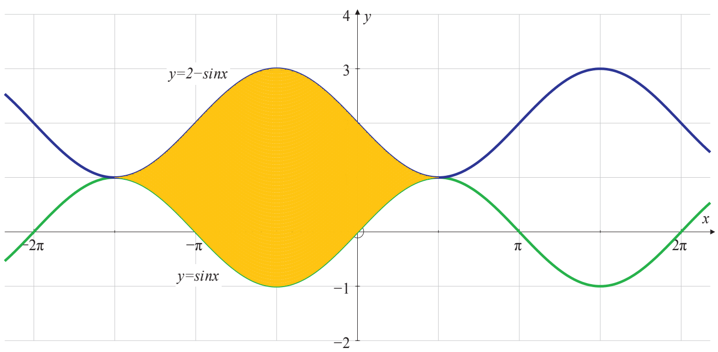


In this case, the numerical answer obtained by the TI-84 Plus model is not very accurate, so we have to be very careful in the interpretation of the results. The TI-Nspire model gives the following results.



Notice that the approximation is much better.

8 Sketching the given curves and shading the enclosed area for one period only gives the following graph.

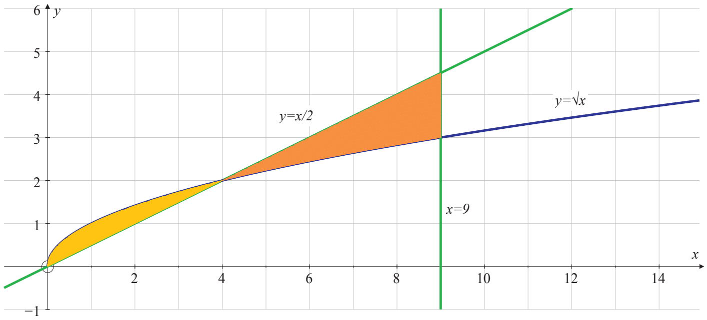


By inspection, we notice that the points of intersection are $\left(-\frac{3\pi}{2}, 1\right)$ and $\left(\frac{\pi}{2}, 1\right)$, so the integral we have to calculate is:

$$\int_{-\frac{3\pi}{2}}^{\frac{\pi}{2}} [(2 - \sin x) - \sin x]\, dx = \int_{-\frac{3\pi}{2}}^{\frac{\pi}{2}} (2 - 2\sin x)\, dx = [2x + 2\cos x]_{-\frac{3\pi}{2}}^{\frac{\pi}{2}}$$

$$= \left(\cancel{2} \times \frac{\pi}{\cancel{2}} + 2\cos\frac{\pi}{2}\right) - \left(\cancel{2} \times \left(-\frac{3\pi}{\cancel{2}}\right) + 2\cos\left(-\frac{3\pi}{2}\right)\right) = \pi + 3\pi = 4\pi$$

9 Sketching the given curves and shading the enclosed area gives the following graph.



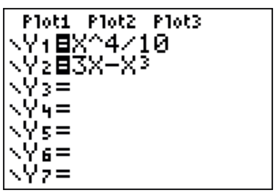
We notice that there are two areas bounded by the curves and that the curves are exchanging the position of being upper and lower. By inspection, we can see that the point of intersection is (4, 2); therefore, to find the total shaded area, we need to find the following integrals and add them together.

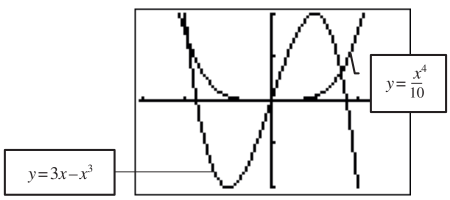
$$\int_0^4 \left(\sqrt{x} - \frac{x}{2}\right) dx = \left[\frac{2}{3}x^{\frac{3}{2}} - \frac{x^2}{4}\right]_0^4 = \left(\frac{2}{3}4^{\frac{3}{2}} - \frac{4^2}{4}\right) - 0 = \frac{16}{3} - 4 = \frac{4}{3}$$

$$\int_4^9 \left(\frac{x}{2} - \sqrt{x}\right) dx = \left[\frac{x^2}{4} - \frac{2}{3}x^{\frac{3}{2}}\right]_4^9 = \left(\frac{9^2}{4} - \frac{2}{3}9^{\frac{3}{2}}\right) - \left(\frac{4^2}{4} - \frac{2}{3}4^{\frac{3}{2}}\right) = \left(\frac{81}{4} - 18\right) - \left(4 - \frac{16}{3}\right) = \frac{9}{4} + \frac{4}{3} = \frac{43}{12}$$

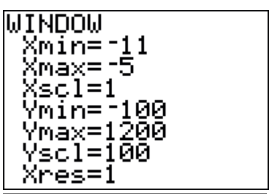
$$\text{Area} = \frac{4}{3} + \frac{43}{12} = \frac{59}{12}$$

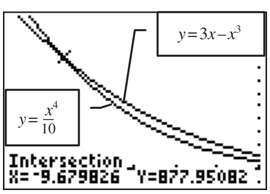
10

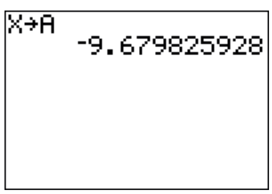


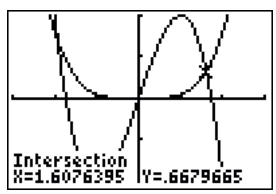


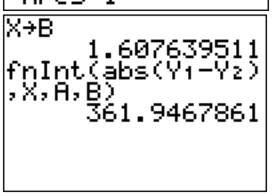
We know that, in general, a quartic function decreases faster than a cubic function. Therefore, apart from the three points of intersection that we can see on the graph, we expect to find one more to the left. To find the fourth point of intersection, we need to adjust the window.







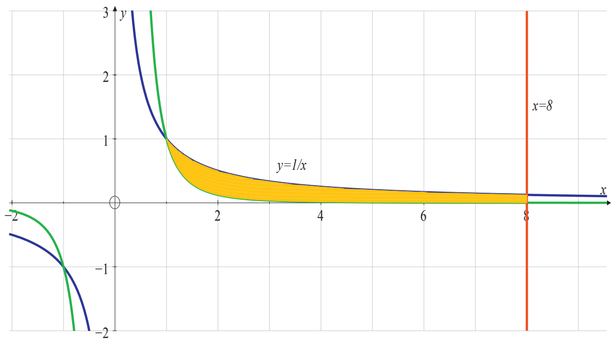




Note (1): By using the absolute value function, we can skip all the points in between and simply calculate the integral from the first point of intersection on the left side until the last point of intersection on the right side. The final answer in the IB exam would be given correct to three significant figures, 362, if not otherwise stated in the question.

Note (2): This question cannot be done algebraically since we cannot solve the equation of the fourth degree to find the boundaries of the integral.

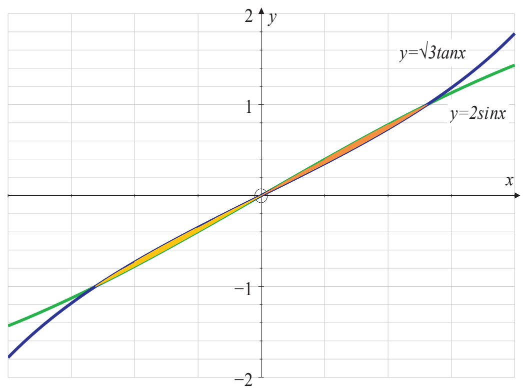
11 Sketching the given curves and shading the enclosed area gives the following graph.



By inspection, we see that the point of intersection is (1, 1). So, the integral we have to calculate is:

$$\int_1^8 \left(\frac{1}{x} - \frac{1}{x^3}\right) dx = \left[\ln|x| + \frac{1}{2x^2}\right]_1^8 = \left(\ln 8 + \frac{1}{128}\right) - \left(\ln 1 + \frac{1}{2}\right) = 3\ln 2 - \frac{63}{128}$$

12 Sketching the given curves for the restricted domain and shading the enclosed area gives the following graph.



To find the points of intersection, we need to solve the following equation:

$$2\sin x = \sqrt{3}\tan x \Rightarrow 2\sin x - \sqrt{3}\frac{\sin x}{\cos x} = 0 \Rightarrow \sin x\left(2 - \frac{\sqrt{3}}{\cos x}\right) = 0 \Rightarrow$$

$$(\sin x = 0) \text{ or } \left(2 - \frac{\sqrt{3}}{\cos x} = 0\right) \Rightarrow (\sin x = 0) \text{ or } \left(\cos x = \frac{\sqrt{3}}{2}\right) \Rightarrow x_1 = 0, x_2 = -\frac{\pi}{6}, x_3 = \frac{\pi}{6}$$

Since we have two areas, one in the first quadrant and one in the third, bounded by the curves, we need to calculate the following integrals:

$$\int_0^{\frac{\pi}{6}} \left(2\sin x - \sqrt{3}\tan x\right) dx = \left[-2\cos x + \sqrt{3}\ln|\cos x|\right]_0^{\frac{\pi}{6}}$$

$$= \left(-2\cos\left(\frac{\pi}{6}\right) + \sqrt{3}\ln\left(\cos\left(\frac{\pi}{6}\right)\right)\right) - \left(-2\cos 0 + \sqrt{3}\ln(\cos 0)\right)$$

$$= -\cancel{2} \times \frac{\sqrt{3}}{\cancel{2}} + \sqrt{3}\ln\left(\frac{\sqrt{3}}{2}\right) + 2 = 2 - \sqrt{3} + \sqrt{3}\left(\frac{1}{2}\ln 3 - \ln 2\right)$$

In a similar way, we find that

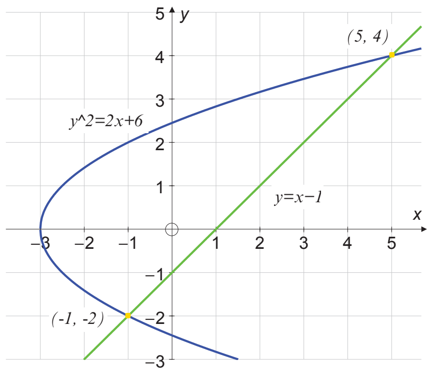
$\int_{-\frac{\pi}{6}}^{0}\left(\sqrt{3}\tan x-2\sin x\right)dx=\left[-\sqrt{3}\ln|\cos x|+2\cos x\right]_{-\frac{\pi}{6}}^{0}=\ldots=2-\sqrt{3}+\sqrt{3}\left(\frac{1}{2}\ln 3-\ln 2\right)$, which we expected since both functions are odd (symmetrical with respect to the origin); therefore, the enclosed areas must each have the same area. So, the final area is obtained by adding those two integrals or multiplying one by 2.

Area $=2\times\left(2-\sqrt{3}+\sqrt{3}\left(\frac{1}{2}\ln 3-\ln 2\right)\right)=4-2\sqrt{3}+\sqrt{3}\,(\ln 3-2\ln 2)$, which is an equivalent form to the one given in the solutions in the textbook.

$\int \tan x\,dx=\int\frac{\sin x}{\cos x}\,dx=\int\frac{-d\,(\cos x)}{\cos x}=-\ln|\cos x|+c,\ c\in\mathbb{R}$

In questions 13–17, we will integrate with respect to y since, in both equations of the curves, the variable x is linear and therefore it is easier to express it in terms of y.

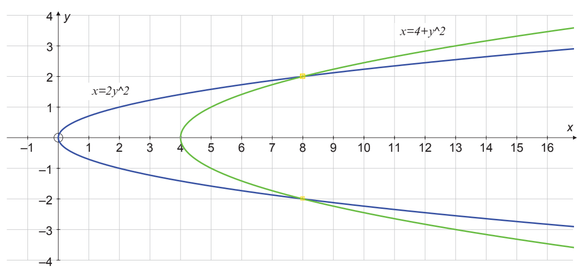
13 We sketch the curves, find the points of intersection and swap the variable of integration.



$$\int_{-2}^{4}\left((y+1)-\left(\frac{1}{2}y^2-3\right)\right)dy=\int_{-2}^{4}\left(y-\frac{1}{2}y^2+4\right)dy=\left(\frac{y^2}{2}-\frac{y^3}{6}+4y\right)\Bigg]_{-2}^{4}$$

$$=\left(8-\frac{32}{3}+16\right)-\left(2+\frac{4}{3}-8\right)=30-12=18$$

14 We sketch the curves, find the points of intersection and swap the variable of integration.



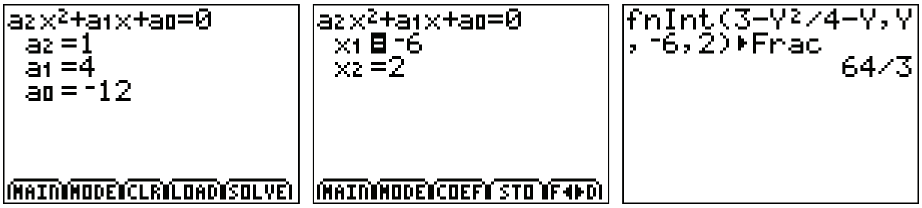
$$\int_{-2}^{2}\left((4+y^2)-(2y^2)\right)dy=\int_{-2}^{2}(4-y^2)\,dy=\left(4y-\frac{y^3}{3}\right)\Bigg]_{-2}^{2}=\left(8-\frac{8}{3}\right)-\left(-8-\frac{-8}{3}\right)=\frac{32}{3}$$

Note: Since the functions are symmetrical with respect to the x-axis, we could have used an integral from 0 to 2 and multiplied it by 2.

15 To use a GDC, we need to express x explicitly in terms of y, and then use the finite integral feature on the calculator.

$$\begin{cases}4x+y^2=12\\ y=x\end{cases}\Rightarrow\begin{cases}x=3-\dfrac{y^2}{4}\\ x=y\end{cases}$$

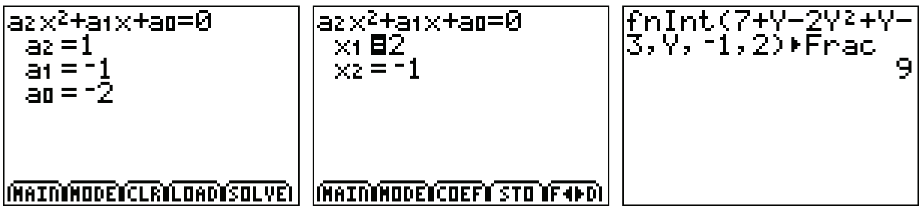
Since the parabola opens downwards, the upper function is the quadratic function.
The final step is to partially solve the simultaneous equations to find the borders of integration. (**Note:** We will use x instead of y on the GDC.)



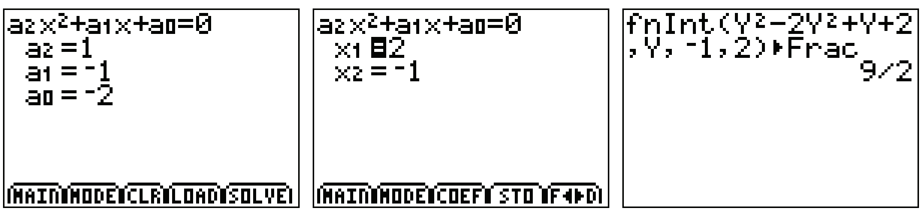
16 Again, we need to express x explicitly in terms of y, and then use the finite integral feature on the GDC.

$$\begin{cases} x - y = 7 \\ x = 2y^2 - y + 3 \end{cases} \Rightarrow \begin{cases} x = 7 + y \\ x = 2y^2 - y + 3 \end{cases}$$

Since the parabola opens upwards, the upper function is the linear function.
The final step is to partially solve the simultaneous equations to find the borders of integration.
$7 + y = 2y^2 - y + 3 \Rightarrow 2y^2 - 2y - 4 = 0 \Rightarrow y^2 - y - 2 = 0$

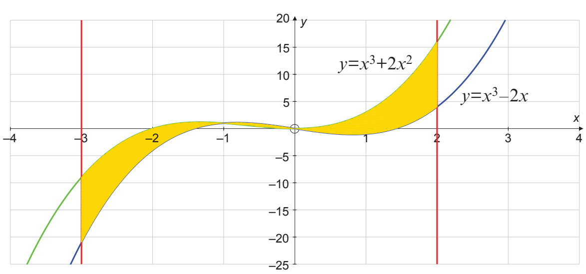


17 Since the variable x is already expressed explicitly in terms of y, we simply need to partially solve the simultaneous equations to find the borders of integration.
$y^2 = 2y^2 - y - 2 \Rightarrow y^2 - y - 2 = 0$

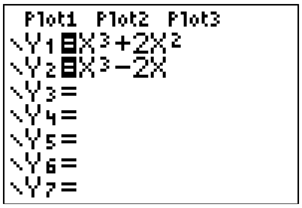


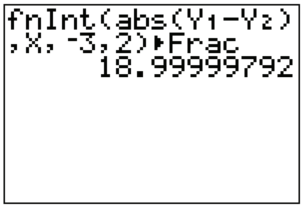
Note: We don't have to simplify the expressions and, if we are not sure which function is the upper one and which is the lower one, we don't need to spend too much time on graphing and identifying. We can simply use the absolute value of the difference of two functions; the result will always be positive and therefore it is the area between the curves.

18



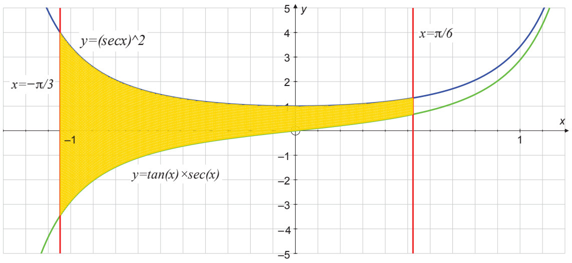
In this case, since the boundaries of integration are given by the vertical lines, we don't even have to sketch the curves. Since there are multiple areas enclosed by the two curves (alternating upper and lower curves), we can simply apply the absolute value function.





The answer provided by the calculator is not exact, but this is quite common for answers provided by a GDC. Calculators use certain numerical algorithms which contain an error visible in the answer. In this case, we give our final answer as 19. Anyhow, the IB policy to use three significant figures will enforce our answer, 19.0.

19



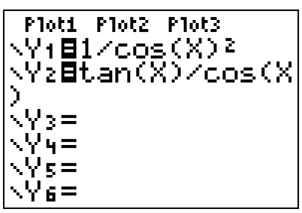
$$\int_{-\frac{\pi}{3}}^{\frac{\pi}{6}} \left(\sec^2 x - \sec x \tan x\right) dx = (\tan x - \sec x)\Big]_{-\frac{\pi}{3}}^{\frac{\pi}{6}}$$

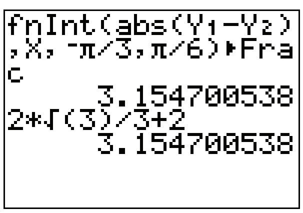
$$= \left(\tan\left(\frac{\pi}{6}\right) - \sec\left(\frac{\pi}{6}\right)\right) - \left(\tan\left(-\frac{\pi}{3}\right) - \sec\left(-\frac{\pi}{3}\right)\right)$$

$$= \frac{\sqrt{3}}{3} - \frac{2\sqrt{3}}{3} + \sqrt{3} + 2 = \frac{2\sqrt{3}}{3} + 2$$

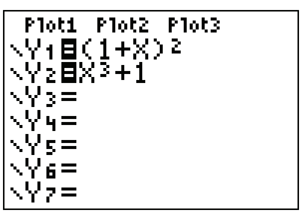
GDC verification:

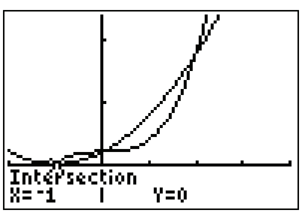
When using a calculator, the angle mode must be radians. Again, we don't have to spend too much time establishing which function is upper or lower since the absolute value will cover that part of the calculation.

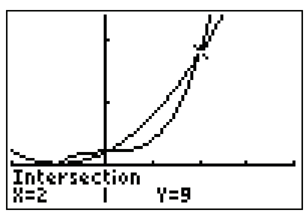


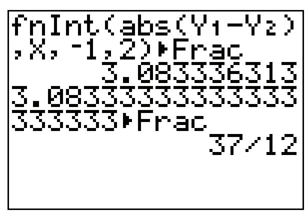


20



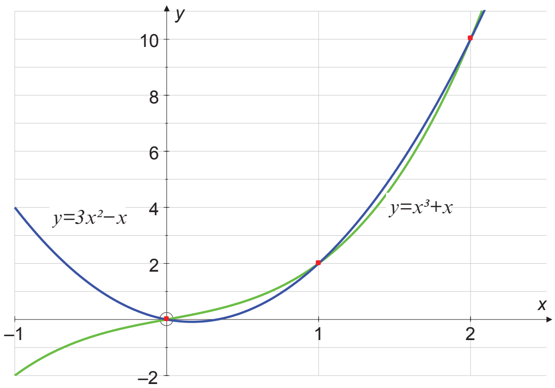






Even though the GDC does not offer an answer in fraction form, due to the algorithm's error, we are able to find the final answer as a fraction by correcting the error.

21 Firstly, we need to sketch the functions and find the points of intersection.



To find the points of intersection, we need to solve the system of simultaneous equations by using the substitution method.

$$\begin{cases} y = x^3 + x \\ y = 3x^2 - x \end{cases} \Rightarrow \begin{cases} y = x^3 + x \\ x^3 + x = 3x^2 - x \end{cases} \Rightarrow \begin{cases} y = x^3 + x \\ x^3 - 3x^2 + 2x = 0 \end{cases} \Rightarrow \begin{cases} y = x^3 + x \\ x(x-1)(x-2) = 0 \end{cases} \Rightarrow$$

$$\begin{cases} y_1 = 0, y_2 = 2, y_3 = 10 \\ x_1 = 0, x_2 = 1, x_3 = 2 \end{cases}$$

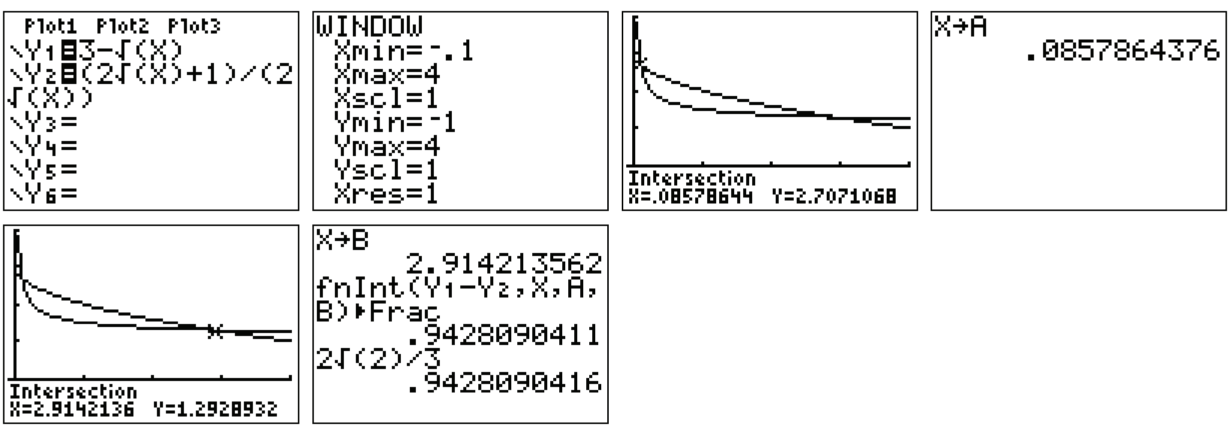
Since there are two regions enclosed by the curves, we need to split the integral into two, exchanging the upper and lower function.

$$\int_0^1 \left(\left(x^3 + x\right) - \left(3x^2 - x\right)\right) dx + \int_1^2 \left(\left(3x^2 - x\right) - \left(x^3 + x\right)\right) dx$$

$$\int_0^1 \left(x^3 - 3x^2 + 2x\right) dx + \int_1^2 \left(-x^3 + 3x^2 - 2x\right) dx = \left(\frac{x^4}{4} - x^3 + x^2\right)\Bigg]_0^1 + \left(-\frac{x^4}{4} + x^3 - x^2\right)\Bigg]_1^2$$

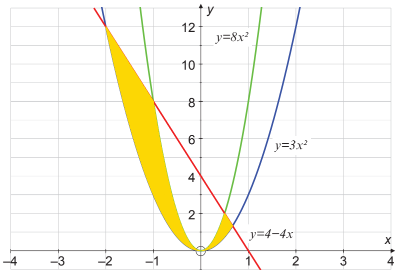
$$= \frac{1}{4} - 1 + 1 + (-4) + 8 - 4 + \frac{1}{4} - 1 + 1 = \frac{1}{2}$$

22



Here, we can conclude that the error obtained by the calculator will approximate the exact value of the area.

23



We need to find the points of intersection of the curves, so that we can establish the boundaries of integration, by solving the corresponding system of simultaneous equations.

$$\begin{cases} y = 3x^2 \\ y = 4-4x \end{cases} \Rightarrow \begin{cases} 4-4x = 3x^2 \\ y = 4-4x \end{cases} \Rightarrow \begin{cases} 3x^2+4x-4=0 \\ y = 4-4x \end{cases} \Rightarrow \begin{cases} x_{1,2} = \dfrac{-4 \pm \sqrt{16+48}}{6} = \dfrac{-4 \pm 8}{6} \\ y = 4-4x \end{cases} \Rightarrow$$

$$\begin{cases} x_1 = -2,\ x_2 = \dfrac{2}{3} \\ y = 4-4x \end{cases}$$

$$\begin{cases} y = 8x^2 \\ y = 4-4x \end{cases} \Rightarrow \begin{cases} 4-4x = 8x^2 \\ y = 4-4x \end{cases} \Rightarrow \begin{cases} 8x^2+4x-4=0 \\ y = 4-4x \end{cases} \Rightarrow \begin{cases} 2x^2+x-1=0 \\ y = 4-4x \end{cases} \Rightarrow$$

$$\begin{cases} x_{1,2} = \dfrac{-1 \pm \sqrt{1+8}}{4} = \dfrac{-1 \pm 3}{4} \\ y = 4-4x \end{cases} \Rightarrow \begin{cases} x_1 = -1,\ x_2 = \dfrac{1}{2} \\ y = 4-4x \end{cases}$$

In both cases, we did not calculate the y-values since, for the integration, we simply need the x-coordinates of the points of intersection.

$$\int_{-2}^{-1}\left(4-4x-3x^2\right)dx + \int_{-1}^{\frac{1}{2}}\left(\underbrace{8x^2-3x^2}_{5x^2}\right)dx + \int_{\frac{1}{2}}^{\frac{2}{3}}\left(4-4x-3x^2\right)dx$$

$$= \left(4x-2x^2-x^3\right)\Big]_{-2}^{-1} + \left(\frac{5x^3}{3}\right)\Bigg]_{-1}^{\frac{1}{2}} + \left(4x-2x^2-x^3\right)\Big]_{\frac{1}{2}}^{\frac{2}{3}}$$

$$= \cancel{-4}\ \cancel{-2} + 1\ \cancel{+8}\ \cancel{+8}\ \cancel{-8} + \frac{5}{24} + \frac{5}{3} + \frac{8}{3} - \frac{8}{9} - \frac{8}{27}\ \cancel{-2} + \frac{1}{2} + \frac{1}{8} = \frac{269}{54}$$

Solution Paper 1 type

24 Firstly, we need to find the equation of the tangent at the point $(1, e)$.

$y = e^x \Rightarrow y' = e^x$

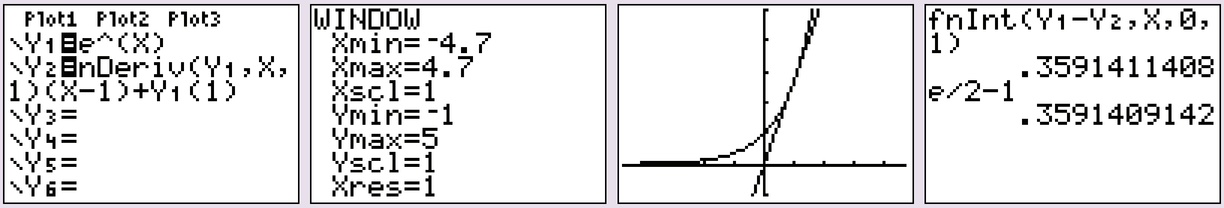
$m = y'(1) = e \Rightarrow$ Equation of tangent: $y = e(x-1)+e \Rightarrow y = ex$

Since $y'' = e^x \Rightarrow y''(1) = e > 0$, we can conclude that the curve is above the tangent; therefore, to find the area of the region enclosed by the curves, we calculate the following integral.

$$\int_0^1\left(e^x - ex\right)dx = \left(e^x - e\frac{x^2}{2}\right)\Bigg]_0^1 = \left(e - \frac{e}{2}\right) - (1-0) = \frac{e}{2} - 1$$

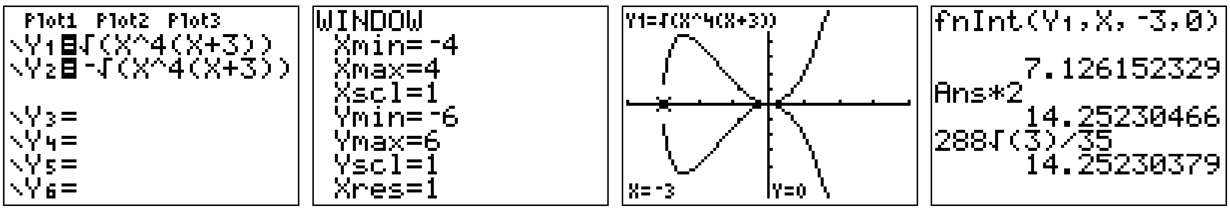
Solution Paper 2 type

24 We can immediately input the equation of the curve and the equation of the tangent by using the graphing function.



For the equation of the tangent to the curve, $y = f(x)$, at the point (x_1, y_1), we were using the following formula: $y = f'(x_1)(x - x_1) + y_1$. If the question does not ask for an exact value, we can solve using a GDC.

25 Since the implicitly defined function is symmetrical with respect to the x-axis (only even powers of y), we simply need to use one of the two branches.



26 Again, the implicitly defined function is symmetrical with respect to the x-axis (only even powers of y). To find the borders of integration, we need to solve the equation by setting $y = 0$.

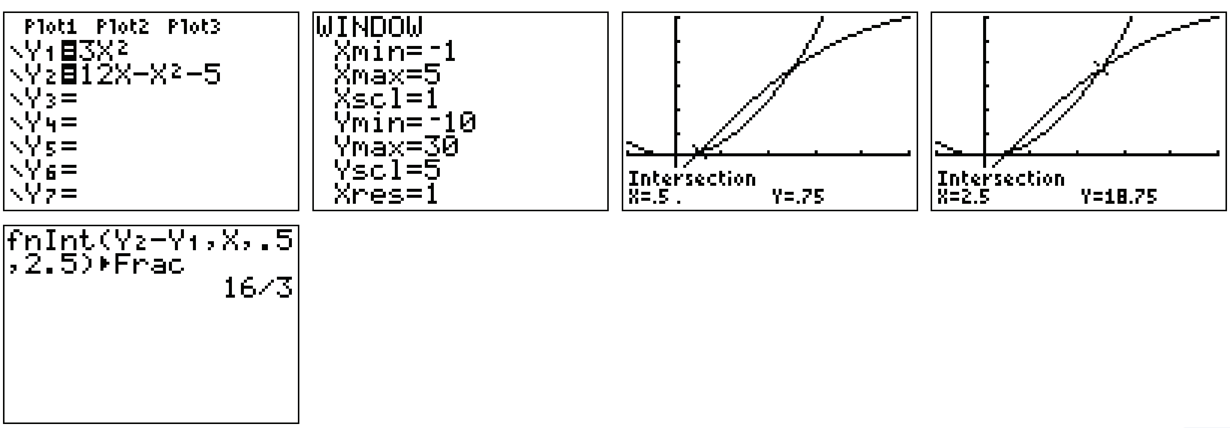
$$0 = 2x^2 - 4x^4 \Rightarrow 2x^2\left(1 - 2x^2\right) = 0 \Rightarrow x_1 = 0,\ x_2 = -\frac{\sqrt{2}}{2},\ x_3 = \frac{\sqrt{2}}{2}$$

Alternatively, since the implicitly defined function is symmetrical with respect to the y-axis (only even powers of x), we can calculate a simpler integral and then multiply it by 4.

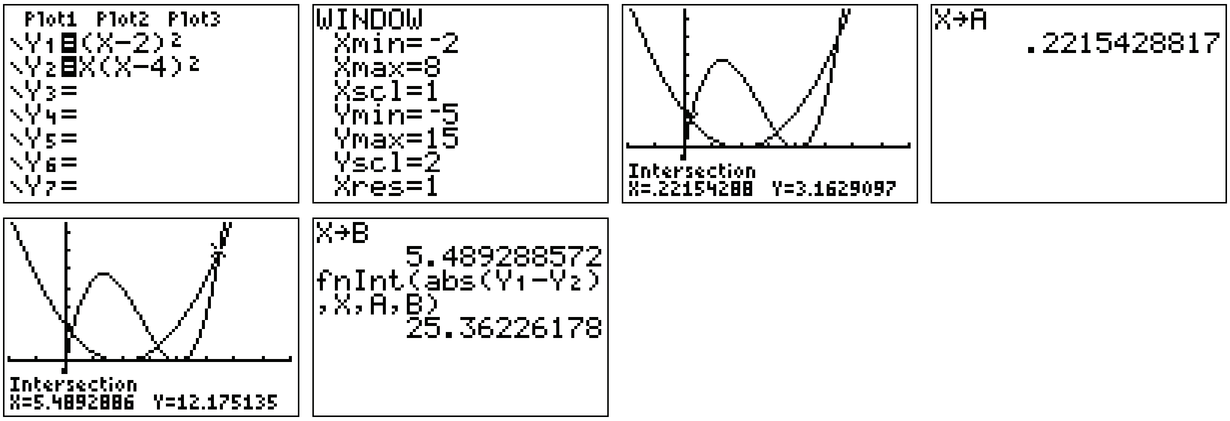
$$4 \times \int_0^{\frac{\sqrt{2}}{2}} \sqrt{2x^2 - 4x^4}\,dx = 4\sqrt{2} \times \int_0^{\frac{\sqrt{2}}{2}} x\sqrt{1 - 2x^2}\,dx = \begin{bmatrix} 1 - 2x^2 = t \\ -4x\,dx = dt \end{bmatrix} = 4\sqrt{2} \times \int_1^0 \sqrt{t} \times \left(-\frac{1}{4}\,dt\right)$$

$$= 4\sqrt{2} \times \int_1^0 \sqrt{t} \times \left(-\frac{1}{4}\,dt\right) = \cancel{4}\sqrt{2} \times \frac{1}{\cancel{4}} \times \int_0^1 \sqrt{t}\,dt = \sqrt{2} \times \left(\frac{2}{3}t^{\frac{3}{2}}\right)\Bigg]_0^1 = \frac{2\sqrt{2}}{3}$$

27 In this question, we can do all the calculations with respect to y, since x is expressed as the subject in both equations. We are going to swap them on a GDC.



28

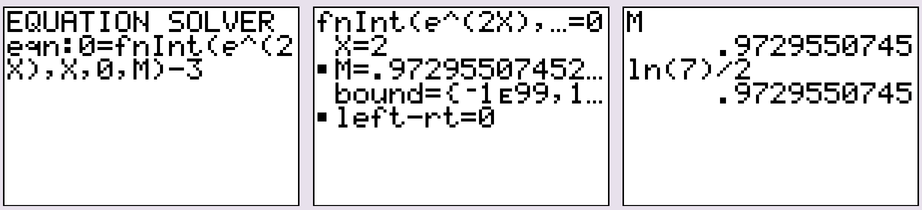


Solution Paper 1 type

29 $\int_0^m e^{2x}\,dx = \left(\frac{1}{2}e^{2x}\right)\Big]_0^m = \frac{1}{2}\left(e^{2m} - 1\right) \Rightarrow \frac{1}{2}\left(e^{2m} - 1\right) = 3 \Rightarrow e^{2m} - 1 = 6 \Rightarrow e^{2m} = 7 \Rightarrow 2m = \ln 7 \Rightarrow m = \frac{\ln 7}{2}$

Solution Paper 2 type

29



We have simply confirmed the result obtained by the first method.

(The value of x is not relevant to this problem.)

Solution Paper 1 type

30 Firstly, we need to find the zeros of the function.

$$x^3 - 4x^2 + 3x = 0 \Rightarrow x(x-1)(x-3) = 0 \Rightarrow x_1 = 0,\ x_1 = 1,\ x_1 = 3$$

$$y'(x) = 3x^2 - 8x + 3 \Rightarrow y'(0) = 3 > 0,\ y'(1) = -2 < 0,\ y'(3) = 6 > 0$$

Therefore, we conclude that the function is increasing at the first and the third zero, whilst it is decreasing at the second zero. To calculate the area enclosed by the function and the x-axis, we need to take into account the fact that the region between the second and the third zero is below the x-axis and hence we need to take the opposite expression.

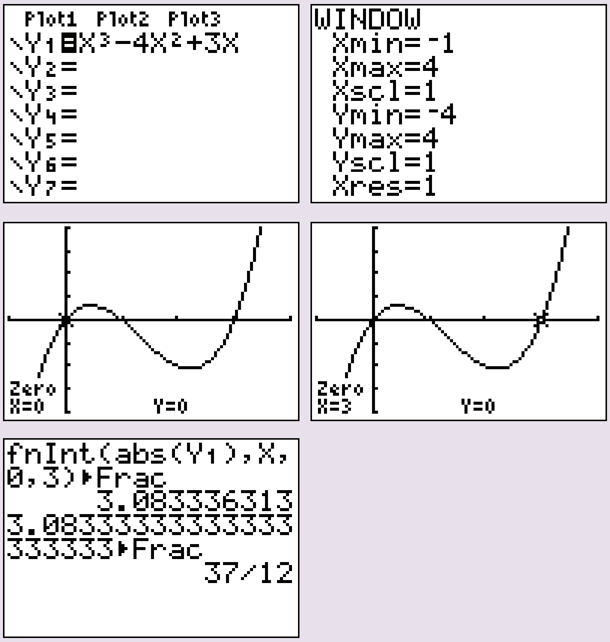
$$\int_0^1 \left(x^3 - 4x^2 + 3x\right)dx + \int_1^3 \left(-x^3 + 4x^2 - 3x\right)dx = \left(\frac{x^4}{4} - \frac{4x^3}{3} + \frac{3x^2}{2}\right)\Bigg]_0^1 + \left(-\frac{x^4}{4} + \frac{4x^3}{3} - \frac{3x^2}{2}\right)\Bigg]_1^3$$

$$= \frac{1}{4} - \frac{4}{3} + \frac{3}{2} - \frac{81}{4} + \frac{108}{3} - \frac{27}{2} + \frac{1}{4} - \frac{4}{3} + \frac{3}{2}$$

$$= -\frac{79}{4} + \frac{100}{3} - \frac{21}{2} = \frac{-237 + 400 - 126}{12} = \frac{37}{12}$$

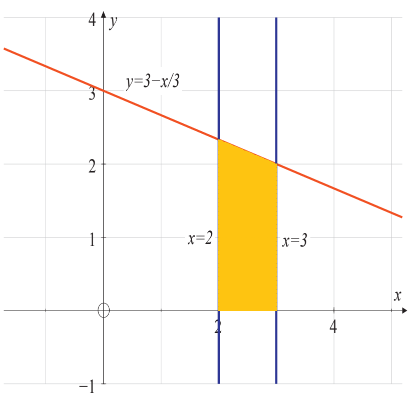
Solution Paper 2 type

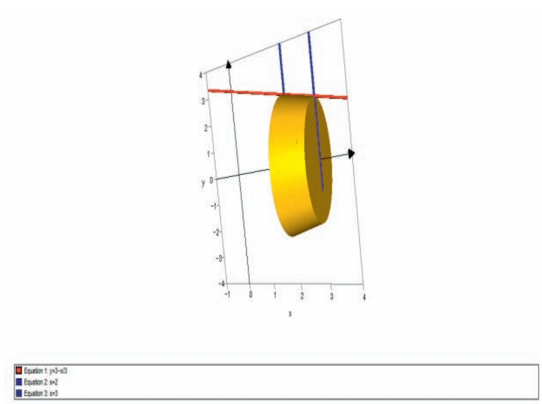
30



Exercise 16.6

1 We begin by sketching the lines and shading the region that is rotated about the *x*-axis.



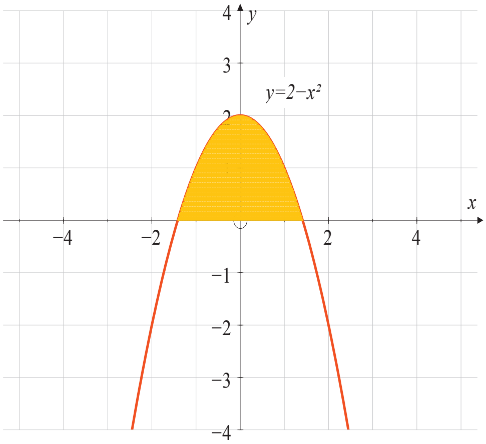


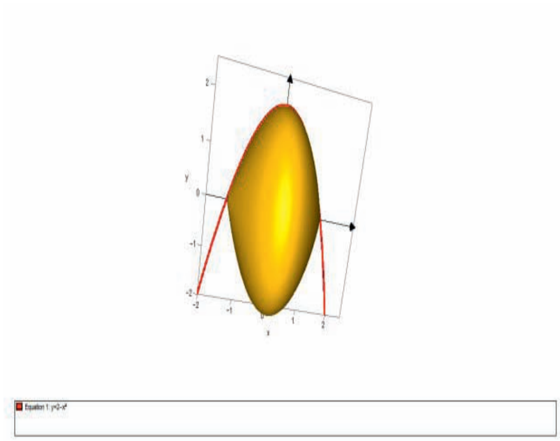
To find the volume, we need to evaluate the following integral:

$$V = \pi\int_2^3 \left(3 - \frac{x}{3}\right)^2 dx = \pi\int_2^3 \left(9 - 2x + \frac{x^2}{9}\right) dx = \pi\left[9x - x^2 + \frac{x^3}{27}\right]_2^3$$

$$= \pi\left((27 - 9 + 1) - \left(18 - 4 + \frac{8}{27}\right)\right) = \frac{127}{27}\pi$$

2 We begin by sketching the parabola and shading the region between the x-axis ($y = 0$) and the parabola that is rotated about the x-axis. By inspection, we find that the parabola intersects the x-axis at the points $\left(-\sqrt{2}, 0\right)$ and $\left(\sqrt{2}, 0\right)$.





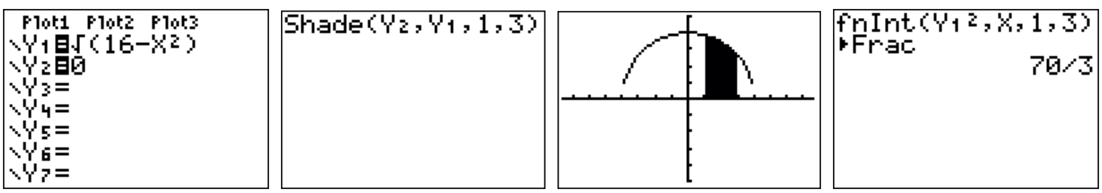
To find the volume of the solid, we need to evaluate the following integral:

$$V = \pi\int_{-\sqrt{2}}^{\sqrt{2}} \left(2 - x^2\right)^2 dx = \pi\int_{-\sqrt{2}}^{\sqrt{2}} \left(4 - 4x^2 + x^4\right) dx = \pi\left[4x - \frac{4}{3}x^3 + \frac{x^5}{5}\right]_{-\sqrt{2}}^{\sqrt{2}}$$

$$= \pi\left(\left(4\sqrt{2} - \frac{4}{3}\times 2\sqrt{2} + \frac{4\sqrt{2}}{5}\right) - \left(-4\sqrt{2} + \frac{4}{3}\times 2\sqrt{2} - \frac{4\sqrt{2}}{5}\right)\right)$$

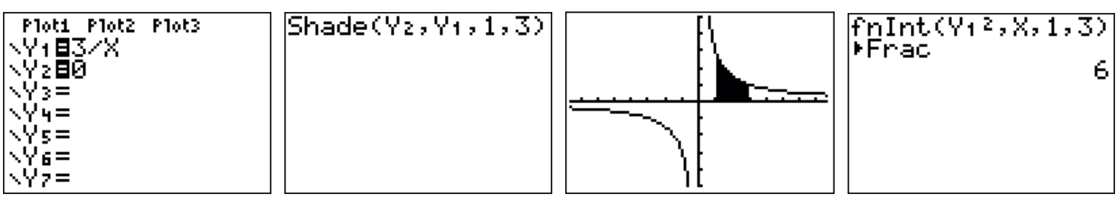
$$= \pi\left(8\sqrt{2} - \frac{16\sqrt{2}}{3} + \frac{8\sqrt{2}}{5}\right) = \frac{64\sqrt{2}}{15}\pi$$

3 We begin by sketching the curves and shading the region that is rotated about the x-axis.



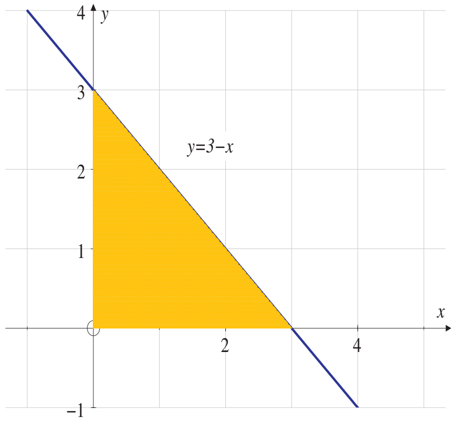
We need to multiply the answer on the final GDC screen by π to obtain the volume. So, the volume is $V = \dfrac{70}{3}\pi$.

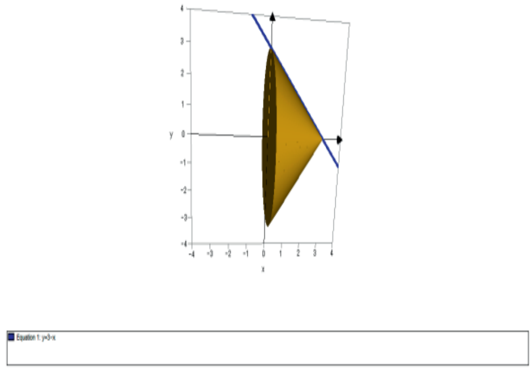
4



We need to multiply the answer on the final GDC screen by π to find the volume. So, the volume is $V = 6\pi$.

5 We begin by sketching the lines and shading the region that is rotated about the x-axis.

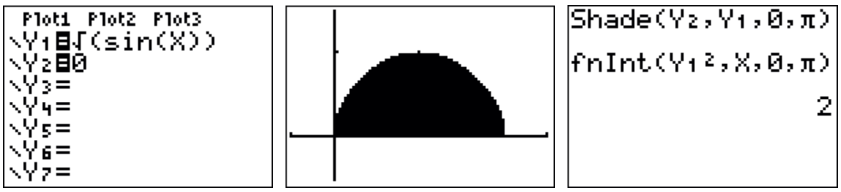




To find the volume of the solid, we need to evaluate the following integral:

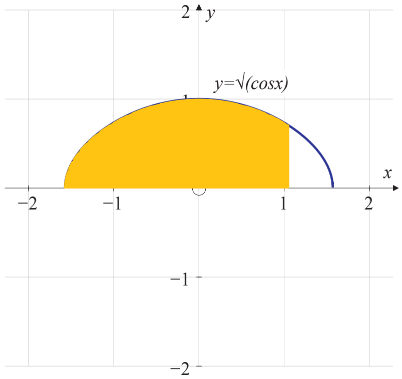
$$V = \pi\int_0^3 (3-x)^2\,dx = \pi\int_0^3 \left(9-6x+x^2\right)dx = \pi\left[9x - \cancel{6}\,3 \times \frac{x^2}{\cancel{2}} + \frac{x^3}{3}\right]_0^3 = \pi\left(\cancel{27} - \cancel{27} + \frac{3^{\cancel{3}\,2}}{\cancel{3}}\right) = 9\pi$$

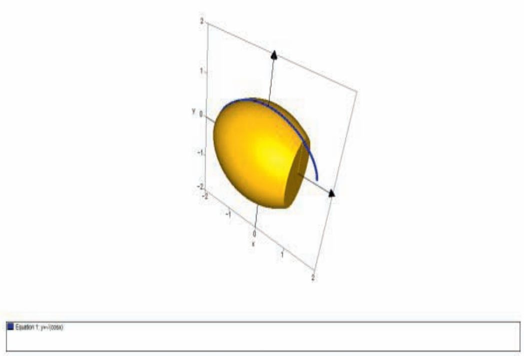
6



So, the volume is 2π.

7 We begin by sketching the cosine curve and shading the region that is rotated about the x-axis.

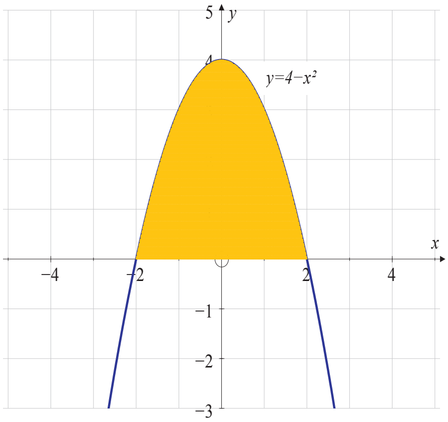


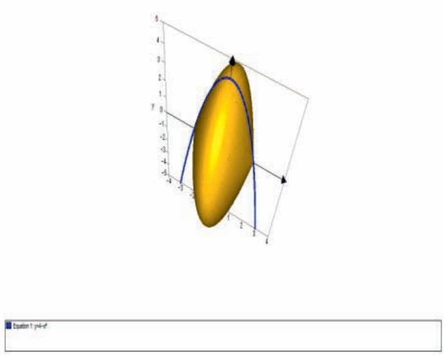


To find the volume of the solid, we need to evaluate the following integral:

$$V = \pi\int_{-\frac{\pi}{2}}^{\frac{\pi}{3}} \left(\sqrt{\cos x}\right)^2 dx = \pi\int_{-\frac{\pi}{2}}^{\frac{\pi}{3}} \cos x\,dx = \pi\left[\sin x\right]_{-\frac{\pi}{2}}^{\frac{\pi}{3}} = \pi\left(\sin\left(\frac{\pi}{3}\right) - \sin\left(-\frac{\pi}{2}\right)\right) = \pi\left(\frac{\sqrt{3}}{2} + 1\right)$$

8 We begin by sketching the parabola and shading the region that is rotated about the x-axis. By inspection, we find that the points of intersection with the x-axis are $(-2, 0)$ and $(2, 0)$.



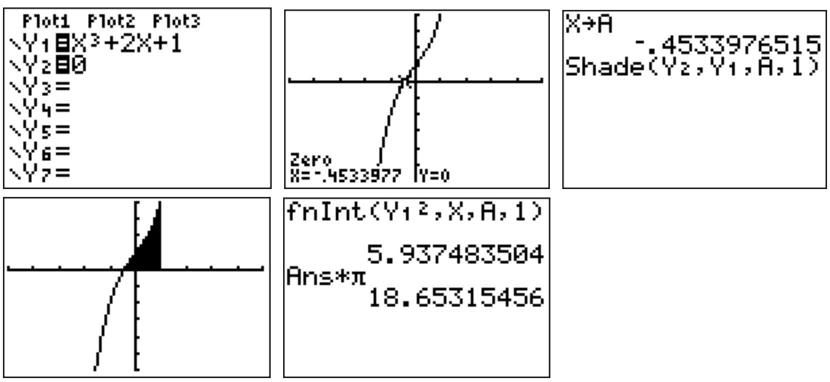


To find the volume, we need to evaluate the following integral:

$$V = \pi\int_{-2}^{2}(4-x^2)^2\,dx = \pi\int_{-2}^{2}(16-8x^2+x^4)\,dx = \pi\left[16x-8\times\frac{x^3}{3}+\frac{x^5}{5}\right]_{-2}^{2}$$

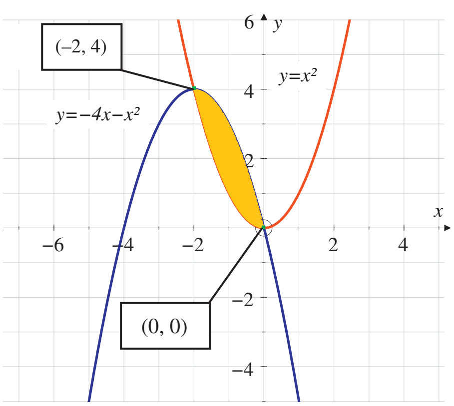
$$= \pi\left(\left(32-\frac{64}{3}+\frac{32}{5}\right)-\left(-32+\frac{64}{3}-\frac{32}{5}\right)\right) = \frac{512\pi}{15}$$

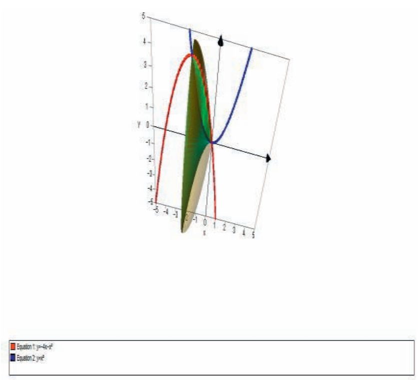
9 We need to find the point of intersection of the cubic curve with the x-axis. We will store the point in the memory to simplify our later calculations.



So, the volume is $V \approx 5.937\pi$ or $V = 18.7$, correct to three significant figures.

10 We begin by sketching the parabolas and shading the region that is rotated about the x-axis. By inspection, we find that the points of intersection of the *curves* are $(-2, 4)$ and $(0, 0)$.





To find the volume, we need to evaluate the integral of the upper function and the integral of the lower function, and then subtract them to obtain the final answer.

$$V_{upper} = \pi\int_{-2}^{0}\left(-4x - x^2\right)^2 dx = \pi\int_{-2}^{0}\left(16x^2 + 8x^3 + x^4\right)dx = \pi\left[16\times\frac{x^3}{3} + 8\,2\times\frac{x^4}{4} + \frac{x^5}{5}\right]_{-2}^{0}$$

$$= \pi\left(0 - \left(16\times\left(-\frac{8}{3}\right) + 2\times16 + \frac{-32}{5}\right)\right) = \pi\left(\frac{128}{3} - 32 + \frac{32}{5}\right) = \frac{256}{15}\pi$$

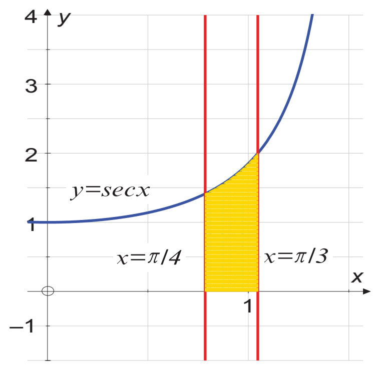
$$V_{lower} = \pi\int_{-2}^{0}\left(x^2\right)^2 dx = \pi\int_{-2}^{0} x^4\,dx = \pi\left[\frac{x^5}{5}\right]_{-2}^{0} = \pi\left(0 - \frac{-32}{5}\right) = \frac{32\pi}{5}$$

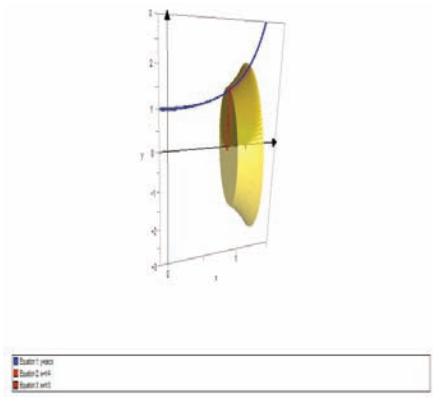
So, the final volume is:

$$V = V_{upper} - V_{lower} = \frac{256}{15}\pi - \frac{32}{5}\pi = \frac{256-96}{15}\pi = \frac{160}{15}\pi = \frac{32}{3}\pi$$

In questions 11–19, we begin by sketching the curves, shading the region that is rotated about the x-axis and then drawing the solid of revolution.

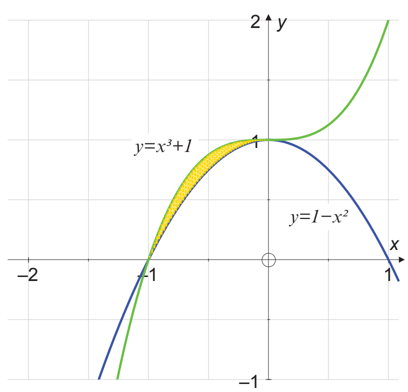
11

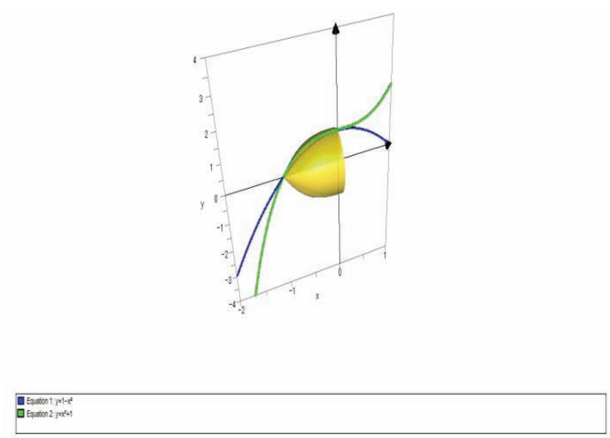




$$V = \pi\int_{\frac{\pi}{4}}^{\frac{\pi}{3}} \sec^2 x\,dx = \pi\left(\tan x\right]_{\frac{\pi}{4}}^{\frac{\pi}{3}}\Bigg) = \pi\left(\tan\left(\frac{\pi}{3}\right) - \tan\left(\frac{\pi}{4}\right)\right) = \pi\left(\sqrt{3} - 1\right)$$

12

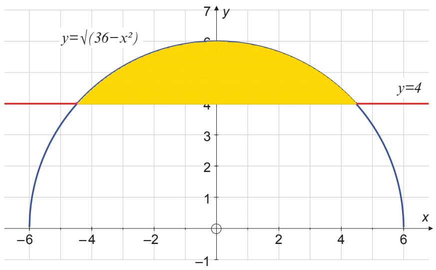


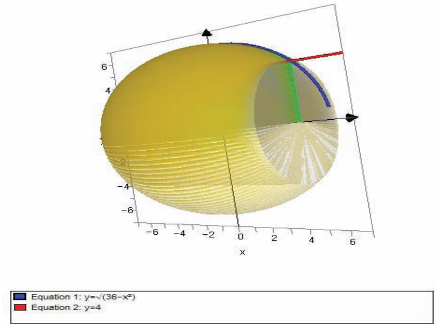


$$V = \pi\int_{-1}^{0}\left(\left(x^3+1\right)^2 - \left(1-x^2\right)^2\right)dx = \pi\int_{-1}^{0}\left(x^6 - x^4 + 2x^3 + 2x^2\right)dx = \pi\left(\frac{x^7}{7} - \frac{x^5}{5} + \frac{x^4}{2} + \frac{2x^3}{3}\right)\Bigg]_{-1}^{0}$$

$$= \pi\left(0 - \left(-\frac{1}{7} + \frac{1}{5} + \frac{1}{2} - \frac{2}{3}\right)\right) = \pi\,\frac{30 - 42 - 105 + 140}{210} = \frac{23\pi}{210}$$

13





Before we find the volume of revolution, we firstly need to find the borders of integration by solving the simultaneous equations.

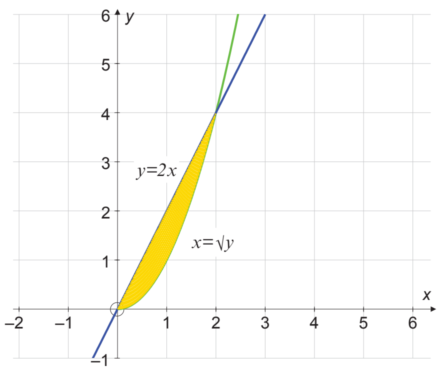
$$\begin{cases} y = \sqrt{36 - x^2} \\ y = 4 \end{cases} \Rightarrow \begin{cases} 4 = \sqrt{36 - x^2} \\ y = 4 \end{cases} \Rightarrow \begin{cases} 16 = 36 - x^2 \\ y = 4 \end{cases} \Rightarrow \begin{cases} x^2 = 20 \\ y = 4 \end{cases} \Rightarrow \begin{cases} x_1 = -2\sqrt{5}, x_2 = 2\sqrt{5} \\ y = 4 \end{cases}$$

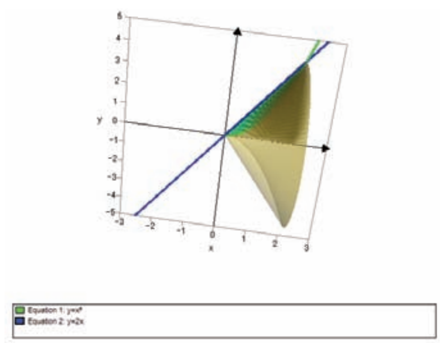
Since both functions are even (symmetrical with respect to the y-axis), we can simply calculate the integral from 0 to $2\sqrt{5}$ and then multiply it by 2.

$$V = 2\pi\int_0^{2\sqrt{5}} \left(\left(\sqrt{36 - x^2}\right)^2 - 4^2\right) dx = 2\pi\int_0^{2\sqrt{5}} \left(20 - x^2\right) dx = 2\pi\left[20x - \frac{x^3}{3}\right]_0^{2\sqrt{5}}$$

$$= 2\pi\left(40\sqrt{5} - \frac{40\sqrt{5}}{3}\right) = \frac{160\pi\sqrt{5}}{3}$$

14



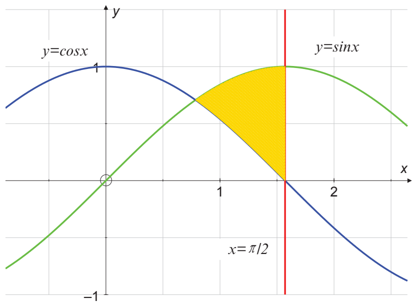


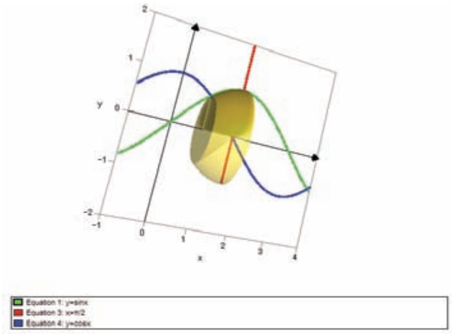
To ease our calculations, we use the fact that $x = \sqrt{y} \Leftrightarrow y = x^2, x \geqslant 0$. By inspection, we can find the borders of integration.

$$V = \pi\int_0^2 \left((2x)^2 - \left(x^2\right)^2\right) dx = \pi\int_0^2 \left(4x^2 - x^4\right) dx = \pi\left[\frac{4x^3}{3} - \frac{x^5}{5}\right]_0^2$$

$$= \pi\left(\frac{32}{3} - \frac{32}{5}\right) = \frac{64\pi}{15}$$

15

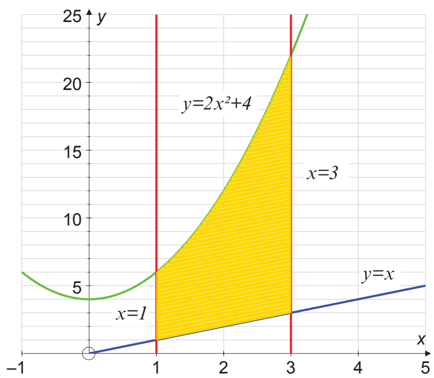


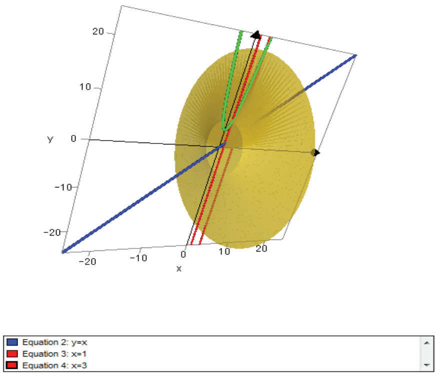


$$V=\pi\int_{\frac{\pi}{4}}^{\frac{\pi}{2}}\left(\sin^2 x-\cos^2 x\right)dx=\pi\int_{\frac{\pi}{4}}^{\frac{\pi}{2}}\left(-\cos 2x\right)dx=\pi\left(-\frac{1}{2}\sin 2x\right)\Bigg]_{\frac{\pi}{4}}^{\frac{\pi}{2}}$$

$$=-\frac{\pi}{2}\left(\underbrace{\sin\pi}_{0}-\underbrace{\sin\frac{\pi}{2}}_{1}\right)=\frac{\pi}{2}$$

16

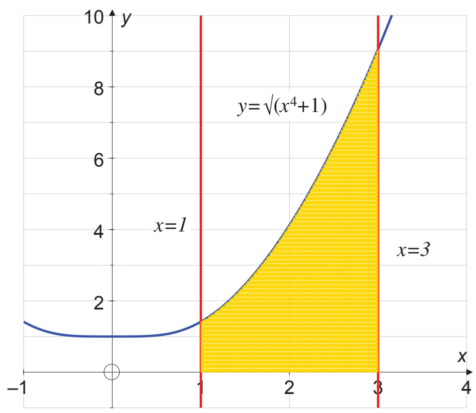


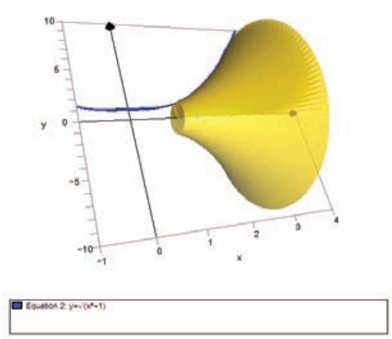


$$V=\pi\int_1^3\left(\left(2x^2+4\right)^2-x^2\right)dx=\pi\int_1^3\left(4x^4+15x^2+16\right)dx=\pi\left(\frac{4x^5}{5}+5x^3+16x\right)\Bigg]_1^3$$

$$=\pi\left(\frac{972}{5}+135+48-\frac{4}{5}-5-16\right)=\frac{1778\pi}{5}$$

17

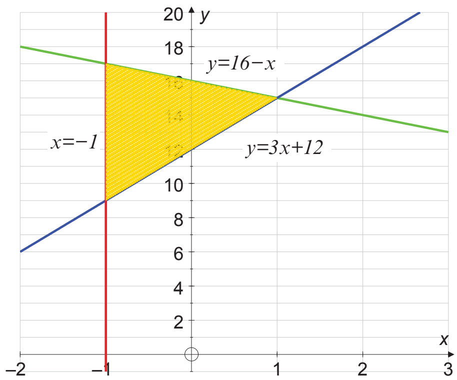


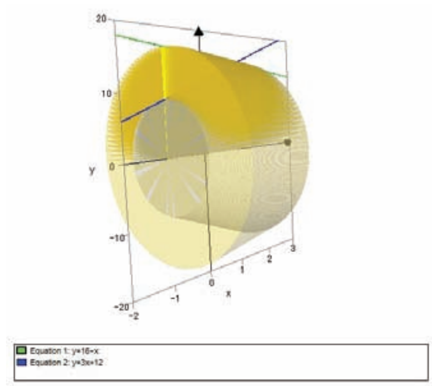


$$V=\pi\int_1^3\left(\sqrt{x^4+1}\right)^2dx=\pi\int_1^3\left(x^4+1\right)dx=\pi\left(\frac{x^5}{5}+x\right)\Bigg]_1^3$$

$$=\pi\left(\frac{243}{5}+3-\frac{1}{5}-1\right)=\frac{252\pi}{5}$$

18 Here, we can find the point of intersection by inspection.



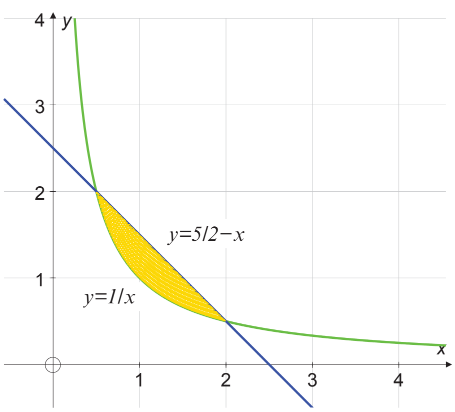


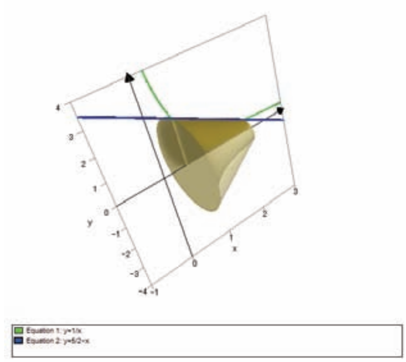
$$V = \pi\int_{-1}^{1}\left((16-x)^2-(3x+12)^2\right)dx = \pi\int_{-1}^{1}\left(-8x^2-104x+112\right)dx$$

$$= \pi\left(-\frac{8x^3}{3}-57x^2+112x\right)\Bigg]_{-1}^{1} = \pi\left(-\frac{8}{3}-57+112-\frac{8}{3}+57+112\right) = \frac{656\pi}{3}$$

19 We can find the points of intersection by solving the simultaneous equations.

$$\begin{cases} y=\dfrac{1}{x} \\ y=\dfrac{5}{2}-x \end{cases} \Rightarrow \begin{cases} y=\dfrac{1}{x} \\ \dfrac{1}{x}=\dfrac{5}{2}-x \Big/ \times 2x \end{cases} \Rightarrow \begin{cases} y=\dfrac{1}{x} \\ 2x^2-5x+2=0 \end{cases} \Rightarrow \begin{cases} y=\dfrac{1}{x} \\ (2x-1)(x-2)=0 \end{cases} \Rightarrow \begin{cases} y_1=2,\ y_2=\dfrac{1}{2} \\ x_1=\dfrac{1}{2},\ x_2=2 \end{cases}$$





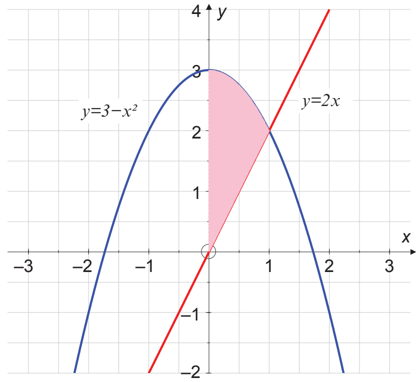
$$V = \pi\int_{\frac{1}{2}}^{2}\left(\left(\frac{5}{2}-x\right)^2-\left(\frac{1}{x}\right)^2\right)dx = \pi\int_{\frac{1}{2}}^{2}\left(\frac{25}{4}-5x+x^2-\frac{1}{x^2}\right)dx = \pi\left(\frac{25}{4}x-\frac{5x^2}{2}+\frac{x^3}{3}+\frac{1}{x}\right)\Bigg]_{\frac{1}{2}}^{2}$$

$$= \pi\left(\frac{25}{2}-10+\frac{8}{3}+\frac{1}{2}-\frac{25}{8}+\frac{5}{8}-\frac{1}{24}-2\right) = \frac{9\pi}{8}$$

An alternative method of solving a rotation about the y-axis involves swapping the variables of integration from x to y, which means that we need to express x in terms of y to find the volume of revolution. So, in questions 20–31, we will use both methods to find the volume of revolution.

Note: The second method, known as the shell method, is described in the HL textbook. In most cases, the shell method is simpler, but it is not part of the IB syllabus (in this cycle).

20



a) $V = \pi\int_0^1 \left((3-x^2)^2 - (2x)^2\right)dx = \pi\int_0^1 \left(9-10x^2+x^4\right)dx = \pi\left(9x - \frac{10x^3}{3} + \frac{x^5}{5}\right)\Big]_0^1$

$= \pi\left(9 - \frac{10}{3} + \frac{1}{5}\right) = \frac{88\pi}{15}$

b) $y = 3 - x^2 \Rightarrow x = \sqrt{3-y},\ y = 2x \Rightarrow x = \frac{y}{2}$

We need to split the region into two, and change the boundaries of integration from 0 to 2 for the first integral, and from 2 to 3 for the second integral.

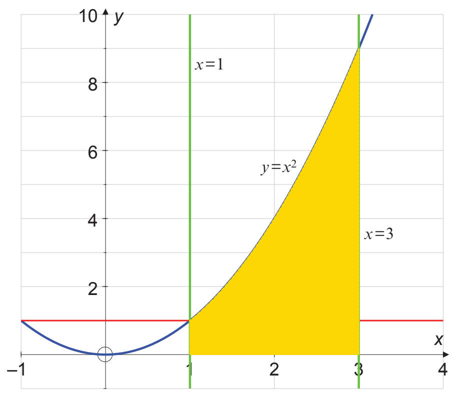
$V = \pi\int_0^2 \left(\frac{y^2}{4}\right)dy + \pi\int_2^3 \left(\sqrt{3-y}\right)^2 dy = \pi\left(\frac{y^3}{12}\right)\Big]_0^2 + \pi\left(3y - \frac{y^2}{2}\right)\Big]_2^3 = \pi\left(\frac{2}{3} + 9 - \frac{9}{2} - 6 + 2\right) = \frac{7\pi}{6}$

The shell method:

$V = 2\pi\int_0^1 \left(x\times(3-x^2-2x)\right)dx = 2\pi\int_0^1 \left(3x - x^3 - 2x^2\right)dx = 2\pi\left(\frac{3x^2}{2} - \frac{x^4}{4} - \frac{2x^3}{3}\right)\Big]_0^1$

$= 2\pi\left(\frac{3}{2} - \frac{1}{4} - \frac{2}{3}\right) = 2\pi\times\frac{18-3-8}{12} = \frac{7\pi}{6}$

21



We need to express x in terms of y: $y = x^2 \Rightarrow x = \sqrt{y}$

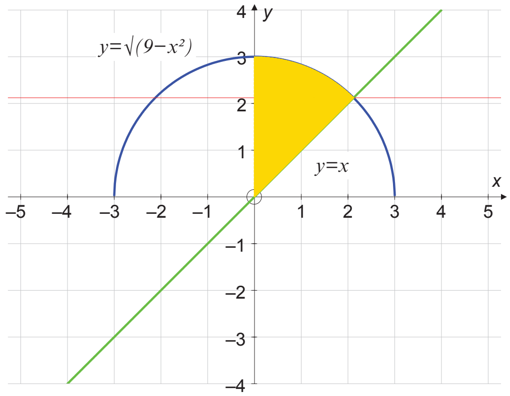
$V = \pi\int_0^1 \left(3^2 - 1^2\right)dy + \pi\int_1^9 \left(3^2 - \left(\sqrt{y}\right)^2\right)dy = \pi\left(\left(8y\right)\Big]_0^1 + \left(9y - \frac{y^2}{2}\right)\Big]_1^9\right)$

$= \pi\left(8 + 81 - \frac{81}{2} - 9 + \frac{1}{2}\right) = 40\pi$

The shell method:

$V = 2\pi\int_1^3 \left(x\left(x^2\right)\right)dx = 2\pi\int_1^3 x^3\,dx = 2\pi\left(\frac{x^4}{4}\right)\Big]_1^3 = 2\pi\left(\frac{81}{4} - \frac{1}{4}\right) = 2\pi\times 20 = 40\pi$

22



Notice that these two curves intersect at the point with equal x- and y-coordinate:

$\sqrt{9-x^2} = x \Rightarrow 9 - x^2 = x^2 \Rightarrow 9 = 2x^2 \Rightarrow x = \sqrt{\frac{9}{2}} = \frac{3\sqrt{2}}{2}$, and that x and y are symmetrical in the equation of the curve; therefore, $x = \sqrt{9-y^2}$.

We need to split the integration into two parts.

$$V = \pi\int_0^{\frac{3\sqrt{2}}{2}} y^2\, dy + \pi\int_{\frac{3\sqrt{2}}{2}}^{3} \left(\sqrt{9-y^2}\right)^2 dy = \pi\left(\left[\left(\frac{y^3}{3}\right)\right]_0^{\frac{3\sqrt{2}}{2}} + \left[\left(9y - \frac{y^3}{3}\right)\right]_{\frac{3\sqrt{2}}{2}}^{3}\right)$$

$$= \pi\left(\frac{9\sqrt{2}}{4} + 27 - 9 - \frac{27\sqrt{2}}{2} + \frac{9\sqrt{2}}{4}\right) = 9\pi\left(2-\sqrt{2}\right)$$

The shell method:

$$V = 2\pi\int_0^{\frac{3\sqrt{2}}{2}} \left(x\left(\sqrt{9-x^2} + x\right)\right) dx = 2\pi\left(\int_0^{\frac{3\sqrt{2}}{2}} \left(x\sqrt{9-x^2}\right) dx - \int_0^{\frac{3\sqrt{2}}{2}} x^2\, dx\right) = *$$

We need to use the substitution method for the first integral.

$$\int_0^{\frac{3\sqrt{2}}{2}} \left(x\sqrt{9-x^2}\right) dx = \begin{bmatrix} t = 9 - x^2 \\ dt = -2x\, dx \end{bmatrix} = \int_9^{\frac{9}{2}} \left(-\frac{1}{2}\sqrt{t}\right) dt = \frac{1}{2}\int_{\frac{9}{2}}^{9} \sqrt{t}\, dt = \frac{1}{2}\left[\left(\frac{2}{3}t^{\frac{3}{2}}\right)\right]_{\frac{9}{2}}^{9}$$

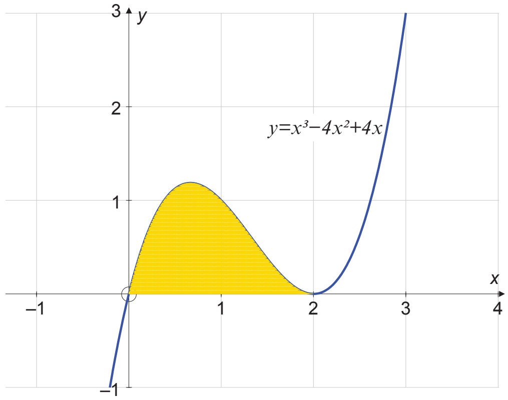
$$= \frac{1}{3}\left(9^{\frac{3}{2}} - \left(\frac{9}{2}\right)^{\frac{3}{2}}\right) = \frac{1}{3}\left(27 - \frac{27\sqrt{2}}{4}\right) = 9 - \frac{9\sqrt{2}}{4}$$

So, we can now proceed with the calculation of the integral.

$$* = 2\pi\left(9 - \frac{9\sqrt{2}}{4} - \left[\left(\frac{1}{3}x^3\right)\right]_0^{\frac{3\sqrt{2}}{2}}\right) = 2\pi\left(9 - \frac{9\sqrt{2}}{4} - \frac{1}{3} \times \frac{27 \times 2\sqrt{2}}{8}\right)$$

$$= 2\pi\left(9 - \frac{9\sqrt{2}}{4} - \frac{9\sqrt{2}}{4}\right) = 9\pi\left(2-\sqrt{2}\right)$$

23

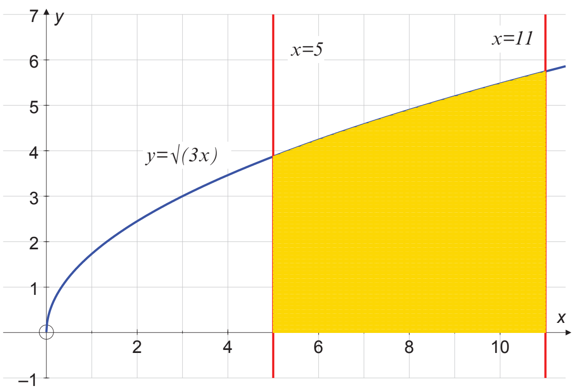


This question cannot be done by swapping the variables, so we will use the shell method.

$$V = 2\pi\int_0^2 \left(x\times\left(x^3-4x^2+4x\right)\right)dx = 2\pi\int_0^2 \left(x^4-4x^3+4x^2\right)dx = 2\pi\left[\left(\frac{x^5}{5}-x^4+\frac{4x^3}{3}\right)\right]_0^2$$

$$= 2\pi\left(\frac{32}{5}-16+\frac{32}{3}\right) = 2\pi\times\frac{96-240+160}{15} = \frac{32\pi}{15}$$

24



To determine the boundaries of integration, we need to find the y-coordinates of the points of intersection of the curve with the vertical lines. We also need to express x in terms of y.

$$x = 5 \Rightarrow y = \sqrt{15},\ x = 11 \Rightarrow y = \sqrt{33},\ y = \sqrt{3x} \Rightarrow x = \frac{y^2}{3}$$

$$V = \pi\int_0^{\sqrt{15}} \left(11^2-5^2\right)dy + \pi\int_{\sqrt{15}}^{\sqrt{33}}\left(11^2-\left(\frac{y^2}{3}\right)^2\right)dy = 96\pi\left(y\right)\Big]_0^{\sqrt{15}} + \pi\left[\left(121y-\frac{1}{9}\times\frac{y^5}{5}\right)\right]_{\sqrt{15}}^{\sqrt{33}}$$

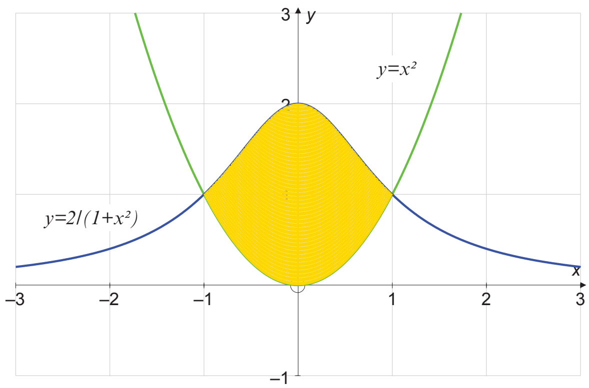
$$= 96\pi\sqrt{15}+121\pi\sqrt{33}-\frac{121\pi\sqrt{33}}{5}-121\pi\sqrt{15}+5\pi\sqrt{15} = \frac{484\pi\sqrt{33}}{5}-20\pi\sqrt{15}$$

The shell method:

$$V = 2\pi\int_5^{11}\left(x\sqrt{3x}\right)dx = 2\pi\sqrt{3}\int_5^{11} x^{\frac{3}{2}}\,dx = 2\pi\sqrt{3}\left[\left(\frac{2}{5}x^{\frac{5}{2}}\right)\right]_5^{11} = \frac{4\pi\sqrt{3}}{5}\left(121\sqrt{11}-25\sqrt{5}\right)$$

$$= \frac{484\pi\sqrt{33}}{5}-20\pi\sqrt{15}$$

25



Since this region is symmetrical about the y-axis, we can take one-half of the region and rotate it about the y-axis. We also need to express x in terms of y for both curves.

$$y = x^2 \Rightarrow x = \sqrt{y},\ y = \frac{2}{1+x^2} \Rightarrow 1 + x^2 = \frac{2}{y} \Rightarrow x = \sqrt{\frac{2}{y} - 1}$$

$$V = \pi\int_0^1 \left(\sqrt{y}\right)^2 dy + \pi\int_1^2 \left(\sqrt{\frac{2}{y} - 1}\right)^2 dy = \pi\int_0^1 y\,dy + \pi\int_1^2 \left(\frac{2}{y} - 1\right) dy$$

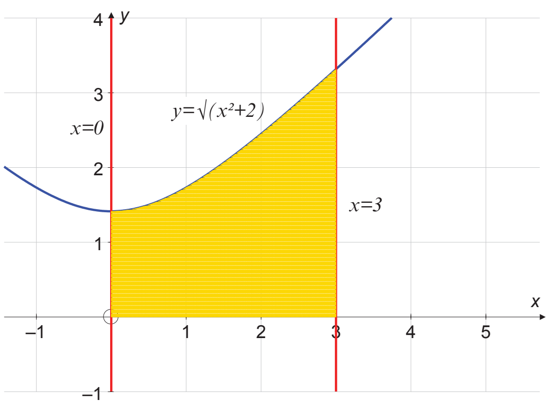
$$= \pi\left(\left(\frac{y^2}{2}\right)\Bigg]_0^1 + (2\ln y - y)]_1^2\right) = \pi\left(\frac{1}{2} + 2\ln 2 - 2 + 1\right) = 2\pi\ln 2 - \frac{\pi}{2}$$

The shell method:

$$V = 2\pi\int_0^1 \left(x \times \left(\frac{2}{1+x^2} - x^2\right)\right) dx = 2\pi\int_0^1 \left(\frac{2x}{1+x^2} - x^3\right) dx = 2\pi\left(\ln\left(1+x^2\right) - \frac{x^4}{4}\right)\Bigg]_0^1$$

$$= 2\pi\left(\ln 2 - \frac{1}{4}\right) = 2\pi\ln 2 - \frac{\pi}{2}$$

26



We need to find the y-coordinates of the points of intersection to determine the boundaries of integration. Also, we need to express x in terms of y.

$$x = 0 \Rightarrow y = \sqrt{2},\ x = 3 \Rightarrow y = \sqrt{11},\ y = \sqrt{x^2 + 2} \Rightarrow x = \sqrt{y^2 - 2}$$

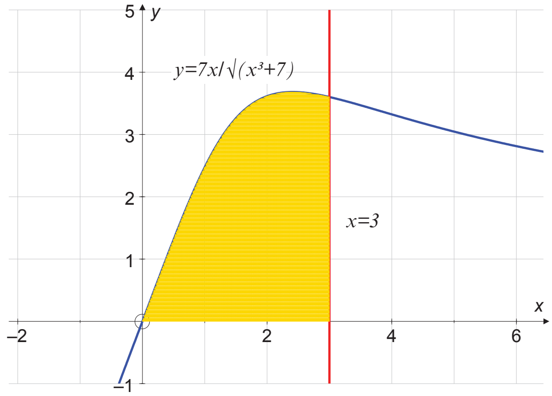
$$V = \pi\int_0^{\sqrt{2}} 3^2\,dy + \pi\int_{\sqrt{2}}^{\sqrt{11}} \left(3^2 - \left(\sqrt{y^2 - 2}\right)^2\right) dy = \pi\left((9y)]_0^{\sqrt{2}} + \left(9y - \frac{y^3}{3} + 2y\right)\Bigg]_{\sqrt{2}}^{\sqrt{11}}\right)$$

$$= \pi\left(\cancel{9\sqrt{2}} + 9\sqrt{11} - \frac{11\sqrt{11}}{3} + 2\sqrt{11} \cancel{-9\sqrt{2}} + \frac{2\sqrt{2}}{3} - 2\sqrt{2}\right) = \frac{2\pi}{3}\left(11\sqrt{11} - 2\sqrt{2}\right)$$

The shell method:

$$V = 2\pi\int_0^3 \left(x\sqrt{x^2+2}\right)dx = \begin{bmatrix} t = x^2 + 2 \\ dt = 2x\,dx \end{bmatrix} = 2\pi\int_2^{11} \sqrt{t} \times \frac{1}{2}\,dt = \pi\left(\frac{2}{3}t^{\frac{3}{2}}\right)\Bigg]_2^{11} = \frac{2\pi}{3}\left(11\sqrt{11} - 2\sqrt{2}\right)$$

27

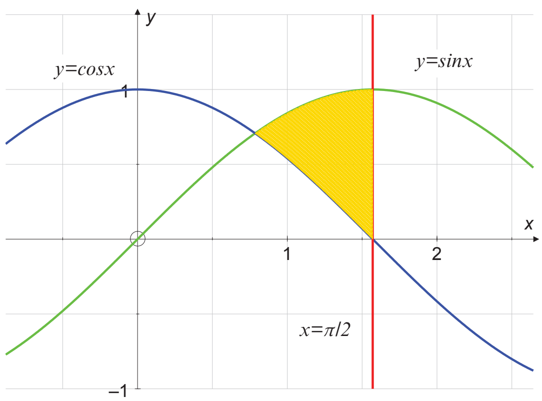


This question cannot be done by swapping the variables, so we will use the shell method.

$$V = 2\pi\int_0^3 \left(x \times \frac{7x}{\sqrt{x^3+7}}\right)dx = 2\pi\int_0^3 \left(\frac{7x^2}{\sqrt{x^3+7}}\right)dx = \begin{bmatrix} t = x^3 + 7 \\ dt = 3x^2 dx \end{bmatrix} = 14\pi\int_7^{34} \frac{\frac{1}{3}dt}{\sqrt{t}} = \frac{14\pi}{3}\left(2\sqrt{t}\right)\Big]_7^{34}$$

$$= \frac{28\pi}{3}\left(\sqrt{34} - \sqrt{7}\right)$$

28



These two curves meet at the point $\left(\frac{\pi}{4}, \frac{\sqrt{2}}{2}\right)$.

We need to express x in terms of y: $y = \sin x \Rightarrow x = \arcsin y$, $y = \cos x \Rightarrow x = \arccos y$

$$V = \pi\int_0^{\frac{\sqrt{2}}{2}} \left(\frac{\pi^2}{4} - (\arccos y)^2\right)dy + \pi\int_{\frac{\sqrt{2}}{2}}^{1} \left(\frac{\pi^2}{4} - (\arcsin y)^2\right)dy$$

In order to find the volume, we firstly need to find the following definite integrals.

$$\int \arcsin^2 x\,dx = \begin{bmatrix} u = \arcsin^2 x & du = \dfrac{2\arcsin x}{\sqrt{1-x^2}}dx \\ dv = dx & v = x \end{bmatrix} = x\arcsin^2 x - \int \frac{2x\arcsin x}{\sqrt{1-x^2}}dx$$

$$= \begin{bmatrix} u = \arcsin x & du = \dfrac{dx}{\sqrt{1-x^2}} \\ dv = \dfrac{2x}{\sqrt{1-x^2}}\,dx & v = -2\sqrt{1-x^2} \end{bmatrix} = x\arcsin^2 x + 2\arcsin x\sqrt{1-x^2} - \int 2\cancel{\sqrt{1-x^2}}\,\frac{dx}{\cancel{\sqrt{1-x^2}}}$$

$$= x\arcsin^2 x + 2\arcsin x\sqrt{1-x^2} - 2x + c,\ c\in\mathbb{R}$$

$$\int \arccos^2 x\,dx = \begin{bmatrix} u = \arccos^2 x & du = \dfrac{2\arccos x}{-\sqrt{1-x^2}}\,dx \\ dv = dx & v = x \end{bmatrix} = x\arccos^2 x + \int \frac{2x\arccos x}{\sqrt{1-x^2}}\,dx$$

$$= \begin{bmatrix} u = \arccos x & du = \dfrac{dx}{-\sqrt{1-x^2}} \\ dv = \dfrac{2x}{\sqrt{1-x^2}}\,dx & v = -2\sqrt{1-x^2} \end{bmatrix} = x\arccos^2 x - 2\arccos x\sqrt{1-x^2} - \int 2\cancel{\sqrt{1-x^2}}\,\frac{dx}{\cancel{\sqrt{1-x^2}}}$$

$$= x\arccos^2 x - 2\arccos x\sqrt{1-x^2} - 2x + c,\ c\in\mathbb{R}$$

So, we can now find the volume of the solid of revolution.

$$V = \pi\int_0^{\frac{\sqrt{2}}{2}}\left(\frac{\pi^2}{4} - (\arccos y)^2\right)dy + \pi\int_{\frac{\sqrt{2}}{2}}^{1}\left(\frac{\pi^2}{4} - (\arcsin y)^2\right)dy$$

$$= \pi\left(\frac{\pi^2}{4}y - y\arccos^2 y + 2\arccos y\sqrt{1-y^2} + 2y\right)\Bigg]_0^{\frac{\sqrt{2}}{2}}$$

$$+ \pi\left(\frac{\pi^2}{4}y - y\arcsin^2 y - 2\arcsin y\sqrt{1-y^2} + 2y\right)\Bigg]_{\frac{\sqrt{2}}{2}}^{1}$$

$$= \pi\left(\frac{\pi^2\sqrt{2}}{8} - \frac{\sqrt{2}}{2}\left(\frac{\pi^2}{16}\right) + 2\times\frac{\pi}{4}\times\frac{\sqrt{2}}{2} + \sqrt{2} - \pi\right) + \pi\left(\left(\frac{\pi^2}{4} - \frac{\pi^2}{4} + 2\right) - \left(\frac{\pi^2\sqrt{2}}{8} - \frac{\sqrt{2}}{2}\left(\frac{\pi^2}{16}\right) - 2\times\frac{\pi}{4}\times\frac{\sqrt{2}}{2} - \sqrt{2}\right)\right)$$

$$= \pi\left(\frac{\pi\sqrt{2}}{2} - \pi + 2\right)$$

The shell method:

$$V = 2\pi\int_{\frac{\pi}{4}}^{\frac{\pi}{2}}(x\sin x - x\cos x)\,dx = *$$

Before we continue, let's find the following integrals.

$$\int x\sin x\,dx = \begin{bmatrix} u = x & du = dx \\ dv = \sin x\,dx & v = -\cos x \end{bmatrix} = -x\cos x + \int \cos x\,dx = -x\cos x + \sin x + c,\ c\in\mathbb{R}$$

$$\int x\cos x\,dx = \begin{bmatrix} u = x & du = dx \\ dv = \cos x\,dx & v = \sin x \end{bmatrix} = x\sin x - \int \sin x\,dx = x\sin x + \cos x + c,\ c\in\mathbb{R}$$

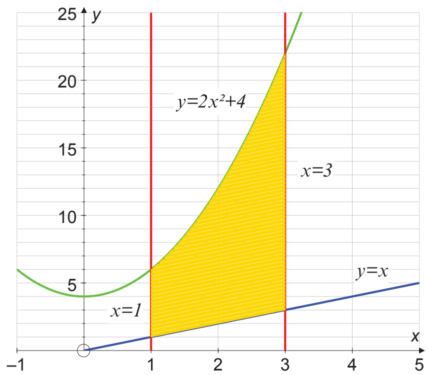
Now, we can continue with the shell method.

$$* = 2\pi(-x\cos x + \sin x - x\sin x - \cos x)\Big]_{\frac{\pi}{4}}^{\frac{\pi}{2}}$$

$$= 2\pi\left(-\frac{\pi}{2}\underbrace{\cos\frac{\pi}{2}}_{0} + \underbrace{\sin\frac{\pi}{2}}_{1} - \frac{\pi}{2}\underbrace{\sin\frac{\pi}{2}}_{1} - \underbrace{\cos\frac{\pi}{2}}_{0} + \frac{\pi}{4}\underbrace{\cos\frac{\pi}{4}}_{\frac{\sqrt{2}}{2}} - \underbrace{\sin\frac{\pi}{4}}_{\frac{\sqrt{2}}{2}} + \frac{\pi}{4}\underbrace{\sin\frac{\pi}{4}}_{\frac{\sqrt{2}}{2}} + \underbrace{\cos\frac{\pi}{4}}_{\frac{\sqrt{2}}{2}}\right)$$

$$= 2\pi\left(1 - \frac{\pi}{2} + \frac{\pi\sqrt{2}}{4}\right) = \pi\left(2 - \pi + \frac{\pi\sqrt{2}}{2}\right)$$

29



$$V = \pi\left(\int_1^3 \left(y^2 - 1\right) dy + \int_3^6 (9-1)\, dy + \int_6^{22}\left(9 - \left(\sqrt{\frac{y}{2} - 2}\right)^2\right)\right) dy$$

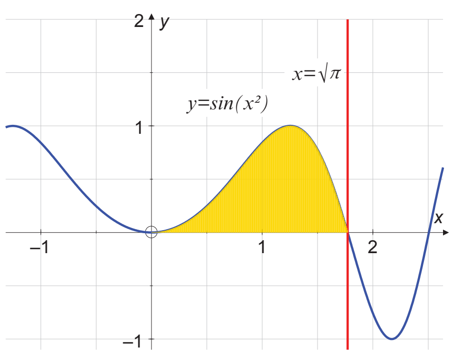
$$= \pi\left(\left(\frac{y^3}{3} - y\right)\Bigg]_1^3 + (8y)\Big]_3^6 + \left(11y - \frac{y^2}{4}\right)\Bigg]_6^{22}\right) = \pi\left(9 - 3 - \frac{1}{3} + 1 + 48 - 24 + 242 - 121 - 66 + 9\right) = \frac{284\pi}{3}$$

The shell method:

$$V = 2\pi\int_1^3 \left(x\left(2x^2 + 4 - x\right)\right) dx = 2\pi\int_1^3 \left(2x^3 + 4x - x^2\right) dx = 2\pi\left(\frac{x^4}{2} + 2x^2 - \frac{x^3}{3}\right)\Bigg]_1^3$$

$$= 2\pi\left(\frac{81}{2} + 18 - 9 - \frac{1}{2} - 2 + \frac{1}{3}\right) = 2\pi\left(47 + \frac{1}{3}\right) = 2\pi \times \frac{142}{3} = \frac{284\pi}{3}$$

30

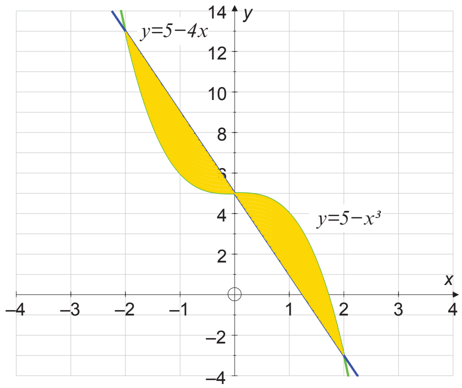


This question cannot be done by swapping the variables, so we will use the shell method.

$$V = 2\pi\int_0^{\sqrt{\pi}} \left(x \sin\left(x^2\right)\right) dx = \left[\begin{array}{c} t = x^2 \\ dt = 2x\, dx \end{array}\right] = 2\pi\int_0^{\pi} \sin t \times \frac{1}{2}\, dt = \pi(-\cos t)\Big]_0^{\pi}$$

$$= \pi\left(-\underbrace{\cos\pi}_{-1} + \underbrace{\cos 0}_{1}\right) = 2\pi$$

31



Firstly, we need to express x in terms of y.

Since the region is symmetrical with respect to the point (0, 5), it is sufficient to calculate the volume of one part of the region and then multiply it by 2.

$$V = 2\pi\int_{-3}^{5}\left(\left(\sqrt[3]{5-y}\right)^2 - \left(\frac{5-y}{4}\right)^2\right)dy \text{ or } V = 2\pi\int_{5}^{13}\left(\left(\sqrt[3]{5-y}\right)^2 - \left(\frac{5-y}{4}\right)^2\right)dy$$

```
fnInt((³√(5-Y))²
-((5-Y)/4)²,Y,-3
,5)
         8.533333701
8.533333333333333
3*2▸Frac
              256/15
```

So, the final volume is: $V = \dfrac{256\pi}{15}$.

The shell method:

$$V = 4\pi\int_{0}^{2}\left(x\left(5-x^3-5+4x\right)\right)dx = 4\pi\int_{0}^{2}\left(4x^2-x^4\right)dx$$

or

$$V = 4\pi\int_{-2}^{0}\left(x\left(5-x^3-5+4x\right)\right)dx = 4\pi\int_{-2}^{0}\left(4x^2-x^4\right)dx$$

```
4fnInt(X(4X-X³),
X,0,2)
         17.06666667
Ans▸Frac
              256/15
```

So, the final volume is: $V = \dfrac{256\pi}{15}$.

Exercise 16.7

Solution Paper 1 type

1 To find the displacement, we are going to evaluate the following integral:

$$\text{Displacement} = \int_0^{10}\left(t^2 - 11t + 24\right)dt = \left[\frac{t^3}{3} - 11\times\frac{t^2}{2} + 24t\right]_0^{10}$$

$$= \left(\frac{1000}{3} - 11\times\frac{\cancel{100}\ 50}{\cancel{2}} + 240\right) - 0 = \frac{1000}{3} - 550 + 240 = \frac{70}{3}\ \text{m}$$

To find the total distance travelled, we need to find the zeros of the parabola and identify where the particle changes direction.

$$v(t) = 0 \Rightarrow t^2 - 11t + 24 = 0 \Rightarrow t = \frac{11 \pm \sqrt{121 - 96}}{2} = \frac{11 \pm 5}{2} \Rightarrow t = 3 \text{ or } t = 8$$

We notice that the particle changes direction twice, so we need to split the integral into three different integrals.

$$\text{Total distance} = \int_0^{3}\left(t^2 - 11t + 24\right)dt + \left|\int_3^{8}\left(t^2 - 11t + 24\right)dt\right| + \int_8^{10}\left(t^2 - 11t + 24\right)dt$$

$$= \left[\frac{t^3}{3} - 11\times\frac{t^2}{2} + 24t\right]_0^{3} + \left[-\frac{t^3}{3} + 11\times\frac{t^2}{2} - 24t\right]_3^{8} + \left[\frac{t^3}{3} - 11\times\frac{t^2}{2} + 24t\right]_8^{10}$$

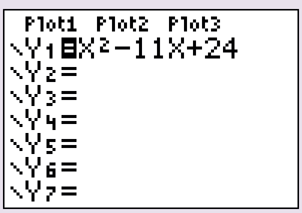
$$= \left(\frac{\cancel{27}\ 9}{\cancel{3}} - 11\times\frac{9}{2} + 72\right) - 0 + \left(-\frac{512}{3} + 11\times\frac{\cancel{64}\ 32}{\cancel{2}} - 192\right) - \left(-9 + \frac{99}{2} - 72\right)$$

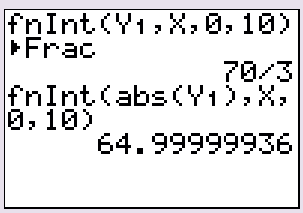
$$+ \left(\frac{1000}{3} - 11\times\frac{\cancel{100}\ 50}{\cancel{2}} + 240\right) - \left(\frac{512}{3} - 352 + 192\right) = \cancel{2}\times\frac{63}{\cancel{2}} - 2\times\frac{32}{3} + \frac{70}{3} = 65\ \text{m}$$

The absolute value was taken only of the middle integral since the parabola is below the x-axis (negative) for the interval between the zeros]3, 8[. Because of the opposite anti-derivatives, we noticed that some anti-derivative values appeared twice in the calculation and so we could simplify the sum.

Solution Paper 2 type

1 It is much easier to solve this question by using a GDC.





The only issue here is the accuracy of a GDC. We need to interpret the final answer as 65 m.

Solution Paper 1 type

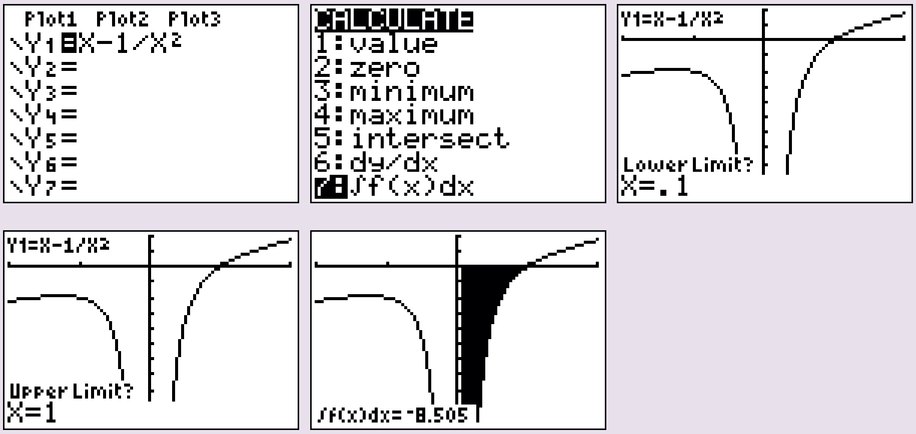
2 To find the displacement, we are going to evaluate the following integral:

$$\int_{0.1}^{1}\left(t - \frac{1}{t^2}\right)dt = \left[\frac{t^2}{2} + \frac{1}{t}\right]_{0.1}^{1} = \left(\frac{1}{2} + 1\right) - \left(\frac{0.01}{2} + \frac{1}{0.1}\right) = 1.5 - 10.005 = -8.505$$

We can see that the particle moves to the left and that the displacement is 8.505 m. However, since there are no zeros within the given interval, the total distance travelled is the same, i.e. 8.505 m.

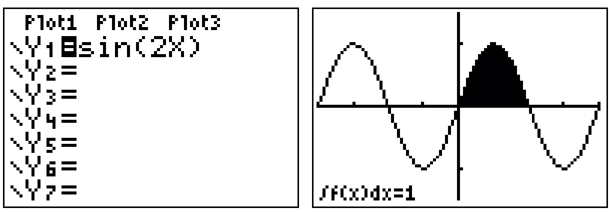
Solution Paper 2 type

2 Again, it is much easier to solve this question by using a GDC. This time we will use the graphing Calculate menu so that we can see the graph and the area, which we can use to confirm the direction of the particle and the conclusion regarding the total distance travelled.



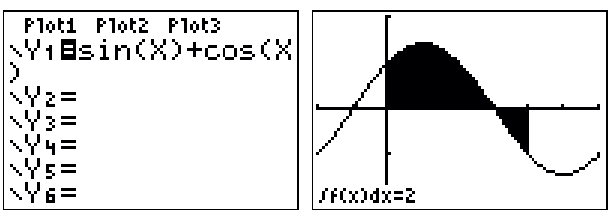
So again, since the whole graph is below the *x*-axis within the given interval, our definite integral is negative and we use the absolute value, i.e. 8.505 m. Since there are no changes in direction, the total distance travelled is the same.

3

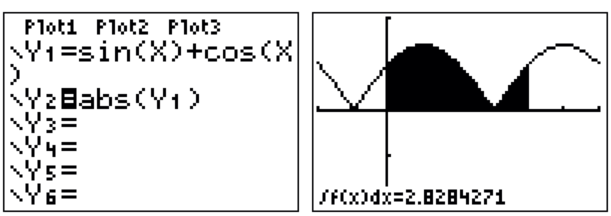


We notice that the whole graph is above the *x*-axis within the given interval, and therefore the displacement and the total distance travelled are the same, i.e. $\int_0^{\frac{\pi}{2}} \sin(2t)\,dt = 1$ m.

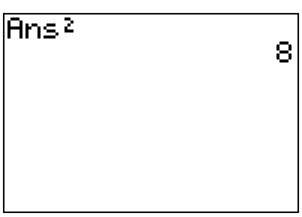
4



So, the displacement is 2 m.

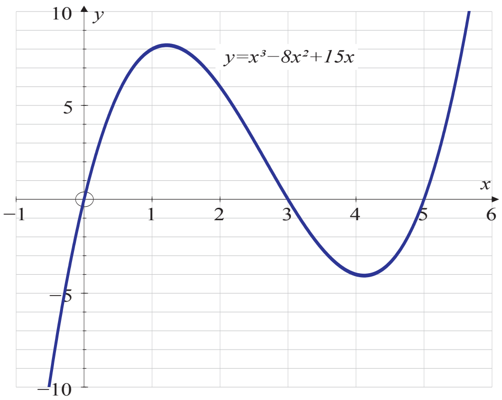


The total distance travelled is 2.83 m, correct to three significant figures. To find the exact value, which we suspect is $2\sqrt{2}$, we need to square the answer.



Since the square is 8, we can conclude that the answer is $\sqrt{8} = 2\sqrt{2}$ m.

5 By looking at the curve $v(t) = t^3 - 8t^2 + 15t = t(t-3)(t-5)$, we notice that within the interval]0, 6[there are two zeros, 3 and 5; therefore, the particle changes direction twice. To find the displacement, we are going to evaluate the integral within that interval; but to find the total distance travelled, we are going to split the integral into three parts and take the absolute value of the middle part, since the curve is below the x-axis in the interval]3, 5[.



$$\text{Displacement} = \int_0^6 \left(t^3 - 8t^2 + 15t\right) dt = \left[\frac{t^4}{4} - 8\times\frac{t^3}{3} + 15\times\frac{t^2}{2}\right]_0^6 = (324 - 576 + 270) - 0 = 18 \text{ m}$$

$$\text{Total distance} = \int_0^3 \left(t^3 - 8t^2 + 15t\right) dt + \left|\int_3^5 \left(t^3 - 8t^2 + 15t\right) dt\right| + \int_5^6 \left(t^3 - 8t^2 + 15t\right) dt$$

$$= \left[\frac{t^4}{4} - 8\times\frac{t^3}{3} + 15\times\frac{t^2}{2}\right]_0^3 + \left[-\frac{t^4}{4} + 8\times\frac{t^3}{3} - 15\times\frac{t^2}{2}\right]_3^5 + \left[\frac{t^4}{4} - 8\times\frac{t^3}{3} + 15\times\frac{t^2}{2}\right]_5^6$$

$$= \left(\frac{81}{4} - 8\times\frac{\cancel{27}\,9}{\cancel{3}} + 15\times\frac{9}{2}\right) - 0 + \left(-\frac{625}{4} + 8\times\frac{125}{3} - 15\times\frac{25}{2}\right) - \left(-\frac{81}{4} + 72 - \frac{225}{2}\right)$$

$$+ \left(\frac{\cancel{1296}\,324}{\cancel{4}} - 8\times\frac{\cancel{216}\,72}{\cancel{3}} + 15\times\frac{\cancel{36}\,18}{\cancel{2}}\right) - \left(\frac{625}{4} - \frac{1000}{3} + \frac{375}{2}\right)$$

$$= \cancel{2}\times\frac{63}{\cancel{4}\,2} - \cancel{2}\times\frac{125}{\cancel{12}\,6} + 18 = \frac{86}{3} \approx 28.7 \text{ m, correct to three significant figures.}$$

Solution Paper 1 type

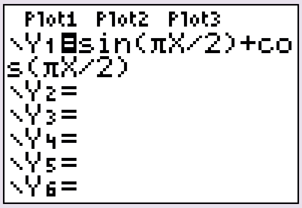
6 We notice that the function is always positive on the given interval and therefore the displacement and the total distance travelled are going to be the same.

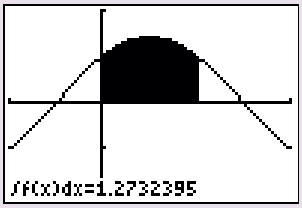
$$\int_0^1 \left(\sin\left(\frac{\pi t}{2}\right) + \cos\left(\frac{\pi t}{2}\right)\right) dt = \left[\frac{2}{\pi}\left(-\cos\left(\frac{\pi t}{2}\right)\right) + \frac{2}{\pi}\sin\left(\frac{\pi t}{2}\right)\right]_0^1$$

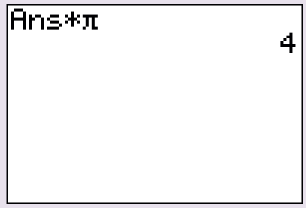
$$= \frac{2}{\pi}\left(-\cos\frac{\pi}{2} + \sin\frac{\pi}{2} + \cos 0 - \sin 0\right)$$

$$= \frac{2}{\pi}(0 + 1 + 1 - 0) = \frac{4}{\pi} \text{ m}$$

Solution Paper 2 type

6 





So, both the displacement and the total distance travelled are 1.27 m, correct to three significant figures. Again, to show that the exact form is $\frac{4}{\pi}$, we can multiply the answer by π.

7 $v(t) = \int a(t)\,dt \Rightarrow v(t) = \int 3\,dt = 3t + c, c \in \mathbb{R}$. To find the value of the constant, we are going to use the initial velocity: $v(0) = 0 \Rightarrow 3 \times 0 + c = 0 \Rightarrow c = 0$. So, the velocity function is: $v(t) = 3t$. Now we proceed to find the displacement and the total distance travelled. Since there is no change in direction, both quantities are the same and we calculate them using the following integral:

$$\int_0^2 3t\,dt = \left[3\frac{t^2}{2}\right]_0^2 = \left(3 \times \frac{2^{\cancel{2}}}{\cancel{2}}\right) - 0 = 6\text{ m}$$

8 $v(t) = \int a(t)\,dt \Rightarrow v(t) = \int (2t-4)\,dt = \cancel{2}\frac{t^2}{\cancel{2}} - 4t + c = t^2 - 4t + c, c \in \mathbb{R}$. To find the value of the constant, we are going to use the initial velocity: $v(0) = 3 \Rightarrow 0^2 - 4 \times 0 + c = 3 \Rightarrow c = 3$. So, the velocity function is: $v(t) = t^2 - 4t + 3$. Now we proceed to find the displacement and the total distance travelled.

$$\text{Displacement} = \int_0^3 \left(t^2 - 4t + 3\right) dt = \left[\frac{t^3}{3} - \cancel{4}2 \times \frac{t^2}{\cancel{2}} + 3t\right]_0^3 = (9 - 18 + 9) - 0 = 0$$

We notice that the zeros of the velocity parabola are 1 and 3; therefore, there is a change in direction at 1 and we need to split the integral into two to calculate the total distance travelled.

$$\text{Total distance travelled} = \int_0^1 \left(t^2 - 4t + 3\right) dt + \int_1^3 \left(-t^2 + 4t - 3\right) dt$$

$$= \left[\frac{t^3}{3} - \cancel{4}2 \times \frac{t^2}{\cancel{2}} + 3t\right]_0^1 + \left[-\frac{t^3}{3} + \cancel{4}2 \times \frac{t^2}{\cancel{2}} - 3t\right]_1^3$$

$$= \left(\frac{1}{3} - 2 \times 1 + 3 \times 1\right) - 0 + (-9 + 18 - 9) - \left(-\frac{1}{3} + 2 \times 1 - 3 \times 1\right) = 2 \times \frac{4}{3} = \frac{8}{3} \approx 2.67\text{ m}$$

9 $v(t) = \int a(t)\,dt \Rightarrow v(t) = \int \sin t\,dt = -\cos t + c$. To find the value of the constant, we are going to use the initial velocity: $v(0) = 0 \Rightarrow -\cos(0) + c = 0 \Rightarrow -1 + c = 0 \Rightarrow c = 1$. So, the velocity function is: $v(t) = 1 - \cos t$. Now we proceed to find the displacement and the total distance travelled. The cosine value is always between -1 and 1 so we see that the velocity function is never negative. There is no change in direction and therefore the displacement and the total distance travelled are the same.

$$\int_0^{\frac{3\pi}{2}} (1 - \cos t)\,dt = [t - \sin t]_0^{\frac{3\pi}{2}} = \left(\frac{3\pi}{2} - \sin\left(\frac{3\pi}{2}\right)\right) - 0 = \frac{3\pi}{2} + 1 \approx 5.71\text{ m}$$

10 $v(t) = \int a(t)\,dt \Rightarrow v(t) = \int \frac{-1}{\sqrt{t+1}}\,dt = -2\sqrt{t+1} + c, c \in \mathbb{R}$. To find the value of the constant, we are going to use the initial velocity: $v(0) = 2 \Rightarrow -2\sqrt{0+1} + c = 2 \Rightarrow -2 + c = 2 \Rightarrow c = 4$. So, the velocity function is: $v(t) = -2\sqrt{t+1} + 4$. Now we proceed to find the displacement and the total distance travelled.

$$\text{Displacement} = \int_0^4 \left(4 - 2\sqrt{t+1}\right) dt = \left[4t - 2 \times \frac{(t+1)^{\frac{3}{2}}}{\frac{3}{2}}\right]_0^4 = \left[4t - \frac{4}{3}(t+1)^{\frac{3}{2}}\right]_0^4$$

$$= \left(16 - \frac{4}{3}(4+1)^{\frac{3}{2}}\right) - \left(0 - \frac{4}{3}\right) = \frac{52 - 20\sqrt{5}}{3} \approx 2.43 \text{ m}$$

By inspection, we can see that the zero is 3, which lies in the interval]0, 4[, so we need to split the integral into two when calculating the total distance travelled.

$$\text{Total distance travelled} = \int_0^3 \left(4 - 2\sqrt{t+1}\right) dt + \int_3^4 \left(2\sqrt{t+1} - 4\right) dt$$

$$= \left[4t - \frac{4}{3}(t+1)^{\frac{3}{2}}\right]_0^3 + \left[\frac{4}{3}(t+1)^{\frac{3}{2}} - 4t\right]_3^4$$

$$= \left(12 - \frac{4}{3}(3+1)^{\frac{3}{2}}\right) - \left(0 - \frac{4}{3}\right) + \left(\frac{4}{3}(4+1)^{\frac{3}{2}} - 16\right) - \left(\frac{4}{3}(3+1)^{\frac{3}{2}} - 12\right)$$

$$= 2\left(12 - \frac{4}{3} \times 8\right) + \frac{4}{3} + \frac{20\sqrt{5} - 48}{3} = \frac{20\sqrt{5} - 36}{3} \approx 2.91 \text{ m}$$

11 $v(t) = \int a(t)\, dt \Rightarrow v(t) = \int \left(6t - \frac{1}{(t+1)^3}\right) dt = \cancel{6}\,3\frac{t^2}{\cancel{2}} - \frac{(t+1)^{-2}}{-2} + c = 3t^2 + \frac{1}{2(t+1)^2} + c, c \in \mathbb{R}$. To find the value of the constant, we are going to use the initial velocity: $v(0) = 2 \Rightarrow 0 + \frac{1}{2} + c = 2 \Rightarrow c = \frac{3}{2}$. So, the velocity function is: $v(t) = 3t^2 + \frac{1}{2(t+1)^2} + \frac{3}{2}$. Now we proceed to find the displacement and the total distance travelled. The velocity function consists of a sum of positive terms and, as such, the velocity function is never negative. There is no change in direction and therefore the displacement and the total distance travelled are the same.

$$\text{Displacement} = \int_0^2 \left(3t^2 + \frac{1}{2(t+1)^2} + \frac{3}{2}\right) dt = \left[t^3 + \frac{(t+1)^{-1}}{2 \times (-1)} + \frac{3t}{2}\right]_0^2 = \left[t^3 - \frac{1}{2(t+1)} + \frac{3t}{2}\right]_0^2$$

$$= \left(8 - \frac{1}{6} + 3\right) - \left(0 - \frac{1}{2}\right) = \frac{68}{6} = \frac{34}{3} \approx 11.3 \text{ m}$$

And, total distance travelled = 11.3 m.

12 $v = 9.8t + 5 \Rightarrow s(t) = \int (9.8t + 5)\, dt = 9.8 \times \frac{t^2}{2} + 5t + c = 4.9t^2 + 5t + c, c \in \mathbb{R}$

$s(0) = 10 \Rightarrow 10 = c$

So, the position of the object at time t is given by: $s(t) = 4.9t^2 + 5t + 10$.

13 $v = 32t - 2 \Rightarrow s(t) = \int (32t - 2)\, dt = 32 \times \frac{t^2}{2} - 2t + c = 16t^2 - 2t + c, c \in \mathbb{R}$

$s(0.5) = 4 \Rightarrow 4 = 16 \times 0.25 - 2 \times 0.5 + c \Rightarrow c = 1$

So, the position of the object at time t is given by: $s(t) = 16t^2 - 2t + 1$.

14 $v = \sin(\pi t) \Rightarrow s(t) = \int \sin(\pi t)\, dt = -\frac{1}{\pi}\cos(\pi t) + c, c \in \mathbb{R}$

$s(0) = 0 \Rightarrow 0 = -\frac{1}{\pi}\underbrace{\cos 0}_{1} + c \Rightarrow c = \frac{1}{\pi}$

So, the position of the object at time t is given by: $s(t) = -\frac{1}{\pi}\cos(\pi t) + \frac{1}{\pi}$.

15 $v = \dfrac{1}{t+2}, t > -2, s(t) = \int \dfrac{dt}{t+2} = \ln(t+2) + c, t > -2$

We were able to omit the absolute value brackets due to the restriction on the domain, $t > -2$.

$s(-1) = \dfrac{1}{2} \Rightarrow \dfrac{1}{2} = \ln(1) + c \Rightarrow c = \dfrac{1}{2}$

So, the position of the object at time t is given by: $s(t) = \ln(t+2) + \dfrac{1}{2}$.

16 $a = e^t \Rightarrow v(t) = \int e^t\, dt = e^t + c, c \in \mathbb{R}; v(0) = 20 \Rightarrow 20 = \underbrace{e^0}_{1} + c \Rightarrow c = 19$

$v(t) = e^t + 19 \Rightarrow s(t) = \int (e^t + 19)\, dt = e^t + 19t + c, c \in \mathbb{R}; s(0) = 5 \Rightarrow 5 = \underbrace{e^0}_{1} + 0 + c \Rightarrow c = 4$

So, the position of the object at time t is given by: $s(t) = e^t + 19t + 4$.

17 $a = 9.8 \Rightarrow v(t) = \int 9.8\, dt = 9.8t + c, c \in \mathbb{R}; v(0) = -3 \Rightarrow -3 = 0 + c \Rightarrow c = -3$

$v(t) = 9.8t - 3 \Rightarrow s(t) = \int (9.8t - 3)\, dt = 9.8 \times \dfrac{t^2}{2} - 3t + c, c \in \mathbb{R}; s(0) = 0 \Rightarrow c = 0$

So, the position of the object at time t is given by: $s(t) = 4.9t^2 - 3t$.

18 $a = -4\sin 2t \Rightarrow v(t) = \int -4\sin 2t\, dt = -4 \times \left(-\dfrac{1}{2}\right)\cos 2t + c, c \in \mathbb{R}; v(0) = 2 \Rightarrow 2 = 2\underbrace{\cos 0}_{1} + c \Rightarrow c = 0$

$v(t) = 2\cos 2t \Rightarrow s(t) = \int 2\cos 2t\, dt = 2 \times \dfrac{1}{2}\sin 2t + c, c \in \mathbb{R}; s(0) = -3 \Rightarrow -3 = \underbrace{\sin 0}_{0} + c \Rightarrow c = -3$

So, the position of the object at time t is given by: $s(t) = \sin(2t) - 3$.

19 $a = \dfrac{9}{\pi^2}\cos\left(\dfrac{3t}{\pi}\right) \Rightarrow v(t) = \int \dfrac{9}{\pi^2}\cos\left(\dfrac{3t}{\pi}\right) dt = \dfrac{9}{\pi^2} \times \dfrac{\pi}{3}\sin\left(\dfrac{3t}{\pi}\right) + c = \dfrac{3}{\pi}\sin\left(\dfrac{3t}{\pi}\right) + c, c \in \mathbb{R}$

$v(0) = 0 \Rightarrow 0 = \dfrac{3}{\pi}\underbrace{\sin 0}_{0} + c \Rightarrow c = 0$

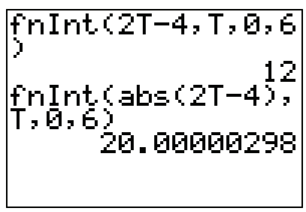
$v(t) = \dfrac{3}{\pi}\sin\left(\dfrac{3t}{\pi}\right) \Rightarrow s(t) = \int \dfrac{3}{\pi}\sin\left(\dfrac{3t}{\pi}\right) dt = \dfrac{3}{\pi} \times \left(-\dfrac{\pi}{3}\right)\cos\left(\dfrac{3t}{\pi}\right) + c = -\cos\left(\dfrac{3t}{\pi}\right) + c, c \in \mathbb{R}$

$s(0) = -1 \Rightarrow -1 = -\underbrace{\cos 0}_{1} + c \Rightarrow c = 0$

So, the position of the object at time t is given by: $s(t) = -\cos\left(\dfrac{3t}{\pi}\right)$.

In questions 20–23, we are going to denote the displacement of a point by s and the total distance travelled by d.

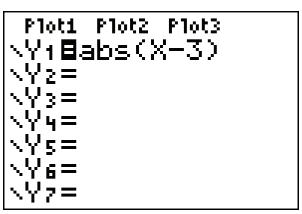
20 $v(t) = 2t - 4 \Rightarrow s = \int_0^6 (2t-4)\, dt, d = \int_0^6 |2t-4|\, dt$

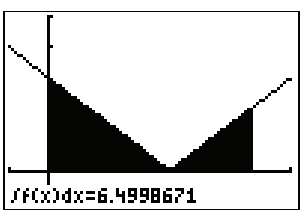


The displacement is 12 m and the total distance travelled is 20 m.

Note: We need to use an absolute value due to the fact that the object changes direction. The calculator's algorithm carries a small error and our final answer reflects this.

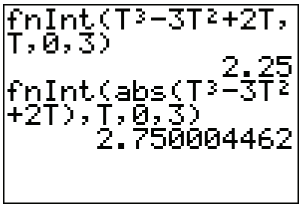
21 $v(t) = |t-3| \Rightarrow s = d = \int_0^5 |t-3|\, dt$





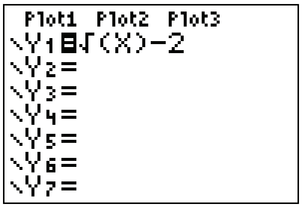
Again, the approximated value given by the GDC is not very accurate due to the error in the algorithm. Our answer should be 6.5 m. In this case, both the displacement and the total distance travelled are 6.5 m since the function is always positive.

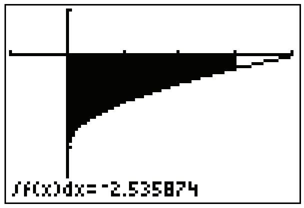
22 $v(t) = t^3 - 3t^2 + 2t \Rightarrow s = \int_0^3 \left(t^3 - 3t^2 + 2t\right) dt,\ d = \int_0^3 \left|t^3 - 3t^2 + 2t\right| dt$



So, the displacement is 2.25 m and the total distance travelled is 2.75 m.

23 $v(t) = \sqrt{t} - 2 \Rightarrow d = \int_0^3 \left(\sqrt{t} - 2\right) dt,\ s = \int_0^3 \left|\sqrt{t} - 2\right| dt$





Since the function is always negative on the given interval, the displacement is −2.54 m, whilst the total distance travelled is 2.54 m.

Note: The displacement can be negative, whilst the distance is always positive.

24 $a(t) = t - 2 \Rightarrow v(t) = \int (t-2)\,dt = \frac{t^2}{2} - 2t + c,\ c \in \mathbb{R};\ v(0) = 0 \Rightarrow 0 = c \Rightarrow v(t) = \frac{t^2}{2} - 2t$

$$d = \int_1^5 \left(\frac{t^2}{2} - 2t\right) dt = \left(\frac{t^3}{6} - t^2\right)\Bigg]_1^5 = \left(\frac{125}{6} - 25\right) - \left(\frac{1}{6} - 1\right) = -\frac{20}{6} = -\frac{10}{3} \approx -3.33 \text{ m}$$

Since the velocity function is quadratic and changes from negative to positive values in the given interval, we need to split the integral into two to calculate the distance travelled.

$$\frac{t^2}{2} - 2t = 0 \Big/ \times 2 \Rightarrow t^2 - 4t = 0 \Rightarrow t(t-4) = 0 \Rightarrow t_1 = 0,\ t_2 = 4$$

$$s = \int_1^4 \left(2t - \frac{t^2}{2}\right) dt + \int_4^5 \left(\frac{t^2}{2} - 2t\right) dt = \left(t^2 - \frac{t^3}{6}\right)\Bigg]_1^4 + \left(\frac{t^3}{6} - t^2\right)\Bigg]_4^5$$

$$= \left(16 - \frac{32}{3} - 1 + \frac{1}{6}\right) + \left(\frac{125}{6} - 25 - \frac{32}{3} + 16\right) = \frac{34}{6} = \frac{17}{3} \approx 5.67 \text{ m}$$

25 $a(t) = \frac{1}{\sqrt{5t+1}} \Rightarrow v(t) = \int \frac{dt}{\sqrt{5t+1}} \Rightarrow v(t) = \frac{2}{5}\sqrt{5t+1} + c,\ v(0) = 2 \Rightarrow 2 = \frac{2}{5} + c \Rightarrow c = \frac{8}{5}$

Since the function $v(t) = \frac{2}{5}\sqrt{5t+1} + \frac{8}{5}$ is always positive, the displacement and the distance travelled are the same.

$$d = s = \int_0^3 \left(\frac{2}{5}\sqrt{5t+1} + \frac{8}{5}\right) dt = \left(\frac{2}{5} \times \frac{2}{15}(5t+1)^{\frac{3}{2}} + \frac{8}{5}t\right)\Bigg]_0^3 = \frac{4}{75} \times (64-1) + \frac{8}{5}(3-0) = \frac{84}{25} + \frac{24}{5}$$

$$= \frac{84+120}{25} = \frac{204}{25} = 8.16 \text{ m}$$

26 $a(t) = -2 \Rightarrow v(t) = -2t + c, v(0) = 3 \Rightarrow c = 3 \Rightarrow v(t) = -2t + 3$

$$d = \int_1^4 (-2t+3)\,dt = \left(-t^2 + 3t\right)\Big]_1^4 = -16 + 12 + 1 - 3 = -6 \text{ m}$$

This is a linear function that has a zero, $\frac{3}{2}$, in the given interval, so, to calculate the total distance travelled, we need to split the integral into two using only positive values.

$$s = \int_1^{\frac{3}{2}} (-2t+3)\,dt + \int_{\frac{3}{2}}^4 (2t-3)\,dt = \left(-t^2+3t\right)\Big]_1^{\frac{3}{2}} + \left(t^2-3t\right)\Big]_{\frac{3}{2}}^4 = -\frac{9}{4} + \frac{9}{2} + 1 - 3 + 16 - 12 - \frac{9}{4} + \frac{9}{2}$$

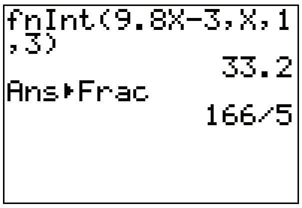
$$= \frac{13}{2} = 6.5 \text{ m}$$

27 $v(t) = 9.8t - 3$

a) $s = \int_1^3 (9.8t - 3)\,dt$

b) $s = \int_1^3 (9.8t - 3)\,dt$

c) $s = \int_1^3 (9.8t - 3)\,dt$



Note: The displacement does not depend on the initial conditions since the displacement is the integral of the velocity function.

28 $s(t) = 50t - 10t^2 + 1000$

a) $v(t) = s'(t) = 50 - 20t$

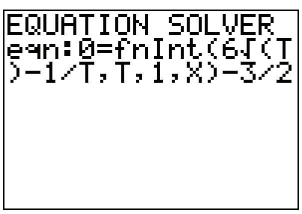
b) The maximum displacement takes place when the object stops and starts returning towards point O.

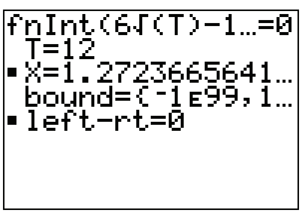
$$v(t) = 0 \Rightarrow 50 - 20t = 0 \Rightarrow t = \frac{5}{2},\ s\left(\frac{5}{2}\right) = 50 \times \frac{5}{2} - 10 \times \left(\frac{5}{2}\right)^2 + 1000 = \frac{2125}{2} = 1062.5 \text{ m}$$

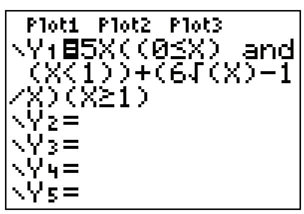
29 $v(t) = \begin{cases} 5t, & 0 \leqslant t < 1 \\ 6\sqrt{t} - \frac{1}{t}, & t \geqslant 1 \end{cases}$

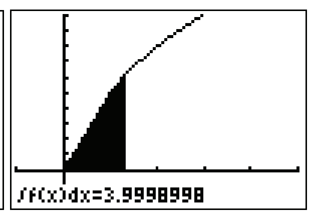
For the distance to be 4 cm, we actually need to look at the integral of the velocity function. The first part of the integral between 0 and 1 is: $\int_0^1 5t\,dt = \left(5\frac{t^2}{2}\right)\Big]_0^1 = \frac{5}{2}$; therefore, we need to find the time after 1 second in which the particle covers another 1.5 cm.

$$\int_1^x \left(6\sqrt{t} - \frac{1}{t}\right)dt = \frac{3}{2}$$









So, the time at which the particle is 4 cm from the starting point is 1.27 seconds.

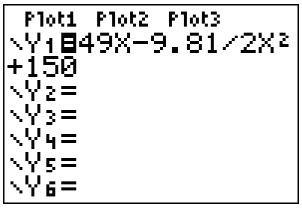
30 The velocity of a projectile fired upwards will be influenced by gravity and hence the deceleration will slow the projectile down. The deceleration we are going to use is $9.81\ \text{m/s}^2$.

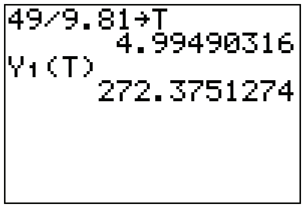
a) $v(t) = 49 - 9.81t \Rightarrow v(t) = 0 \Rightarrow 49 - 9.81t = 0 \Rightarrow t = \dfrac{49}{9.81} \approx 4.995$; so, we can say 5 seconds.

b) To find the maximum height, we need to find the height formula.

$$v(t) = 49 - 9.81t \Rightarrow h(t) = \int (49 - 9.81t)\,dt = 49t - 9.81\frac{t^2}{2} + c$$

$$h(0) = 150 \Rightarrow c = 150 \Rightarrow h(t) = 49t - 9.81\frac{t^2}{2} + 150$$

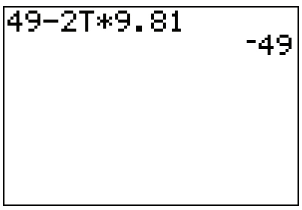




So, the maximum height is 272.4 metres.

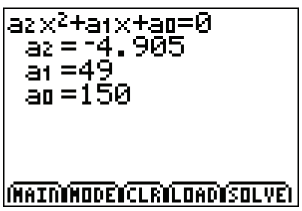
c) Since the parabola is symmetrical with respect to the vertical axis of symmetry that passes through the vertex, we can say that the time taken to reach the maximum height will be doubled. So, the answer is approximately 10 seconds.

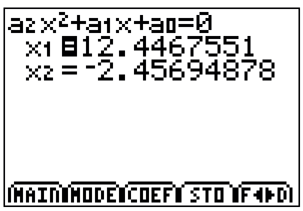
d)



The velocity will be -49 m/s.

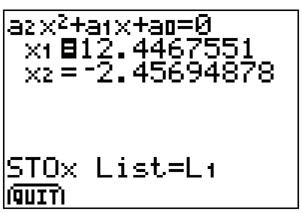
e)

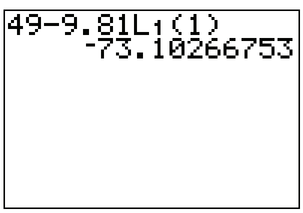




So, the projectile will take 12.4 seconds to hit the ground.

f)





So, the speed at impact is 73.1 m/s.

Exercise 16.8

1 $x^{-3}dy = 4y\,dx \Rightarrow \dfrac{dy}{y} = 4x^3dx \Rightarrow \displaystyle\int \frac{dy}{y} = \int 4x^3\,dx \Rightarrow \ln|y| = 4 \times \frac{x^4}{4} + c,\ c \in \mathbb{R} \Rightarrow$

$|y| = e^{x^4+c} = Ce^{x^4},\ C \in \mathbb{R}^+$

$y(0) = 3 \Rightarrow 3 = C \times e^0 \Rightarrow C = 3$

So, the final answer is: $|y| = 3e^{x^4} \Rightarrow y = \pm 3e^{x^4}$.

2 $\frac{dy}{dx} = xy \Rightarrow \frac{dy}{y} = x\,dx \Rightarrow \int \frac{dy}{y} = \int x\,dx \Rightarrow \ln|y| = \frac{x^2}{2} + c, c \in \mathbb{R} \Rightarrow |y| = Ce^{\frac{x^2}{2}}, C \in \mathbb{R}^+$

$y(0) = 1 \Rightarrow |1| = C \times e^0 \Rightarrow C = 1$

So, the final answer is: $y = \pm e^{\frac{x^2}{2}}$.

3 $y' - xy^2 = 0 \Rightarrow \frac{dy}{dx} = xy^2 \Rightarrow \frac{dy}{y^2} = x\,dx \Rightarrow \int \frac{dy}{y^2} = \int x\,dx \Rightarrow -\frac{1}{y} + c = \frac{x^2}{2} \Rightarrow c - \frac{x^2}{2} = \frac{1}{y} \Rightarrow$

$y = \frac{2}{c - x^2}, c \in \mathbb{R}$

$y(1) = 2 \Rightarrow 2 = \frac{2}{c-1} \Rightarrow c - 1 = \frac{2}{2} \Rightarrow c = 2$

So, the final answer is: $y = \frac{2}{2 - x^2}$.

Note: We were not using the usual algebraic properties for the constant c due to the fact that c can be any real number.

4 $y' - y^2 = 0 \Rightarrow \frac{dy}{dx} = y^2 \Rightarrow \frac{dy}{y^2} = dx \Rightarrow \int \frac{dy}{y^2} = \int dx \Rightarrow -\frac{1}{y} + c = x \Rightarrow y = \frac{1}{c - x}, c \in \mathbb{R}$

$y(2) = 1 \Rightarrow 1 = \frac{1}{c-2} \Rightarrow c - 2 = 1 \Rightarrow c = 3$

So, the final answer is: $y = \frac{1}{3 - x}$.

5 $\frac{dy}{dx} - e^y = 0 \Rightarrow \frac{dy}{dx} = e^y \Rightarrow e^{-y}dy = dx \Rightarrow \int e^{-y}\,dy = \int dx \Rightarrow -e^{-y} + c = x \Rightarrow c - x = e^{-y} \Rightarrow$

$-y = \ln(c - x) \Rightarrow y = -\ln(c - x) = \ln\left(\frac{1}{c - x}\right), c \in \mathbb{R}$

$y(0) = 1 \Rightarrow 1 = \ln\left(\frac{1}{c}\right) \Rightarrow -1 = \ln c \Rightarrow c = \frac{1}{e}$

So, the final answer is: $y = \ln\left(\frac{1}{\frac{1}{e} - x}\right) = \ln\left(\frac{e}{1 - ex}\right) = 1 - \ln(1 - ex)$.

6 $y' e^{y-x} = 1 \Rightarrow \frac{dy}{dx} \times \frac{e^y}{e^x} = 1 \Rightarrow e^y dy = e^x dx \Rightarrow \int e^y\,dy = \int e^x\,dx \Rightarrow e^y = e^x + c \Rightarrow$

$y = \ln(e^x + c), c \in \mathbb{R}$

7 $\frac{dy}{dx} = y^{-2}x + y^{-2} \Rightarrow \frac{dy}{dx} = \frac{x+1}{y^2} \Rightarrow y^2 dy = (x+1)\,dx \Rightarrow \int y^2\,dy = \int (x+1)\,dx \Rightarrow$

$\frac{y^3}{3} = \frac{x^2}{2} + x + c \Rightarrow y^3 = \frac{3x^2}{2} + 3x + c \Rightarrow y = \sqrt[3]{\frac{3x^2}{2} + 3x + c}$

$y(0) = 1 \Rightarrow 1 = \sqrt[3]{c} \Rightarrow c = 1$

So, the final answer is: $y = \sqrt[3]{\frac{3x^2}{2} + 3x + 1}$.

Note: The answer in the textbook is written in a different form. When we expand the expression given in the book, we can simplify it to the result given here.

8 $x\,dy - y^2dx = -dy \Rightarrow (x+1)\,dy = y^2dx \Rightarrow \dfrac{dy}{y^2} = \dfrac{dx}{x+1} \Rightarrow \int \dfrac{dy}{y^2} = \int \dfrac{dx}{x+1} \Rightarrow$

$-\dfrac{1}{y} + c = \ln|x+1| \Rightarrow \underbrace{c}_{\ln C} - \ln|x+1| = \dfrac{1}{y} \Rightarrow \ln\left(\dfrac{C}{|x+1|}\right) = \dfrac{1}{y} \Rightarrow y = \dfrac{1}{\ln\left(\dfrac{C}{|x+1|}\right)}, C \in \mathbb{R}^+$

$y(0) = 1 \Rightarrow 1 = \dfrac{1}{\ln C} \Rightarrow C = e$

So, the final answer is: $y = \dfrac{1}{\ln\left(\dfrac{e}{|x+1|}\right)} = \dfrac{1}{1 - \ln|x+1|}$.

9 $y^2dy - x\,dx = dx - dy \Rightarrow \left(y^2+1\right)dy = (x+1)\,dx \Rightarrow \int \left(y^2+1\right)dy = \int (x+1)\,dx \Rightarrow$

$\dfrac{y^3}{3} + y = \dfrac{x^2}{2} + x + c, c \in \mathbb{R}$

$y(0) = 3 \Rightarrow \dfrac{27}{3} + 3 = c \Rightarrow c = 12$

So, the final answer is: $\dfrac{y^3}{3} + y = \dfrac{x^2}{2} + x + 12 \Rightarrow 2y^3 + 6y = 3x^2 + 6x + 72.$

10 $yy' = xy^2 + x \Rightarrow y\dfrac{dy}{dx} = x\left(y^2+1\right) \Rightarrow \dfrac{y\,dy}{y^2+1} = x\,dx \Rightarrow \int \dfrac{\frac{1}{2}d\left(y^2+1\right)}{y^2+1} = \int x\,dx \Rightarrow$

$\dfrac{1}{2}\ln\left(y^2+1\right) = \dfrac{x^2}{2} + c \Rightarrow \ln\left(y^2+1\right) = x^2 + c, c \in \mathbb{R}$

$y(0) = 0 \Rightarrow \ln 1 = c \Rightarrow c = 0$

So, the final answer is: $\ln\left(y^2+1\right) = x^2 \Rightarrow y^2 + 1 = e^{x^2}$.

11 $\dfrac{dy}{dx} = y^2x + x \Rightarrow \dfrac{dy}{dx} = x\left(y^2+1\right) \Rightarrow \dfrac{dy}{y^2+1} = x\,dx \Rightarrow \int \dfrac{dy}{y^2+1} = \int x\,dx \Rightarrow$

$\arctan y = \dfrac{x^2}{2} + c, c \in \mathbb{R}$

12 $y' = \dfrac{xy - y}{y+1} \Rightarrow \dfrac{dy}{dx} = \dfrac{y(x-1)}{y+1} \Rightarrow \dfrac{(y+1)\,dy}{y} = (x-1)\,dx \Rightarrow \int\left(1 + \dfrac{1}{y}\right)dy = \int (x-1)\,dx \Rightarrow$

$y + \ln|y| = \dfrac{x^2}{2} - x + c, c \in \mathbb{R}$

$y(2) = 1 \Rightarrow 1 + \ln 1 = \dfrac{4}{2} - 2 + c \Rightarrow c = 1$

So, the final answer is: $y + \ln|y| = \dfrac{x^2}{2} - x + 1$.

13 $e^{x-y}dy = x\,dx \Rightarrow e^{-y}dy = xe^{-x}dx \Rightarrow \int e^{-y}\,dy = \int xe^{-x}\,dx \begin{bmatrix} u = x & du = dx \\ dv = e^{-x}dx & v = -e^{-x} \end{bmatrix} \Rightarrow$

$-e^{-y} + c = -xe^{-x} + \int e^{-x}\,dx \Rightarrow -e^{-y} = -xe^{-x} - e^{-x} - c \Rightarrow e^{-y} = e^{-x}(x+1) + c \Rightarrow$

$\dfrac{1}{e^y} = \dfrac{x + 1 + ce^x}{e^x} \Rightarrow e^y = \dfrac{e^x}{x + 1 + ce^x} \Rightarrow y = x - \ln\left|x + 1 + ce^x\right|, c \in \mathbb{R}$

14 $y' = xy^2 - x - y^2 + 1 \Rightarrow \frac{dy}{dx} = x\left(y^2 - 1\right) - \left(y^2 - 1\right) \Rightarrow \frac{dy}{dx} = \left(y^2 - 1\right)(x - 1) \Rightarrow$

$$\frac{dy}{y^2 - 1} = (x-1)\,dx \Rightarrow \left(\frac{\frac{1}{2}}{y-1} - \frac{\frac{1}{2}}{y+1}\right)dy = (x-1)\,dx \Rightarrow \frac{1}{2}\int\left(\frac{1}{y-1} - \frac{1}{y+1}\right)dy = \int (x-1)\,dx \Rightarrow$$

$$\frac{1}{2}\left(\ln|y-1| - \ln|y+1|\right) = \frac{x^2}{2} - x + c \Rightarrow \ln\left|\frac{y-1}{y+1}\right| = x^2 - 2x + c \Rightarrow \frac{y-1}{y+1} = Ce^{x^2 - 2x}, C \in \mathbb{R}^+$$

15 $xy \ln xy' = (y+1)^2 \Rightarrow \frac{y\,dy}{(y+1)^2} = \frac{dx}{x \ln x} \Rightarrow \int \frac{y\,dy}{(y+1)^2} = \int \frac{dx}{x \ln x} \Rightarrow$

$$\int \frac{(y+1) - 1\,dy}{(y+1)^2} = \int \frac{d(\ln x)}{\ln x} \Rightarrow \int\left(\frac{1}{y+1} - \frac{1}{(y+1)^2}\right)dy = \ln|\ln x| + c \Rightarrow$$

$$\ln|y+1| + \frac{1}{y+1} = \ln|\ln x| + c \Rightarrow (y+1)\ln|y+1| + 1 = (y+1)(\ln|\ln x| + c),\, c \in \mathbb{R}$$

16 $\frac{dy}{dx} = \frac{1 + 2y^2}{y \sin x} \Rightarrow \frac{y\,dy}{1 + 2y^2} = \frac{dx}{\sin x} \Rightarrow \int \frac{\frac{1}{4}d\left(1 + 2y^2\right)}{1 + 2y^2} = \int \csc x\,dx \Rightarrow *$

In order to find the solution, we need to solve the integral of the cosecant function, which is not an easy one. The method we are going to use to solve it is called the universal trigonometric substitution:

$$t = \tan\left(\frac{x}{2}\right) \Rightarrow dt = \sec^2\left(\frac{x}{2}\right) \times \frac{1}{2}\,dx = \left(\tan^2\left(\frac{x}{2}\right) + 1\right) \times \frac{1}{2}\,dx \Rightarrow dx = \frac{2}{t^2 + 1}\,dt$$

This universal substitution also uses the following trigonometric identities:

$\tan x = \frac{2t}{1 - t^2}$, $\sin t = \frac{2t}{1 + t^2}$ and $\cos t = \frac{1 - t^2}{1 + t^2}$, which can be proved by using the double angle formulae and the formulae for converting trigonometric functions. There is also a further trigonometric identity that we are going to use in this question:

$$\tan^2\left(\frac{x}{2}\right) = \frac{\sin^2\left(\frac{x}{2}\right)}{\cos^2\left(\frac{x}{2}\right)} = \frac{\frac{1 - \cos x}{2}}{\frac{1 + \cos x}{2}} = \frac{(1 - \cos x)/\times(1 + \cos x)}{(1 + \cos x)/\times(1 + \cos x)} = \frac{1 - \cos^2 x}{(1 + \cos x)^2} = \frac{\sin^2 x}{(1 + \cos x)^2} \Rightarrow$$

$$\tan\left(\frac{x}{2}\right) = \frac{\sin x}{1 + \cos x}$$

Now we can proceed with finding the integral of the cosecant function.

$$\int \csc x\,dx = \left[\begin{array}{c} t = \tan\left(\frac{x}{2}\right) \Rightarrow \csc x = \frac{1 + t^2}{2t} \\ dx = \frac{2}{t^2 + 1}\,dt \end{array}\right] = \int \frac{1 + t^2}{2t} \times \frac{2}{t^2 + 1}\,dt = \int \frac{1}{t}\,dt = \ln|t| + c$$

$$= \ln\left|\tan\left(\frac{x}{2}\right)\right| + c = \ln\left|\frac{\sin x}{\cos x + 1}\right| + c,\, c \in \mathbb{R}$$

Now we can continue with the solution of the original question.

$$* \Rightarrow \frac{1}{4}\ln\left(1 + 2y^2\right) = \ln\left|\frac{\sin x}{\cos x + 1}\right| + c \Rightarrow 1 + 2y^2 = C\frac{\sin^4 x}{(1 + \cos x)^4}, C \in \mathbb{R}^+$$

17 $\frac{dy}{dx} = x\sqrt{\frac{1-y^2}{1-x^2}} \Rightarrow \frac{dy}{\sqrt{1-y^2}} = \frac{x\,dx}{\sqrt{1-x^2}} \Rightarrow \int \frac{dy}{\sqrt{1-y^2}} = \frac{-\frac{1}{2}d\left(1-x^2\right)}{\sqrt{1-x^2}} \Rightarrow$

$\arcsin y = -\frac{1}{2} \times 2\sqrt{1-x^2} + c \Rightarrow \arcsin y = -\sqrt{1-x^2} + c, c \in \mathbb{R}$

$y(0) = 0 \Rightarrow \arcsin 0 = -\sqrt{1} + c \Rightarrow c = 1$

So, the final answer is: $\arcsin y = 1 - \sqrt{1-x^2}$.

18 $y'\left(1+e^x\right) = e^{x-y} \Rightarrow \frac{dy}{dx}\left(1+e^x\right) = \frac{e^x}{e^y} \Rightarrow e^y dy = \frac{e^x dx}{1+e^x} \Rightarrow \int e^y\,dy = \int \frac{d\left(1+e^x\right)}{1+e^x} \Rightarrow$

$e^y = \ln\left(1+e^x\right) + c, c \in \mathbb{R}$

$y(1) = 0 \Rightarrow e^0 = \ln\left(1+e^1\right) + c \Rightarrow c = 1 - \ln(1+e)$

So, the final answer is: $e^y = \ln\left(1+e^x\right) + 1 - \ln(1+e) \Rightarrow y = \ln\left(\ln\left(\frac{1+e^x}{1+e}\right) + 1\right)$.

19 $(y+1)\,dy = \left(x^2 y - y\right)dx \Rightarrow (y+1)\,dy = y\left(x^2-1\right)dx \Rightarrow \frac{(y+1)\,dy}{y} = \left(x^2-1\right)dx \Rightarrow$

$\int\left(1+\frac{1}{y}\right)dy = \int\left(x^2-1\right)dx \Rightarrow y + \ln|y| = \frac{x^3}{3} - x + c, c \in \mathbb{R}$

$y(3) = 1 \Rightarrow 1 + \underbrace{\ln(1)}_{0} = \frac{27}{3} - 3 + c \Rightarrow c = -5$

So, the final answer is: $y + \ln|y| = \frac{x^3}{3} - x - 5$.

20 $\cos y\,dx + \left(1+e^{-x}\right)\sin y\,dy = 0 \Rightarrow \tan y\,dy = \frac{-dx}{1+e^{-x}} \Rightarrow \int \tan y\,dy = \int \frac{\left(e^{-x} - e^{-x} - 1\right)dx}{1+e^{-x}}$

$\int \tan y\,dy = -\int \frac{d\left(1+e^{-x}\right)}{1+e^{-x}} - \int dx \Rightarrow -\ln|\cos y| + c = -\ln\left(1+e^{-x}\right) - x \Rightarrow$

$\ln\left(1+e^{-x}\right) + x + c = \ln|\cos y| \Rightarrow \cos y = C\left(1+e^{-x}\right)e^x = C\left(e^x+1\right), C \in \mathbb{R}^+$

In the last line, we were able to omit the absolute value of the cosine expression since the right side is always positive.

$y(0) = \frac{\pi}{4} \Rightarrow \underbrace{\cos\left(\frac{\pi}{4}\right)}_{\frac{1}{\sqrt{2}}} = C(1+1) \Rightarrow C = \frac{1}{2\sqrt{2}} = \frac{\sqrt{2}}{4}$

So, the final answer is: $\cos y = \frac{\sqrt{2}}{4}\left(e^x+1\right)$.

21 $xy' - y = 2x^2 y \Rightarrow xy' = y\left(2x^2+1\right) \Rightarrow \frac{dy}{y} = \frac{\left(2x^2+1\right)dx}{x} \Rightarrow \int \frac{dy}{y} = \int\left(2x + \frac{1}{x}\right)dx \Rightarrow$

$\ln|y| = x^2 + \ln|x| + c, c \in \mathbb{R} \Rightarrow |y| = C|x|\,e^{x^2}, C \in \mathbb{R}^+$

$y(1) = 1 \Rightarrow 1 = C \times e \times 1 \Rightarrow C = \frac{1}{e}$

So, the final answer is: $|y| = |x|\,e^{x^2-1}$.

22 $xy\,dx + e^{-x^2}\left(y^2-1\right)dy = 0 \Rightarrow xy\,dx = e^{-x^2}\left(1-y^2\right)dy \Rightarrow xe^{x^2}dx = \frac{\left(1-y^2\right)dy}{y} \Rightarrow$

$\int e^{x^2} \times \frac{1}{2}d\left(x^2\right) = \int\left(\frac{1}{y} - y\right)dy \Rightarrow \frac{e^{x^2}}{2} = \ln|y| - \frac{y^2}{2} + c, c \in \mathbb{R}$

$y(0)=1 \Rightarrow \frac{1}{2}=\ln(1)-\frac{1}{2}+c \Rightarrow c=1$

So, the final answer is: $\frac{e^{x^2}}{2}=\ln|y|-\frac{y^2}{2}+1.$

23 $(1+\tan y)\,y'=x^2+1 \Rightarrow (1+\tan y)\,dy=(x^2+1)\,dx \Rightarrow \int(1+\tan y)\,dy=\int(x^2+1)\,dx \Rightarrow$

$y-\ln|\cos y|=\frac{x^3}{3}+x+c,\ c\in\mathbb{R}$

24 $\frac{dy}{dt}=\frac{te^t}{y\sqrt{y^2+1}} \Rightarrow y\sqrt{y^2+1}\,dy=te^t dt \Rightarrow \int y\sqrt{y^2+1}\,dy=\int te^t\,dt \Rightarrow$

To find the first integral, we need to use the substitution method, and to find the second integral we need to use integration by parts.

$$\int y\sqrt{y^2+1}\,dy=\begin{bmatrix} y^2+1=s \\ 2y\,dy=ds \end{bmatrix}=\int\sqrt{s}\times\frac{1}{2}\,ds=\frac{1}{2}\times\frac{2}{3}s^{\frac{3}{2}}+c=\frac{1}{3}(y^2+1)^{\frac{3}{2}}+c_1$$

$$\int te^t\,dt=\begin{bmatrix} u=t & du=dt \\ dv=e^t dt & v=e^t \end{bmatrix}=te^t-\int e^t\,dt=te^t-e^t+c_2$$

So, the final answer is: $\frac{1}{3}(y^2+1)^{\frac{3}{2}}=te^t-e^t+c,\ c\in\mathbb{R}.$

25 $y\sec\theta\,dy=e^y\sin^2\theta\,d\theta \Rightarrow ye^{-y}dy=\sin^2\theta\cos\theta\,d\theta \Rightarrow \int ye^{-y}\,dy=\int\sin^2\theta\,d(\sin\theta) \Rightarrow$

We are going to find the first integral by using integration by parts.

$$\int ye^{-y}\,dy=\begin{bmatrix} u=y & du=dy \\ dv=e^{-y}dy & v=-e^{-y} \end{bmatrix}=-ye^{-y}+\int e^{-y}\,dy=-ye^{-y}-e^{-y}+c,\ c\in\mathbb{R}$$

So, the final answer is: $-e^{-y}(y+1)=\frac{\sin^3\theta}{3}+c,\ c\in\mathbb{R}.$

26 $x\cos x=(2y+e^{3y})\,y' \Rightarrow x\cos x\,dx=(2y+e^{3y})\,dy \Rightarrow \int x\cos x\,dx=\int(2y+e^{3y})\,dy$

To find the first integral, we need to use integration by parts.

$$\int x\cos x\,dx=\begin{bmatrix} u=x & du=dx \\ dv=\cos x\,dx & v=\sin x \end{bmatrix}=x\sin x-\int\sin x\,dx=x\sin x+\cos x+c,\ c\in\mathbb{R}$$

$x\sin x+\cos x=y^2+\frac{1}{3}e^{3y}+c,\ c\in\mathbb{R}$

$y(0)=0 \Rightarrow 0+\underbrace{\cos 0}_{1}=0^2+\frac{1}{3}e^0+c \Rightarrow c=\frac{2}{3}$

So, the final answer is: $x\sin x+\cos x=y^2+\frac{1}{3}e^{3y}+\frac{2}{3} \Rightarrow 3x\sin x+3\cos x=3y^2+e^{3y}+2.$

27 $\frac{dy}{dx}=e^x-2x \Rightarrow dy=(e^x-2x)\,dx \Rightarrow \int dy=\int(e^x-2x)\,dx \Rightarrow y=e^x-x^2+c,\ c\in\mathbb{R}$

$y(0)=3 \Rightarrow 3=e^0+c \Rightarrow c=2$

So, the final answer is: $y=e^x-x^2+2.$

28 a) $\frac{dT}{dt} = m(T-21) \Rightarrow \frac{dT}{T-21} = m\,dt \Rightarrow \ln|T-21| = mt + c, c \in \mathbb{R} \Rightarrow T - 21 = e^{mt+c}$

$T = 21 + Ce^{mt}, C \in \mathbb{R}^+$

b) Now, we need to solve the simultaneous equations to find m and C.

i) $\begin{cases} T(0) = 99 \\ T(15) = 69 \end{cases} \Rightarrow \begin{cases} C + 21 = 99 \\ Ce^{15m} + 21 = 69 \end{cases} \Rightarrow \begin{cases} C = 78 \\ 78e^{15m} = 48 \end{cases} \Rightarrow \begin{cases} C = 78 \\ e^{15m} = \frac{\cancel{48}\,8}{\cancel{78}\,13} \end{cases} \Rightarrow$

$\begin{cases} C = 78 \\ 15m = \ln\left(\frac{8}{13}\right) \end{cases} \Rightarrow \begin{cases} C = 78 \\ m = \frac{1}{15}\ln\left(\frac{8}{13}\right) \end{cases}$

ii) $T(t) = 78e^{\frac{1}{15}\ln\left(\frac{8}{13}\right)t} + 21 \Rightarrow 39 = 78e^{\frac{1}{15}\ln\left(\frac{8}{13}\right)t} + 21 \Rightarrow 18 = 78e^{\frac{1}{15}\ln\left(\frac{8}{13}\right)t} \Rightarrow \frac{3}{13} = e^{\frac{1}{15}\ln\left(\frac{8}{13}\right)t} \Rightarrow$

$\ln\left(\frac{3}{13}\right) = \frac{1}{15}\ln\left(\frac{8}{13}\right)t \Rightarrow t = \frac{15 \times (\ln 3 - \ln 13)}{\ln 8 - \ln 13} \approx 45.3$ minutes

Chapter 17

Exercise 17.1

1 **a)** Discrete, since the number of words in a spelling test is finite.

b) Continuous; the amount of water is measured in litres which is correct to a given accuracy but, in fact, it could be any value within the given interval.

c) Continuous; the amount of time is measured in minutes or seconds and it could be any value within a given interval of accuracy.

d) Discrete; even though the number of bacteria could be very large, it is limited with respect to the volume it occupies.

e) Continuous; the amount of CO is measured as a volume that could be any value within a given interval of accuracy.

f) Continuous, since the amount of vaccine is measured as a volume.

g) Discrete, since the heart rate (per minute) is always measured as an exact number of heart beats within the given period of time.

h) Continuous, since the pressure is a measure that can take on any value within the given interval of accuracy.

i) Continuous, since the distance travelled is a measure that can take on any value within the given interval of accuracy.

j) Discrete, since the scores in the league cannot be any other values than integers up to the single larger integer; therefore, the total score is finite.

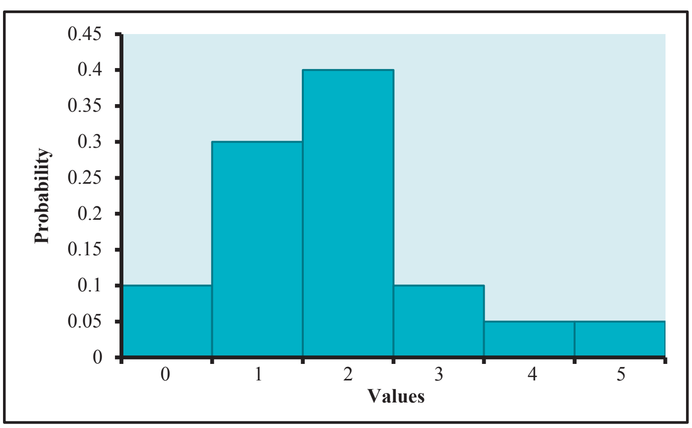
k) Continuous, since the height can take on any value within the given interval of accuracy.

l) Continuous, since the strength can take on any value within the given interval of accuracy.

m) Discrete, since the number of overdue books cannot exceed the number of books in the library, that is, the number is finite.

2 **a)** Since the sum of all the probabilities should be 1, we get the following:
$0.1 + 0.3 + P(2) + 0.1 + 0.05 + 0.05 = 1 \Rightarrow 0.6 + P(2) = 1 \Rightarrow P(2) = 0.4$

b)

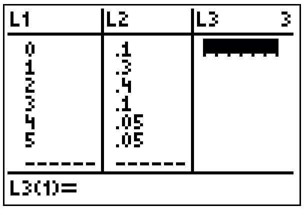


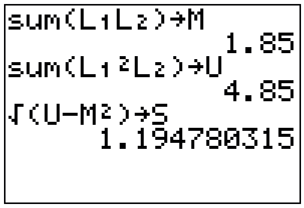
c) $\mu = \sum_i x_i p_i \Rightarrow \mu = 0 \times 0.1 + 1 \times 0.3 + 2 \times 0.4 + 3 \times 0.1 + 4 \times 0.05 + 5 \times 0.05 = 1.85$

$$\sigma^2 = \left(\sum_i x_i^2 p_i\right) - \mu^2 = \left(0^2 \times 0.1 + 1^2 \times 0.3 + 2^2 \times 0.4 + 3^2 \times 0.1 + 4^2 \times 0.05 + 5^2 \times 0.05\right) - 1.85^2$$

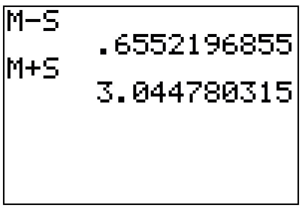
$$= 4.85 - 3.4225 = 1.4275 \Rightarrow \sigma = \sqrt{1.4275} = 1.19$$

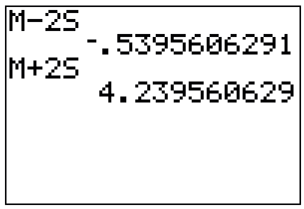
Or, by using a calculator:





d) $\mu \pm \sigma = 1.85 \pm 1.19 \Rightarrow [0.66, 3.04]$ and $\mu \pm 2\sigma = 1.85 \pm 2 \times 1.19 \Rightarrow [-0.54, 4.24]$





e) $\mu = \sum_i z_i p_i \Rightarrow \mu = 1 \times 0.1 + 2 \times 0.3 + 3 \times 0.4 + 4 \times 0.1 + 5 \times 0.05 + 6 \times 0.05 = 2.85$

$$\sigma^2 = \left(\sum_i z_i^2 p_i\right) - \mu^2 = \left(1^2 \times 0.1 + 2^2 \times 0.3 + 3^2 \times 0.4 + 4^2 \times 0.1 + 5^2 \times 0.05 + 6^2 \times 0.05\right) - 2.85^2$$

$$= 9.55 - 8.1225 = 1.4275 \Rightarrow \sigma = \sqrt{1.4275} = 1.19$$

f) $\mathrm{E}(Y + b) = \mathrm{E}(Y) + b$; $\mathrm{V}(Y + b) = \mathrm{V}(Y)$

3 **a)** Since the sum of all the probabilities should be 1, we get the following:
$0.14 + 0.11 + \mathrm{P}(15) + 0.26 + 0.23 = 1 \Rightarrow 0.74 + \mathrm{P}(15) = 1 \Rightarrow \mathrm{P}(15) = 0.26$

b) $\mathrm{P}(x = 12 \text{ or } x = 20) = \mathrm{P}(12) + \mathrm{P}(20) = 0.14 + 0.23 = 0.37$

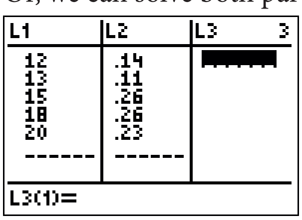
c) $\mathrm{P}(x \leqslant 18) = \mathrm{P}(12) + \mathrm{P}(13) + \mathrm{P}(15) + \mathrm{P}(18) = 0.14 + 0.11 + 0.26 + 0.26 = 0.77$, or we can use the complementary event; therefore, $\mathrm{P}(x \leqslant 18) = 1 - \mathrm{P}(20) = 1 - 0.23 = 0.77$.

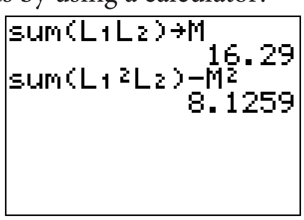
d) $\mathrm{E}(X) = \sum_i x_i p_i \Rightarrow \mu = 12 \times 0.14 + 13 \times 0.11 + 15 \times 0.26 + 18 \times 0.26 + 20 \times 0.23 = 16.29$

e) $\mathrm{V}(X) = \left(\sum_i x_i^2 p_i\right) - \mu^2 \Rightarrow$

$$\mathrm{V}(X) = \left(12^2 \times 0.14 + 13^2 \times 0.11 + 15^2 \times 0.26 + 18^2 \times 0.26 + 20^2 \times 0.23\right) - 16.29^2 = 8.1259$$

Or, we can solve both parts by using a calculator:





f) $\mathrm{E}(Y) = \sum_i y_i p_i \Rightarrow \mu = 2 \times 0.14 + 2.5 \times 0.11 + 3.5 \times 0.26 + 5 \times 0.26 + 6 \times 0.23 = 4.145$

$$V(Y) = \left(\sum_i y_i^2 p_i\right) - \mu^2 \Rightarrow$$

$$V(Y) = \left(2^2 \times 0.14 + 2.5^2 \times 0.11 + 3.5^2 \times 0.26 + 5^2 \times 0.26 + 6^2 \times 0.23\right) - 4.145^2 = 2.071\,475$$

g) $E(aX+b) = aE(X)+b;\ V(aX+b) = a^2V(X)$

4 **a)** At least two patients means 2, 3, 4 or 5, but in this case it would be easier to use the complementary event, which is 0 or 1; therefore, we get:

$$P(x \geqslant 2) = 1 - P(x \leqslant 1) = 1 - (0.002 + 0.029) = 1 - 0.031 = 0.969$$

b) The majority in a group of five would be 3, 4 or 5; therefore, we need to find:

$$P(x \leqslant 2) = P(x \leqslant 1) + P(2) = 0.031 + 0.132 = 0.163$$

c) $E(X) = \sum_i x_i p_i \Rightarrow$

$= 0 \cdot\ 0.002 + 1 \cdot\ 0.029 + 2 \cdot\ 0.132 + 3 \cdot\ 0.309 + 4 \cdot\ 0.36 + 5 \cdot\ 0.168 = 3.5$, which means that we expect that three or four patients will benefit from the treatment.

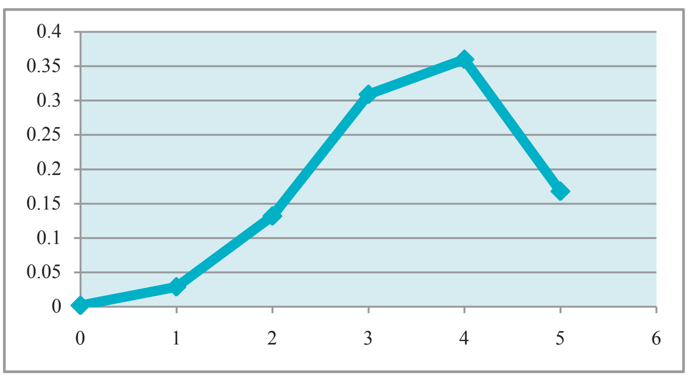
d) $\sigma^2 = \left(\sum_i x_i^2 p_i\right) - \mu^2$

$$= \left(0^2 \times 0.002 + 1^2 \times 0.029 + 2^2 \times 0.132 + 3^2 \times 0.309 + 4^2 \times 0.36 + 5^2 \times 0.168\right) - 3.5^2$$

$= 13.298 - 12.25 = 1.048 \Rightarrow \sigma = \sqrt{1.048} = 1.023\,72$; therefore, 1.02 (3 s.f.).

e) $\mu \pm \sigma = 3.5 \pm 1.02 \Rightarrow [2.48, 4.52]$ and $\mu \pm 2\sigma = 3.5 \pm 2 \times 1.02 \Rightarrow [1.46, 5.54]$

From the empirical rule, we know that $P(\mu - \sigma \leqslant x \leqslant \mu + \sigma) = 0.683$. Since the set of values is of a discrete nature, we can identify the border values as 3 and 4; so, we calculate the probability of the given model: $P(3 \leqslant x \leqslant 4) = 0.669$. The answer is quite close to the result suggested by the empirical rule. The second probability from the empirical rule is $P(\mu - 2\sigma \leqslant x \leqslant \mu + 2\sigma) = 0.954$, and in the probability model we get $P(2 \leqslant x \leqslant 5) = 0.969$, which again is very close to the suggested result.



5

x	12	14	16	18
P(x)	$6k$	$7k$	$8k$	$9k$

The sum of all the probabilities is 1, so we obtain the following:

$$6k + 7k + 8k + 9k = 1 \Rightarrow 30k = 1 \Rightarrow k = \frac{1}{30}$$

6 **a)** Since the sum of all the probabilities should be 1, we get the following:

$$\frac{3}{20}+\frac{7}{30}+k+\frac{3}{10}+\frac{13}{60}=1 \Rightarrow \frac{\cancel{54}\,9}{\cancel{60}\,10}+k=1 \Rightarrow k=\frac{1}{10}$$

b) $\mathrm{P}(x>10)=\mathrm{P}(x=15)+\mathrm{P}(x=20)+\mathrm{P}(x=25)=\frac{1}{10}+\frac{3}{10}+\frac{13}{60}=\frac{37}{60}$

Or again, we can use the complementary event and a calculation that does not involve the result from the previous part, so there is no possible mistake to carry through.

$$\mathrm{P}(x>10)=1-(\mathrm{P}(x=5)+\mathrm{P}(x=10))=1-\frac{3}{20}-\frac{7}{30}=1-\frac{23}{60}=\frac{37}{60}$$

c) $\mathrm{P}(5<x\leqslant 20)=\mathrm{P}(x=10)+\mathrm{P}(x=15)+\mathrm{P}(x=20)=\frac{7}{30}+\frac{1}{10}+\frac{3}{10}=\frac{19}{30}$

Or again, we can use the complementary event and a calculation that does not involve the result from part **b** and hence there is no possible mistake to carry through.

$$\mathrm{P}(5<x\leqslant 20)=1-(\mathrm{P}(x=5)+\mathrm{P}(x=25))=1-\frac{3}{20}-\frac{13}{60}=1-\frac{\cancel{22}\,11}{\cancel{60}\,30}=\frac{19}{30}$$

d) $\mathrm{E}(X)=\sum_i x_i p_i \Rightarrow \mu=\cancel{5}\times\frac{3}{\cancel{20}\,4}+\cancel{10}\times\frac{7}{\cancel{30}\,3}+\cancel{15}\,3\times\frac{1}{\cancel{10}\,2}+\cancel{20}\,2\times\frac{3}{\cancel{10}}+\cancel{25}\,5\times\frac{13}{\cancel{60}\,12}$

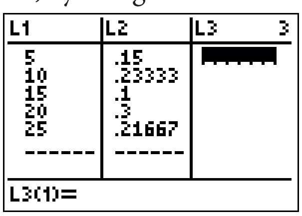
$$=\frac{3}{4}+\frac{7}{3}+\frac{3}{2}+6+\frac{65}{12}=\frac{9+28+18+72+65}{12}=\frac{192}{12}=16$$

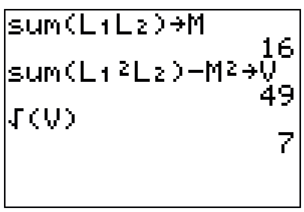
$$\mathrm{V}(X)=\sum_i x_i^2 p_i-\mu^2 \Rightarrow$$

$$\mathrm{V}(X)=\cancel{25}\,5\times\frac{3}{\cancel{20}\,4}+\cancel{100}\,10\times\frac{7}{\cancel{30}\,3}+\cancel{225}\,45\times\frac{1}{\cancel{10}\,2}+\cancel{400}\,40\times\frac{3}{\cancel{10}}+\cancel{625}\,125\times\frac{13}{\cancel{60}\,12}-16^2$$

$$=\frac{15}{4}+\frac{70}{3}+\frac{45}{2}+120+\frac{1625}{12}-256=49 \Rightarrow \sigma=\sqrt{49}=7$$

Or, by using a calculator:





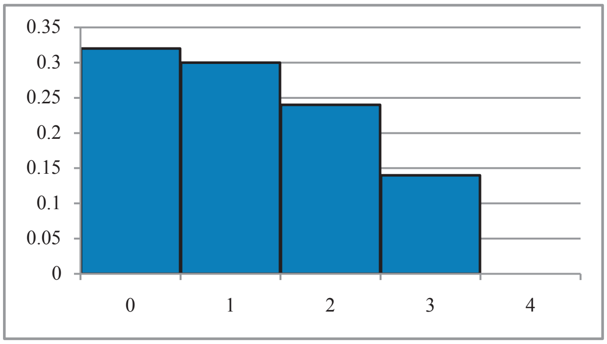
7 **a)** $\mathrm{P}(y=0)=k(16-0^2)=16k;\ \mathrm{P}(y=1)=k(16-1^2)=15k;\ \mathrm{P}(y=2)=k(16-2^2)=12k;$

$\mathrm{P}(y=3)=k(16-3^2)=7k;\ \mathrm{P}(y=4)=k(16-4^2)=0$

Since the sum of all probabilities is 1, we obtain the following:

$$16k+15k+12k+7k+0=1 \Rightarrow 50k=1 \Rightarrow k=\frac{1}{50}$$

b)



c) The calculation is simpler if we use the complementary event:

$$P(1 \leqslant y \leqslant 3) = 1 - (P(y = 0) + P(y = 4)) = 1 - \frac{8}{25} - 0 = \frac{17}{25} = 0.68$$

d) $$E(Y) = \sum_i y_i p_i \Rightarrow \mu = 0 \times \frac{8}{25} + 1 \times \frac{3}{10} + 2 \times \frac{6}{25} + 3 \times \frac{7}{50} + 4 \times 0 = \frac{15 + 24 + 21}{50} = \frac{60}{50} = \frac{6}{5} = 1.2$$

$$V(Y) = \sum_i y_i^2 p_i - \mu^2 \Rightarrow$$

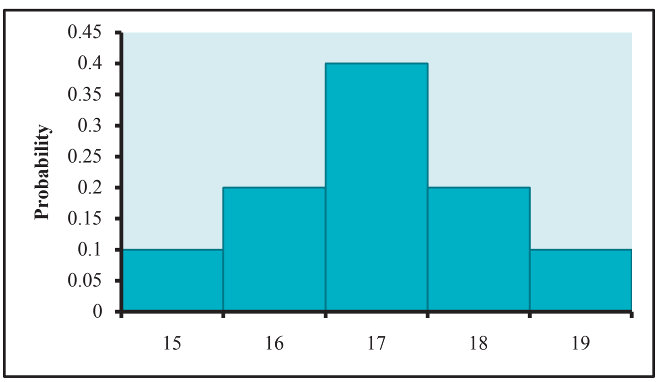
$$V(Y) = \left(0 + 1 \times \frac{3}{10} + 4 \times \frac{6}{25} + 9 \times \frac{7}{50} + 0\right) - \left(\frac{6}{5}\right)^2 = \frac{15 + 48 + 63}{50} - \frac{36}{25} = \frac{54}{50} = \frac{27}{25} = 1.08$$

8 **a)** We are going to use a probability distribution table:

x	15	16	17	18	19
P(x)	0.1	0.2	0.4	$2p$	p

Since the sum of all the probabilities is 1, we obtain the following:

$$0.1 + 0.2 + 0.4 + 2p + p = 1 \Rightarrow 3p = 0.3 \Rightarrow p = 0.1$$



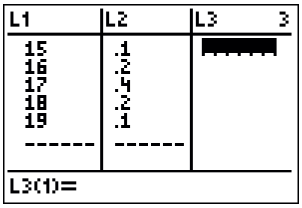
The distribution is symmetrical.

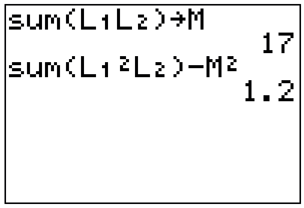
b) $$E(X) = \sum_i x_i p_i \Rightarrow \mu = 15 \times 0.1 + 16 \times 0.2 + 17 \times 0.4 + 18 \times 0.2 + 19 \times 0.1 = 17$$

$$V(X) = \sum_i x_i^2 p_i - \mu^2 \Rightarrow$$

$$V(X) = (15^2 \times 0.1 + 16^2 \times 0.2 + 17^2 \times 0.4 + 18^2 \times 0.2 + 19^2 \times 0.1) - 17^2 = 1.2$$

Or, by using a calculator:





9 a) $\mu = \sum_i x_i p_i \Rightarrow \mu = 0 \times 0.1 + 1 \times 0.4 + 2 \times 0.2 + 3 \times 0.15 + 4 \times 0.1 + 5 \times 0.05 = 1.9$

$\sigma^2 = \sum_i x_i^2 p_i - \mu^2 \Rightarrow \sigma^2 = (0 \times 0.1 + 1 \times 0.4 + 4 \times 0.2 + 9 \times 0.15 + 16 \times 0.1 + 25 \times 0.05) - 1.9^2$

$= 5.4 - 3.61 = 1.79 \Rightarrow \sigma = \sqrt{1.79} \approx 1.34$

b) $\mathrm{P}(\mu - 2\sigma \leqslant X \leqslant \mu + 2\sigma) \approx 0.95 \Rightarrow \mu \pm 2\sigma = 1.79 \pm 2 \times 1.34 = -0.89$ or 4.47; therefore, we would say that between 0 and 4 laptops are sold 95% of the time.

c) Yes, since there is a 5% chance of that happening.

10

x	2	3	4	5	6	7
P(x)	$\frac{1}{4}$	$\frac{1}{16}$	$\frac{1}{64}$	$\frac{1}{256}$	$\frac{1}{1024}$	k

$$\sum \mathrm{P}(x) = 1 \Rightarrow \overbrace{\frac{1}{4} + \frac{1}{16} + \frac{1}{64} + \frac{1}{256} + \frac{1}{1024}}^{\text{geometric sequence}} + k = 1 \Rightarrow k = 1 - \left(\frac{1}{4} \times \frac{1 - \left(\frac{1}{4}\right)^5}{1 - \frac{1}{4}} \right)$$

$$= 1 - \frac{1}{\cancel{4}} \times \frac{1 - \frac{1}{1024}}{\frac{3}{\cancel{4}}} = 1 - \frac{\frac{\cancel{1023}\,341}{1024}}{\cancel{3}} = \frac{1024 - 341}{1024} = \frac{683}{1024} \approx 0.667$$

$$\mathrm{E}(X) = \sum_i x_i p_i \Rightarrow \mu = \cancel{2} \times \frac{1}{\cancel{4}\,2} + 3 \times \frac{1}{16} + \cancel{4} \times \frac{1}{\cancel{64}\,16} + 5 \times \frac{1}{256} + 6 \times \frac{1}{1024} + 7 \times \frac{683}{1024}$$

$$= \frac{512 + 192 + 64 + 20 + 6 + 4781}{1024} = \frac{5575}{1024} \approx 5.44$$

11 a) $\sum \mathrm{P}(y) = 1 \Rightarrow 0.1 + 0.11 + k + (k-1)^2 = 1 \Rightarrow 0.21 + k + k^2 - 2k + \cancel{1} = \cancel{1} \Rightarrow$

$k^2 - k + 0.21 = 0 \Rightarrow (k - 0.3)(k - 0.7) = 0 \Rightarrow k = 0.3$ or $k = 0.7$

b) For $k = 0.3 \Rightarrow \mu = \sum_i x_i p_i \Rightarrow \mu = 0 \times 0.1 + 1 \times 0.11 + 2 \times 0.3 + 3 \times 0.49 = 2.18$

For $k = 0.7 \Rightarrow \mu = \sum_i x_i p_i \Rightarrow \mu = 0 \times 0.1 + 1 \times 0.11 + 2 \times 0.7 + 3 \times 0.09 = 1.78$

12 a) Since we draw a ball three times, the number of red balls, X, can be 0, 1, 2 and 3. The probability that one drawn ball is red is: $p = \frac{8}{8+4} = \frac{\cancel{8}\,2}{\cancel{12}\,3} = \frac{2}{3}$.

Now, let's calculate the corresponding probabilities:

$$\mathrm{P}(X=0)=\left(\frac{1}{3}\right)^3=\frac{1}{27},\ \mathrm{P}(X=1)=\binom{3}{1}\left(\frac{1}{3}\right)^2\frac{2}{3}=\frac{2}{9},\ \mathrm{P}(X=2)=\binom{3}{2}\frac{1}{3}\left(\frac{2}{3}\right)^2=\frac{4}{9},$$

and $\mathrm{P}(X=3)=\left(\frac{2}{3}\right)^3=\frac{8}{27}$

x	0	1	2	3
$\mathrm{P}(X=x)$	$\frac{1}{27}$	$\frac{2}{9}$	$\frac{4}{9}$	$\frac{8}{27}$

b) $\mathrm{E}(X)=\sum_i x_i p_i \Rightarrow \mu = 0\times\frac{1}{27}+1\times\frac{2}{9}+2\times\frac{4}{9}+3\times\frac{8}{27}=\frac{2+8+8}{9}=\frac{18}{9}=2$

13 a)

y	0	1	2	3	4
$\mathrm{P}(Y=y)$	$4k$	$3k$	$2k$	k	0

$$\sum \mathrm{P}(y)=1 \Rightarrow 4k+3k+2k+k=1 \Rightarrow 10k=1 \Rightarrow k=\frac{1}{10}$$

b) $\mathrm{P}(1\leqslant y<3)=\mathrm{P}(y=1)+\mathrm{P}(y=2)=\frac{3}{10}+\frac{1}{5}=\frac{5}{10}=\frac{1}{2}$

14 The following table represents a probability mass function of the variable X.

x	45	46	47	48	49	50	51	52	53	54	55
P(x)	0.05	0.08	0.12	0.15	0.25	0.20	0.05	0.04	0.03	0.02	0.01

a)

x	45	46	47	48	49	50	51	52	53	54	55
$\mathrm{P}(X\leqslant x)$	0.05	0.13	0.25	0.40	0.65	0.85	0.90	0.94	0.97	0.99	1.00

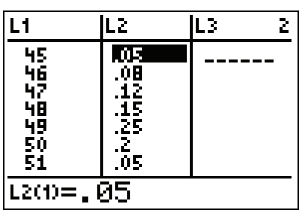
b) If all the ticket holders that show up are to be accommodated, no more than 50 passengers can show up. Therefore, we can use the cumulative distribution function directly from the table: $\mathrm{P}(X\leqslant 50)=0.85$

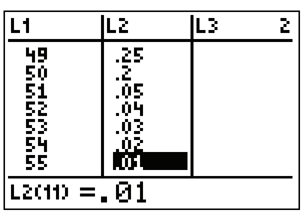
c) This is the complementary event of the event in part **b**.

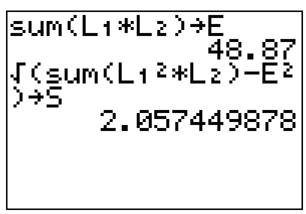
$\mathrm{P}(X\geqslant 51)=1-\mathrm{P}(X\leqslant 50)=1-0.85=0.15$

d) $\mathrm{E}(X)=45\cdot 0.05+46\cdot 0.08+47\cdot 0.12+48\cdot 0.15+49\cdot 0.25+50\cdot 0.20+51\cdot 0.05$
$+52\cdot 0.04+53\cdot 0.03+54\cdot 0.02+55\cdot 0.01=2.25+3.68+5.64+7.2+12.25+10+2.55$
$+2.08+1.59+1.08+0.55=48.87$

e) When we have a distribution with a lot of possible outcomes, it is much easier to use a GDC.

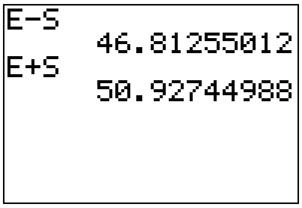






So, the standard deviation is 2.06 (correct to three significant figures).

f)



So, we need to calculate: $P(47 \leqslant X \leqslant 51) = 0.12 + 0.15 + 0.25 + 0.20 + 0.05 = 0.77$

Alternatively, we can use the cumulative distribution function from the second table:

$P(47 \leqslant X \leqslant 50) = P(X \leqslant 51) - P(X \leqslant 46) = 0.90 - 0.13 = 0.77$

15 The following table represents a probability mass function of the variable X.

x	0	1	2	3	4	5	6
P(x)	0.08	0.15	0.22	0.27	0.20	0.05	0.03

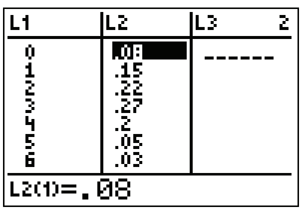
a)

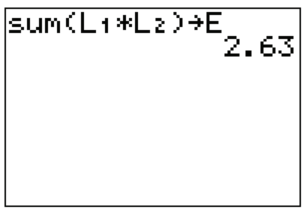
x	0	1	2	3	4	5	6
$P(X \leqslant x)$	0.08	0.23	0.45	0.72	0.92	0.97	1.00

b) $P(X \leqslant 3) = 0.72$

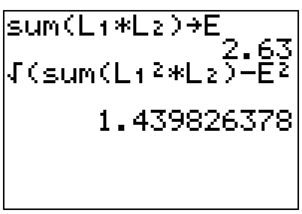
c) To obtain a free line means that there will be at least one free line; therefore, we can use the cumulative distribution function: $P(X \leqslant 5) = 0.97$.

d)





e)



So, the standard deviation is 1.44 (correct to three significant figures).

16 **a)** Since 90% of the batteries have an acceptable voltage: $P(x = 1) = 0.9$.

b) $P(x = 2) = \underbrace{0.1}_{\text{unacceptable}} \cdot \underbrace{0.9}_{\text{acceptable}} = 0.09$

c) $P(x = 3) = \underbrace{0.1}_{\text{unacceptable}} \cdot \underbrace{0.1}_{\text{unacceptable}} \cdot \underbrace{0.9}_{\text{acceptable}} = 0.009$

d) The fourth tested battery, as well as the previous three batteries, should be unacceptable, whilst the fifth battery should be acceptable.

e) $P(X = x) = \overbrace{\underbrace{0.1}_{\text{unacceptable}} \times \underbrace{0.1}_{\text{unacceptable}} \times \ldots \times \underbrace{0.1}_{\text{unacceptable}}}^{x-1} \times \underbrace{0.9}_{\text{acceptable}} = (0.1)^{x-1} \times 0.9$

17 **a)** If the flashlight needs two batteries, just selecting one battery is not enough for the flashlight to work; therefore, $\mathrm{P}(x = 1) = 0$.

b) Both batteries should be acceptable: $\mathrm{P}(x = 2) = \underbrace{0.9}_{\text{acceptable}} \cdot \underbrace{0.9}_{\text{acceptable}} = 0.81$

c) Two batteries should be acceptable for the flashlight to work:

$$\mathrm{P}(x = 3) = \underbrace{0.1}_{\text{unacceptable}} \cdot \underbrace{0.9}_{\text{unacceptable}} \cdot \underbrace{0.9}_{\text{acceptable}} + \underbrace{0.9}_{\text{unacceptable}} \cdot \underbrace{0.1}_{\text{unacceptable}} \cdot \underbrace{0.9}_{\text{acceptable}} = 0.162$$

d) The fourth battery depends on the previous three batteries. If there is no acceptable battery in the previous three, the fourth battery should be acceptable; but, if there was one acceptable battery in the previous three, the fourth battery should be unacceptable. The fifth battery must be acceptable in both cases.

e) In the first $x - 1$ tested batteries, there must be one acceptable battery and the last battery should be acceptable as well. One acceptable battery in the first $x - 1$ can occur in $x - 1$ different ways:

$$\mathrm{P}(X = x) = (x-1)(0.1)^{x-2} \times 0.9^2 = (x-1)(0.1)^{x-2} \times 0.81$$

18

Score	1	2	3	4
Probability	$\frac{1}{2}$	$\frac{1}{5}$	$\frac{1}{5}$	$\frac{1}{10}$
Number of counters player receives	4	5	15	n

Let the variable X represent the number of counters the player receives.

$$\mathrm{E}(X) = \frac{1}{2} \times 4 + \frac{1}{5} \times 5 + \frac{1}{5} \times 15 + \frac{1}{10} \times n = 6 + \frac{n}{10} \Rightarrow 6 + \frac{n}{10} = 9 \Rightarrow \frac{n}{10} = 3 \Rightarrow n = 30$$

19 **a)** **i)** There are four different ways of obtaining a sum of 9: (3, 6), (4, 5), (5, 4) and (6, 3). We know that there are a total of 36 possible outcomes; so,

$$\mathrm{P}(A = 9) = \frac{4}{36} = \frac{1}{9}$$

ii) For both Alan and Belle to obtain a score of 9, it means that both must obtain one of the four outcomes listed in **i**.

$$\mathrm{P}(A = 9) \cdot \mathrm{P}(B = 9) = \frac{1}{9} \cdot \frac{1}{9} = \frac{1}{81}$$

Note: Whenever we say 'and' we multiply the probabilities, whilst whenever we say 'or' we add the probabilities.

b) **i)** There are 11 different scores that Alan and Belle can obtain. We can list all of them and calculate the probability.

$$\mathrm{P}(A = B) = 2 \times \left(\frac{1}{36}\right)^2 + 2 \times \left(\frac{2}{36}\right)^2 + 2 \times \left(\frac{3}{36}\right)^2 + 2 \times \left(\frac{4}{36}\right)^2 + 2 \times \left(\frac{5}{36}\right)^2 + \left(\frac{6}{36}\right)^2$$
$$= \frac{1}{36^2}\left(2 \cdot \left(1^2 + 2^2 + 3^2 + 4^2 + 5^2\right) + 6^2\right) = \frac{1}{36^2}(110 + 36) = \frac{73}{648}$$

ii) Since all the dice are fair, the probability that Alan's score exceeds Belle's score, and vice versa, are equal. Let's call these events x. The sum of all three events is 1.

$$\mathrm{P}(A = B) + \underbrace{\mathrm{P}(A > B)}_{x} + \underbrace{\mathrm{P}(A < B)}_{x} = 1 \Rightarrow \frac{73}{648} + 2x = 1 \Rightarrow 2x = \frac{575}{648} \Rightarrow x = \frac{575}{1296}$$

c) i) If the largest number score is 1, then all four dice must score 1: $\mathrm{P}(X \leqslant 1) = \left(\frac{1}{6}\right)^4$.

If the largest number is less than or equal to 2, then a combination of 1 or 2 on each dice is favourable; therefore: $\mathrm{P}(X \leqslant 2) = \left(\frac{2}{6}\right)^4$.

In a similar way, we can discuss all the remaining possibilities:

$\mathrm{P}(X \leqslant 3) = \left(\frac{3}{6}\right)^4$, $\mathrm{P}(X \leqslant 4) = \left(\frac{4}{6}\right)^4$, $\mathrm{P}(X \leqslant 5) = \left(\frac{5}{6}\right)^4$ and $\mathrm{P}(X \leqslant 6) = \left(\frac{6}{6}\right)^4$

In summary, we can write: $\mathrm{P}(X \leqslant x) = \left(\frac{x}{6}\right)^4$, $x = 1, 2, ..., 6$.

ii) To complete the table, we are going to use the equations relating the probability mass function and the cumulative distribution function.

$$\mathrm{P}(x = k) = \mathrm{P}(X \leqslant k) - \mathrm{P}(X \leqslant k-1),\ k = 2, ..., 6 \text{ and } \mathrm{P}(x = 1) = \mathrm{P}(X \leqslant 1)$$

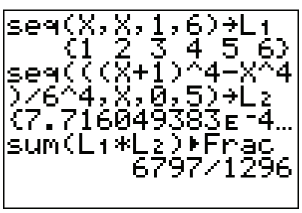
$$\mathrm{P}(x = 3) = \mathrm{P}(X \leqslant 3) - \mathrm{P}(X \leqslant 2) = \left(\frac{3}{6}\right)^4 - \left(\frac{2}{6}\right)^4 = \frac{81-16}{1296} = \frac{65}{1296}$$

$$\mathrm{P}(x = 4) = \mathrm{P}(X \leqslant 4) - \mathrm{P}(X \leqslant 3) = \left(\frac{4}{6}\right)^4 - \left(\frac{3}{6}\right)^4 = \frac{256-81}{1296} = \frac{175}{1296}$$

$$\mathrm{P}(x = 5) = \mathrm{P}(X \leqslant 5) - \mathrm{P}(X \leqslant 4) = \left(\frac{5}{6}\right)^4 - \left(\frac{4}{6}\right)^4 = \frac{625-256}{1296} = \frac{369}{1296}$$

x	1	2	3	4	5	6
$\mathrm{P}(X = x)$	$\frac{1}{1296}$	$\frac{15}{1296}$	$\frac{65}{1296}$	$\frac{175}{1296}$	$\frac{369}{1296}$	$\frac{671}{1296}$

iii) This can be done very quickly on a GDC by using the List menu. In the first list we input the dice scores, whilst in the second list we input the probabilities by using the formula for the cumulative distribution function.



So, the expected value is $\frac{6797}{1296}$.

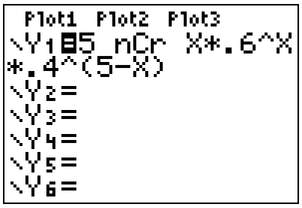
20 $n = 10, \sum_{i=1}^{10} x_i^2 = 1341$ and $s_n = 6.9$, $s_n^2 = \frac{\sum_{i=1}^{10} x_i^2}{n} - \bar{x}^2 \Rightarrow 6.9^2 = \frac{1341}{10} - \bar{x}^2 \Rightarrow$

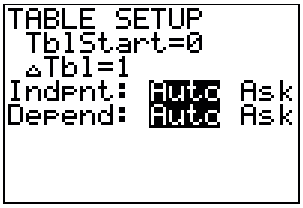
$\bar{x}^2 = 134.1 - 47.61 = 86.49 \Rightarrow \bar{x} = \sqrt{86.49} = 9.3$

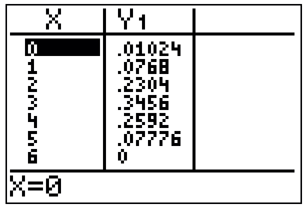
Exercise 17.2

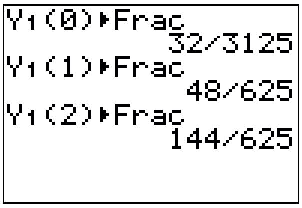
1 $X \sim \text{B}(n = 5,\ p = 0.6)$

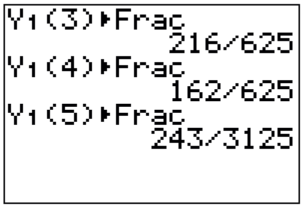
a) The easiest way to calculate the probabilities given by the formula is by using a calculator. We can use either the Function menu or the List menu.



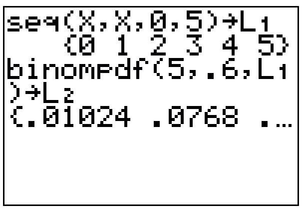


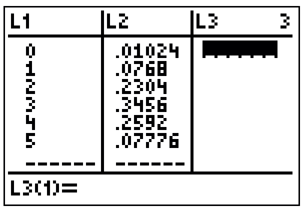






Or:

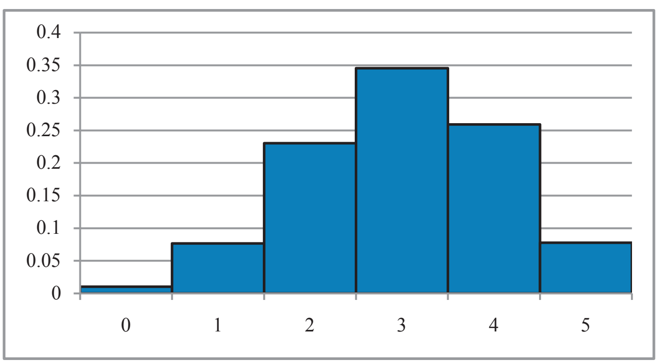




So, the table would be:

x	0	1	2	3	4	5
P(x)	$\frac{32}{3125}$	$\frac{48}{625}$	$\frac{144}{625}$	$\frac{216}{625}$	$\frac{162}{625}$	$\frac{243}{3125}$

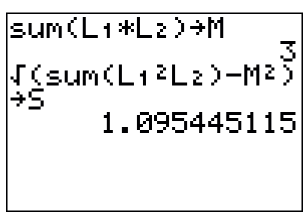
b)



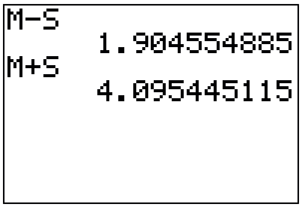
c) **i)** $\mu = np \Rightarrow \mu = 5 \times 0.6 = 3$

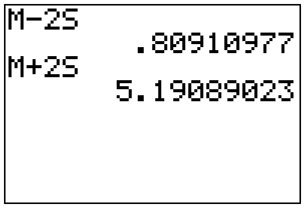
$\sigma = \sqrt{npq} \Rightarrow \sigma = \sqrt{5 \times 0.6 \times 0.4} = \sqrt{1.2} \approx 1.095$

ii) Again, the easiest way would be by using a calculator.



d)

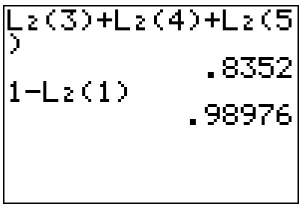




e) Now, we need to calculate the following probabilities: $\mathrm{P}(2 \leqslant x \leqslant 4)$ and $\mathrm{P}(1 \leqslant x \leqslant 5)$.

Whilst the first one can be calculated directly, the second would be easier to calculate by using the complementary event.

Notice that the index of the list member is one more than the value of the random variable.



We notice that the probability within one standard deviation, 0.8352, is much higher than the empirical one, 0.6827. The probability within two standard deviations, 0.989 76, is fairly close to the empirical one, 0.9545.

2 Let X be the number of respondents in favour of the decision. $X \sim \mathrm{B}(n = 20, p = 0.6)$.

a) $\mathrm{P}(X = 5) = \binom{20}{5} 0.6^5 \times 0.4^{15} \approx 0.001\,294\,4935$

b) $\mathrm{P}(X = 0) = \binom{20}{0} 0.6^0 \times 0.4^{20} \approx 0.000\,000\,010\,995$

c) This problem is easier to solve by using the complementary event; therefore,

$\mathrm{P}(X \geqslant 1) = 1 - \mathrm{P}(X = 0) = 1 - 0.000\,000\,010\,995 = 0.999\,999\,909\,005.$

d) Even though we can use the direct form, it is again easier to calculate the probability by using the complementary event; therefore,

$$\mathrm{P}(X \geqslant 2) = 1 - \mathrm{P}(X \leqslant 1) = 1 - \mathrm{P}(X = 0) - \mathrm{P}(X = 1)$$

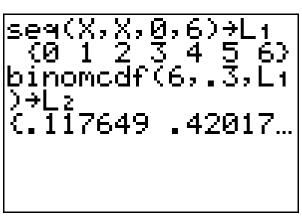
$$= 1 - 0.000\,000\,010\,995 - 0.000\,000\,329\,85$$

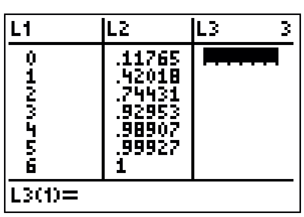
$$= 0.999\,999\,6592.$$

e) $\mu = np \Rightarrow \mu = 20 \times 0.6 = 12$

$\sigma = \sqrt{npq} \Rightarrow \sigma = \sqrt{20 \times 0.6 \times 0.4} = \sqrt{4.8} \approx 2.19$

3 **a)** In this part, we are going to use the binomial cumulative distribution function on a calculator.

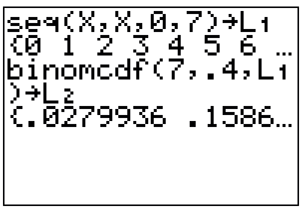


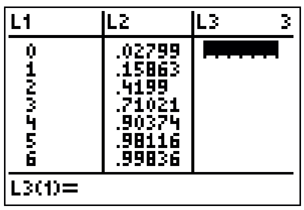


b)

Number of successes x	List the values of x	Write the probability statement	Explain it, if needed	Find the required probability
At most 3	0, 1, 2, 3	$P(x \leqslant 3)$	$P(x \leqslant 3)$	0.929 53
At least 3	3, 4, 5, 6	$P(x \geqslant 3)$	$1 - P(x \leqslant 2)$	0.255 69
More than 3	4, 5, 6	$P(x > 3)$	$1 - P(x \leqslant 3)$	0.070 47
Fewer than 3	0, 1, 2	$P(x \leqslant 2)$	$P(x \leqslant 2)$	0.744 31
Between 3 and 5 (inclusive)	3, 4, 5	$P(3 \leqslant x \leqslant 5)$	$P(x \leqslant 5) - P(x \leqslant 2)$	0.254 961
Exactly 3	3	$P(x = 3)$	$P(x = 3)$	0.185 22

4 **a)**





In the second GDC screen, we can see the cumulative probability values are all given except that for $x = 7$, but that one (which is the last term) is equal to 1.

b)

Number of successes x	List the values of x	Write the probability statement	Explain it, if needed	Find the required probability
At most 3	0, 1, 2, 3	$P(x \leqslant 3)$	$P(x \leqslant 3)$	0.710 208
At least 3	3, 4, 5, 6, 7	$P(x \geqslant 3)$	$1 - P(x \leqslant 2)$	0.580 096
More than 3	4, 5, 6, 7	$P(x > 3)$	$1 - P(x \leqslant 3)$	0.289 792
Fewer than 3	0, 1, 2	$P(x \leqslant 2)$	$P(x \leqslant 2)$	0.419 904
Between 3 and 5 (inclusive)	3, 4, 5	$P(3 \leqslant x \leqslant 5)$	$P(x \leqslant 5) - P(x \leqslant 2)$	0.561 2544
Exactly 3	3	$P(x = 3)$	$P(x = 3)$	0.290 304

5 **a)** This is not a binomial distribution since we don't have a sequence of several independent trials with equal probabilities.

b) If we choose the balls with replacement, then the trials become independent, so a sequence of three such trials with equal probabilities is a binomial distribution.

c) $Y \sim \text{B}\left(3, \frac{5}{8}\right)$

$$\text{P}(Y = y) = \binom{3}{y}\left(\frac{5}{8}\right)^{y}\left(\frac{3}{8}\right)^{3-y}, \; y = 0, 1, 2, 3$$

y	0	1	2	3
$\text{P}(Y = y)$	0.0527	0.2637	0.4395	0.2441

d) It is easier to calculate the complementary event, which is all three green balls.

$\text{P}(y \leqslant 2) = 1 - \text{P}(y = 3) = 1 - 0.2441 = 0.7559$

e) $\text{E}(Y) = np \Rightarrow \text{E}(Y) = 3 \times \frac{5}{8} = \frac{15}{8} = 1.875$

f) $\text{V}(Y) = npq \Rightarrow \text{V}(Y) = 3 \times \frac{5}{8} \times \frac{3}{8} = \frac{45}{64} = 0.703\,125$

g) The complementary event of some green balls would be no green balls will be chosen; therefore, we calculate the probability: $\text{P}(y \geqslant 1) = 1 - \text{P}(y = 0) = 1 - 0.0527 = 0.9473$.

6 Since Nick guesses every single question, the probability that he chooses the correct answer from five possible answers per question is $\frac{1}{5}$; therefore, the distribution is $X \sim \text{B}\left(10, \frac{1}{5}\right)$.

a)
```
binompdf(10,1/5,
0)
         .1073741824
```

b)–c)
```
binomcdf(10,1/10
,5)
         .9998530974
1-binomcdf(10,.2
,0)
         .8926258176
```

d) $\text{E}(X) = np \Rightarrow \text{E}(X) = 10 \times \frac{1}{5} = 2$

7 There are 10 houses and in each we have an alarm system that is 98% reliable, so the distribution is $X \sim \text{B}(10, 0.98)$.

a) $\text{P}(X = 10)$ **b)** $\text{P}(X \geqslant 5) = 1 - \text{P}(X \leqslant 4)$ **c)** $\text{P}(X \leqslant 8)$

```
binompdf(10,.98,
10)
         .8170728069
1-binomcdf(10,.9
8,4)
         .9999999875
```

```
binomcdf(10,.98,
8)
         .0161776407
```

Notice that the result in part **b** can be interpreted as 1.

8 Let X be the number of readers over 30 years of age. $X \sim \text{B}(15, 0.4)$.

a)–b)
```
1-binomcdf(15,.4
,9)→A
         .0338333029
binompdf(15,.4,1
0)→B
         .0244856421
```

c)
```
binomcdf(15,.4,1
0)→C
         .9906523392
A-B+C
                   1
```

We define the events by the letters A, B and C. Notice that the events satisfy the following relationship: $A \cap C = B$, $A \cup C = U$, where U is the universal set. Now, by using the addition formula for two sets, we calculate the probability of their union:
$\mathrm{P}(A \cup C) = \mathrm{P}(A) + \mathrm{P}(C) - \mathrm{P}(A \cap C) \Rightarrow 1 = \mathrm{P}(A) + \mathrm{P}(C) - \mathrm{P}(B)$.

9 Let X be the number of defective hard disks. $X \sim \mathrm{B}(50, 0.015)$.

a) $\mathrm{E}(X) = np \Rightarrow \mathrm{E}(X) = 50 \times 0.015 = 0.75$

b) $\mathrm{P}(X = 3) = \binom{50}{3} 0.015^3 \times 0.985^{47} \approx 0.032\,5112$

c) It is easier to calculate the probability that more than one hard disk is defective by considering the complementary event, i.e. one or no hard disk is defective.

$\mathrm{P}(X > 1) = 1 - \mathrm{P}(X \leqslant 1) = 1 - \mathrm{P}(X = 0) - \mathrm{P}(X = 1) = 1 - 0.469\,690 - 0.357\,632 = 0.172\,678$

10 Let X be the number of 'metallic grey' cars. $X \sim \mathrm{B}(20, 0.1)$.

a) $\mathrm{P}(X \geqslant 5) = 1 - \mathrm{P}(X \leqslant 4) = 1 - 0.956\,8255 = 0.043\,1745$

b) $\mathrm{P}(X \leqslant 6) = 0.997\,614$

c) $\mathrm{P}(X > 5) = 1 - \mathrm{P}(X \leqslant 5) = 0.011\,2531$

d) $\mathrm{P}(4 \leqslant X \leqslant 6) = \mathrm{P}(X \leqslant 6) - \mathrm{P}(X \leqslant 3) = 0.997\,614 - 0.867\,047 = 0.130\,567$

e) If more than 15 are not 'metallic grey', then at least 4 are 'metallic grey'.

$\mathrm{P}(X \leqslant 4) = 0.956\,826$

f) $\mathrm{E}(X) = np \Rightarrow \mathrm{E}(X) = 100 \times 0.1 = 10$

g) $\sigma = \sqrt{npq} \Rightarrow \sigma = \sqrt{100 \times 0.1 \times 0.9} = \sqrt{9} = 3$

h) According to the empirical rule, $\mathrm{P}(\mu - 2\sigma \leqslant X \leqslant \mu + 2\sigma) \approx 0.95$, and, in the probability model, we can calculate: $10 - 2 \times 3 \leqslant X \leqslant 10 + 2 \times 3 \Rightarrow 4 \leqslant X \leqslant 16$. So, $a = 4$ and $b = 16$.

11 Let X be the number of dogs with health insurance. $X \sim \mathrm{B}(100, 0.03)$.

a) $\mathrm{E}(X) = np \Rightarrow \mathrm{E}(X) = 100 \times 0.03 = 3$

b) $\mathrm{P}(X = 5) = \binom{100}{5} 0.03^5 0.97^{95} = 0.101\,308$

c) $\mathrm{P}(X \geqslant 11) = 1 - \mathrm{P}(X \leqslant 10) = 1 - 0.999\,785\,0751 = 0.000\,214\,925$

12 Let X be the number of heads observed. $X \sim \mathrm{B}(5, 0.5)$.

a)

	A	B	C	D
◆		=binompdf(		
1	0	0.03125		
2	1	0.15625		
3	2	0.3125		
4	3	0.3125		
5	4	0.15625		
6	5	0.03125		

B6 =0.03125

b) From the table, we can read that $\mathrm{P}(X = 0) = 0.031\,25$.

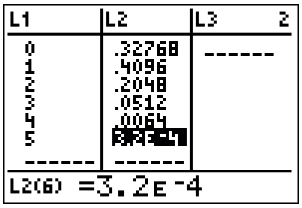
c) From the table, we can read that $\mathrm{P}(X = 5) = 0.031\,25$.

d) $P(X \geqslant 1) = 1 - P(X = 0) = 1 - 0.031\,25 = 0.968\,75$

e) If at least one tail is observed, then at most 4 heads are observed:

$P(X \leqslant 4) = 1 - P(X = 5) = 1 - 0.031\,25 = 0.968\,75$

f) Since 2 heads are observed in every 10 tosses, the probability is 0.2.



From the table, we can read that $P(X = 0) = 0.032\,768$.

From the table, we can read that $P(X = 5) = 0.000\,32$.

$P(X \geqslant 1) = 1 - P(X = 0) = 1 - 0.327\,68 = 0.672\,32$

If at least one tail is observed, then at most 4 heads are observed:

$P(X \leqslant 4) = 1 - P(X = 5) = 1 - 0.000\,32 = 0.999\,68$

13 Let X be the number of hits observed. $X \sim B\left(6, \frac{2}{5}\right)$.

a) $$P(X = 4) = \binom{6}{4}\left(\frac{2}{5}\right)^4\left(\frac{3}{5}\right)^2 = \cancel{15}\,3 \times \frac{16 \times 9}{5^{\cancel{6}\,5}} = \frac{432}{3125}$$

b) John has to miss with his first two throws and then he needs to hit the target with his third throw.

$$P(B) = \left(\frac{3}{5}\right)^2 \times \frac{2}{5} = \frac{18}{125}$$

Solution Paper 1 type

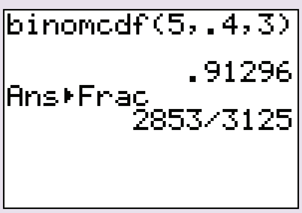
14 Let X be the number of days Alice watched the news. $X \sim B\left(5, \frac{2}{5}\right)$.

$$P(X \leqslant 3) = 1 - (P(X \leqslant 4) + P(X \leqslant 5)) = 1 - \left(\binom{5}{4}\left(\frac{2}{5}\right)^4\left(\frac{3}{5}\right) + \binom{5}{5}\left(\frac{2}{5}\right)^5\right) = 1 - \frac{16}{3125}(15 + 2) = \frac{3125 - 272}{3125} = \frac{2853}{3125}$$

We used the complementary event since it has less elementary outcomes and is therefore easier to calculate.

Solution Paper 2 type

14 We are going to use the binomial cumulative distribution function.



15 Let X be the number of cells that fail within a year. $X \sim \mathrm{B}\left(10, \frac{4}{5}\right)$.

a) $\mathrm{P}(X = 10) = \binom{10}{10}\left(\frac{4}{5}\right)^{10} = \frac{4^{10}}{5^{10}} \approx 0.107$

b) This is complementary to the event in part **a**.

$\mathrm{P}(X \leqslant 9) = 1 - \mathrm{P}(X = 10) = 1 - \frac{4^{10}}{5^{10}} \approx 0.893$

c) Again, we are going to use the complementary event since the calculation is much simpler.

If we have n cells, the probability that all n cells fail is: $\mathrm{P}(X = n) = \left(\frac{4}{5}\right)^n$.

$\mathrm{P}(X \leqslant n-1) = 1 - \mathrm{P}(X = n) = 0.95 \Rightarrow 1 - \left(\frac{4}{5}\right)^n = 0.95 \Rightarrow \frac{1}{20} = \left(\frac{4}{5}\right)^n \Rightarrow$

$n(\ln 4 - \ln 5) = -\ln 20 \Rightarrow n = \frac{\ln 20}{\ln 5 - \ln 4} \approx 13.4$

So, we need at least 14 cells.

Exercise 17.3

1 $X \sim \mathrm{P}_O(m = 3)$

a) $\mathrm{P}(x = 5) = \frac{e^{-3} \times 3^5}{5!} \approx 0.100\,82$

```
e^(-3)*3^5/5!
         .1008188134
poissonpdf(3,5)
         .1008188134
```

b) $\mathrm{P}(x < 5) = \mathrm{P}(x = 0) + P(x = 1) + \mathrm{P}(x = 2) + \mathrm{P}(x = 3) + \mathrm{P}(x = 4)$

$$= \frac{e^{-3} \times 3^0}{0!} + \frac{e^{-3} \times 3^1}{1!} + \frac{e^{-3} \times 3^2}{2!} + \frac{e^{-3} \times 3^3}{3!} + \frac{e^{-3} \times 3^4}{4!} = e^{-3} \times \left(1 + 3 + \frac{9}{2} + \frac{9}{2} + \frac{27}{8}\right) \approx 0.8153$$

```
e^(-3)(13+27/8)
         .8152632445
poissoncdf(3,4)
         .8152632446
```

c) In this case, we need to use the complementary event:

$\mathrm{P}(x \geqslant 5) = 1 - \mathrm{P}(x < 5) = 1 - e^{-3} \times \frac{131}{8} \approx 0.1847$

```
1-e^(-3)*131/8
         .1847367555
1-poissoncdf(3,4
)
         .1847367554
```

d) We notice that the event $x \geqslant 5$ is a complete subset of the event $x \geqslant 3$; therefore, the conditional probability is to be calculated as follows:

$$P(x \geqslant 5 | x \geqslant 3) = \frac{P(x \geqslant 5)}{P(x \geqslant 3)} = \frac{1 - P(x \leqslant 4)}{1 - P(x \leqslant 2)} \approx 0.3203$$

```
(1-poissoncdf(3,
4))/(1-poissoncd
f(3,2))
      .3202731946
```

2 $X \sim P_O(m = 5)$

a) $P(x = 5) = \dfrac{e^{-5} \times 5^5}{5!} \approx 0.1755$

```
e^(-5)*5^5/5!
      .1754673698
poissonpdf(5,5)
      .1754673698
```

b) $P(x < 4) = P(x = 0) + P(x = 1) + P(x = 2) + P(x = 3)$

$$= \frac{e^{-5} \times 5^0}{0!} + \frac{e^{-5} \times 5^1}{1!} + \frac{e^{-5} \times 5^2}{2!} + \frac{e^{-5} \times 5^3}{3!} = e^{-5} \times \left(1 + 5 + \frac{25}{2} + \frac{125}{6}\right) \approx 0.2650$$

```
e^(-5)*(6+25/2+1
25/6)
      .2650259153
poissoncdf(5,3)
      .2650259153
```

c) In this case, we need to use the complementary event:

$$P(x \geqslant 4) = 1 - P(x < 4) = 1 - e^{-5} \times \frac{118}{3} \approx 0.7350$$

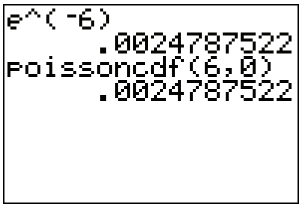
```
1-e^(-5)*118/3
      .7349740847
1-poissoncdf(5,3
)
      .7349740847
```

d) $P(x \leqslant 6 | x \geqslant 4) = \dfrac{P(4 \leqslant x \leqslant 6)}{P(x \geqslant 4)} = \dfrac{P(x \leqslant 6) - P(x \leqslant 3)}{P(x \geqslant 4)} \approx 0.6764$

```
(poissoncdf(5,6)
-poissoncdf(5,3)
)/(1-poissoncdf(
5,3))
      .6764286771
```

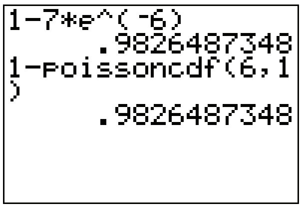
3 $X \sim P_O\,(m = 6)$

a) $P\,(x = 0) = e^{-6} \times \frac{6^0}{0\,!} \approx 0.0025$



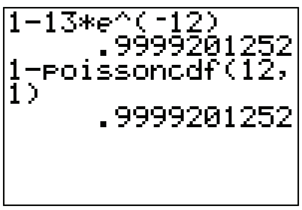
b) We have 'at least' in the question and therefore we are going to use the complementary event.

$P\,(x \geqslant 2) = 1 - P\,(x \leqslant 1) = 1 - (e^{-6} + e^{-6} \times 6) = 1 - 7e^{-6} \approx 0.9826$



c) We are now going to change the Poisson's variable to the new parameter and use the complementary event, as in the previous part.

$Y \sim P_o\,(m = 12) \Rightarrow P\left(y \geqslant 2\right) = 1 - P\left(y \leqslant 1\right) = 1 - \left(e^{-12} + e^{-12} \times 12\right) = 1 - 13e^{-12} \approx 0.9999$

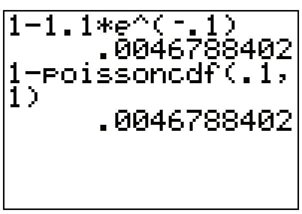


4 Let X be the number of unacceptable DVDs. $X \sim P_O\,(m = 0.1)$.

a) $P\,(x = 0) = e^{-0.1} \approx 0.9048$

b) 'More than one error' calls again for the complementary event.

$P\,(x > 1) = 1 - P\,(x \leqslant 1) = 1 - \left(e^{-0.1} + e^{-0.1} \times 0.1\right) = 1 - 1.1e^{-0.1} \approx 0.004\,679$



c) This is an event that combines two independent events; therefore, the probability of the event that both DVDs have no error is the product of the probabilities that each has no error.

$P\,(C) = P\,(x = 0) \times P\,(x = 0) = (P\,(x = 0))^2 = \left(e^{-0.1}\right)^2 = e^{-0.2} \approx 0.8187$

5 **a)** If there are 0.024 serious injuries and deaths per million vehicle-kilometres, then there are 24 serious injuries and deaths per billion kilometres. $X \sim P_O\,(m = 24)$.

i) $\mathrm{P}(x \leqslant 15) \approx 0.0344$

ii) $\mathrm{P}(x \geqslant 20) = 1 - \mathrm{P}(x \leqslant 19) \approx 0.8197$

```
poissoncdf(24,15
)
          .034400094
1-poissoncdf(24,
19)
        .8197394858
```

b) The rate for light motor vehicles was 0.036 per million kilometres, so there were 36 serious injuries and deaths per billion kilometres. $Y \sim \mathrm{P_O}(m = 36)$.

i) $\mathrm{P}(y \leqslant 15) \approx 0.000\,0653$

ii) $\mathrm{P}(y \geqslant 20) = 1 - \mathrm{P}(x \leqslant 19) \approx 0.9986$

```
poissoncdf(36,15
)
  6.528528136E-5
1-poissoncdf(36,
19)
        .9985779307
```

6 Let X be the number of passengers that arrive at the security checkpoint per 10-minute period. $X \sim \mathrm{P_O}(m = 8)$.

a) $\mathrm{P}(x = 8) = e^{-8} \times \dfrac{8^8}{8!} \approx 0.1396$

b) $\mathrm{P}(x \leqslant 5) \approx 0.1912$

c) $\mathrm{P}(x \geqslant 4) = 1 - \mathrm{P}(x \leqslant 3) \approx 0.9576$

```
poissonpdf(8,8)
      .139586532
poissoncdf(8,5)
     .1912360621
1-poissoncdf(8,3
)
      .957619888
```

7 The variable is the same as in the previous question.

a) Since the period has been extended to 20 minutes, we expect twice the number of passengers.

$Y \sim \mathrm{P_O}(m = 16) \Rightarrow \mathrm{P}(y = 3) = e^{-16} \times \dfrac{16^3}{3!} \approx 0.000\,076\,82$

b) Since we are looking for three passengers to arrive at the checkpoint in two independent 10-minute periods, we need to multiply the individual probabilities.

$\mathrm{P}(x = 3) \times \mathrm{P}(x = 3) = \mathrm{P}^2(x = 3) \approx 0.000\,8195$

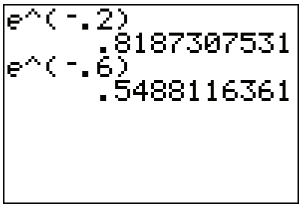
```
poissonpdf(16,3)

  7.682401261E-5
poissonpdf(8,3)²

  8.194561345E-4
```

8 Let X be the number of hits per second. $X \sim \text{P}_\text{O}\,(m = 0.2)$.

a) $\text{P}\,(x = 0) = e^{-0.2} \approx 0.8187$

b) Since the period of hits has been extended to 3 seconds, we have to change the parameter of Poisson's variable to: $m = 3 \cdot\ 0.2 = 0.6$. $Y \sim \text{P}_\text{O}\,(m = 0.6)$.

$\text{P}\,(y = 0) = e^{-0.6} \approx 0.5488$



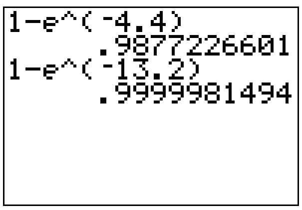
Note: We arrive at the same result if we consider those three seconds as three independent events, in which there are no hits in any second; therefore: $\text{P}\,(y = 0) = (\text{P}\,(x = 0))^3 = \left(e^{-0.2}\right)^3 = e^{-0.6}$.

9 Let X be the number of faults per square metre. $X \sim \text{P}_\text{O}\,(m = 4.4)$.

a) $\text{P}\,(x \geqslant 1) = 1 - \text{P}\,(x = 0) = 1 - e^{-4.4} \approx 0.9877$

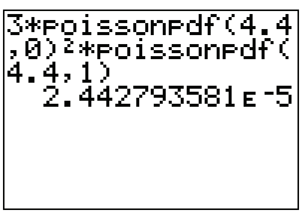
b) Since we are trying to find the number of faults in 3 square metres, we need to change the parameter of Poisson's variable: $m = 3 \cdot\ 4.4 = 13.2$. $Y \sim \text{P}_\text{O}\,(m = 13.2)$.

$\text{P}\,(y \geqslant 1) = 1 - \text{P}\,(y = 0) = 1 - e^{-13.2} \approx 0.999\,998$



c) If three pieces contain only one fault, the other two pieces contain no fault.

$\text{P}\,(C) = 3 \times (\text{P}\,(x = 0))^2 \times \text{P}\,(x = 1) \approx 0.000\,0244$

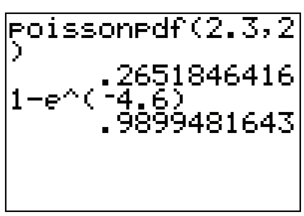


10 Let X be the number of flaws per metre. $X \sim \text{P}_\text{O}\,(m = 2.3)$.

a) $\text{P}\,(x = 2) = e^{-2.3} \times \dfrac{2.3^2}{2\,!} \approx 0.2652$

b) Given that we are now looking for the number of flaws in 2 metres of the wire, we need to change the parameter of Poisson's variable: $m = 2 \cdot\ 2.3 = 4.6$. $Y \sim \text{P}_\text{O}\,(m = 4.6)$.

$\text{P}\,(y \geqslant 1) = 1 - \text{P}\,(y = 0) = 1 - e^{-4.6} \approx 0.9899$

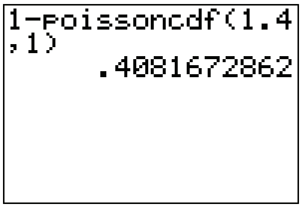


11 **a)** Since the rate is 15 patients per hour, we expect a quarter of 15 within 15 minutes; therefore, the Poisson variable will have a parameter of: $m = \frac{15}{4} = 3.75$. $X \sim \mathrm{P_O}\,(m = 3.75)$.

$\mathrm{P}\,(x = 6) = e^{-3.75} \times \frac{3.75^6}{6!} \approx 0.0908$

b) Given that the first operator fails to answer 1% of the calls and he/she receives 20 calls, the number of expected unanswered calls is 0.2. In a similar way, the number of expected unanswered calls for the second operator is 1.2. Combined, the total expected number of unanswered calls is 1.4. $Y \sim \mathrm{P_O}\,(m = 1.4)$.

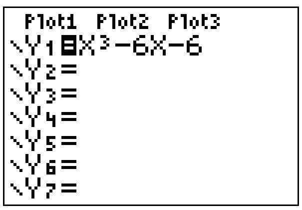
$\mathrm{P}\,(y \geqslant 2) = 1 - \mathrm{P}\,(y \leqslant 1) \approx 0.4082$

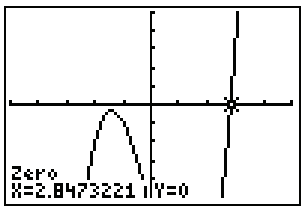


12 **a)** $X \sim \mathrm{P_O}\,(\mu)$

$\mathrm{P}\,(x = 3) = \mathrm{P}\,(x = 0) + \mathrm{P}\,(x = 1) \Rightarrow e^{-\mu}\frac{\mu^3}{3!} = e^{-\mu} + e^{-\mu} \times \mu \,\Big/ \div e^{-\mu} \Rightarrow \frac{\mu^3}{6} = 1 + \mu \Rightarrow \mu^3 - 6\mu - 6 = 0$

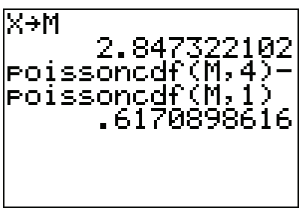
To solve this cubic equation, we need to use a GDC.





b) Once we calculate the value of the parameter, we store it in the GDC's memory.

$\mathrm{P}\,(2 \leqslant x \leqslant 4) = \mathrm{P}\,(x \leqslant 4) - \mathrm{P}\,(x \leqslant 1) \approx 0.6171$

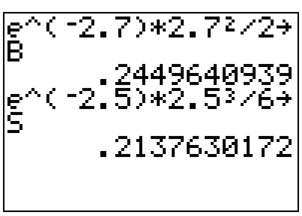


13 **a)** **i)** Let B be the number of mistakes made by Mr Brown. $B \sim \mathrm{P_O}\,(m = 2.7)$.

$\mathrm{P}\,(B = 2) = e^{-2.7}\frac{2.7^2}{2!} \approx 0.2450$

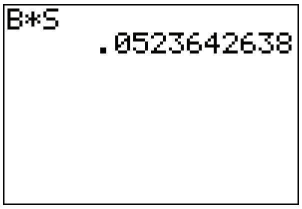
ii) Let S be the number of mistakes made by Mr Smith. $S \sim \mathrm{P_O}\,(m = 2.5)$.

$\mathrm{P}\,(S = 3) = e^{-2.5}\frac{2.5^3}{3!} \approx 0.2138$



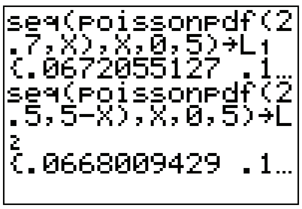
iii) Since both of them have to make that many mistakes and those events occur independently, we need to multiply the individual probabilities.

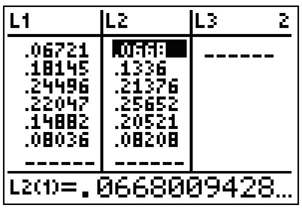
$\mathrm{P}(B = 2) \cdot \ \mathrm{P}(S = 3) = 0.052\,36$



b) If we combine Mr Brown and Mr Smith and the total number of mistakes is five, all the possible outcomes are given in the following table.

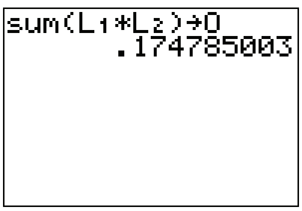
$B = x$	0	1	2	3	4	5
$\mathrm{P}(B = x)$	0.0672	0.1815	0.2450	0.2205	0.1488	0.0804
$S = y$	5	4	3	2	1	0
$\mathrm{P}(S = y)$	0.0668	0.1336	0.2138	0.2565	0.2052	0.0821





So, the probability that both of them made a combined total of five mistakes is:

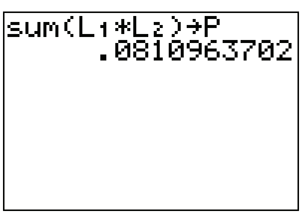
$$\mathrm{P}(B + S = 5) = \sum_{k=0}^{5} \mathrm{P}(B = k) \times \mathrm{P}(S = 5 - k) \approx 0.1748$$

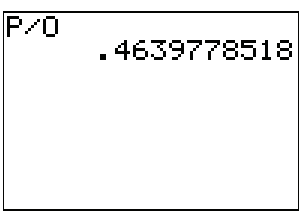


We also notice that in the first three outcomes in the table Mr Brown made fewer mistakes than Mr Smith.

$$\mathrm{P}((B < S) \cap (B + S = 5)) = \sum_{k=0}^{3} \mathrm{P}(B = k) \times \mathrm{P}(S = 5 - k) \approx 0.0811$$

$$\mathrm{P}(B < S | \, B + S = 5) = \frac{\mathrm{P}((B < S) \cap (B + S = 5))}{\mathrm{P}(B + S = 5)} = \frac{0.0811}{0.1728} \approx 0.464$$





Exercise 17.4

1 $f(x) = \begin{cases} kx^2 + \dfrac{3}{2} & 0 \leqslant x \leqslant 1 \\ 0 & \text{otherwise} \end{cases}$

a) $\int_0^1 f(x)\,dx = 1 \Rightarrow \int_0^1 \left(kx^2 + \frac{3}{2}\right) dx = 1 \Rightarrow \left(\frac{kx^3}{3} + \frac{3x}{2}\right)\Bigg]_0^1 = 1 \Rightarrow \frac{k}{3} + \frac{3}{2} = 1 \Rightarrow k = -\frac{3}{2}$

b) $\mathrm{P}(x > 0.5) = \int_{0.5}^1 \left(-\frac{3}{2}x^2 + \frac{3}{2}\right) dx = \left(\frac{-x^3}{2} + \frac{3x}{2}\right)\Bigg]_{0.5}^1 = -\frac{1}{2} + \frac{3}{2} + \frac{1}{16} - \frac{3}{4} = \frac{5}{16}$

c) $\mathrm{P}(0 < x < 0.5) = 1 - \mathrm{P}(x > 0.5) = 1 - \frac{5}{16} = \frac{11}{16}$

Note: We could have calculated the integral from 0 to 0.5, but, if we make no mistakes, it is easier to use the previously calculated results.

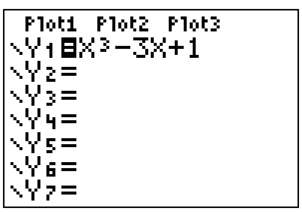
d) The mode is the most frequent observation, and, as such, it is the maximum of the probability density function. In this case, it is $x = 0$ since the vertex of the parabola lies at $\left(0, \frac{3}{2}\right)$. Now, we calculate the mean value.

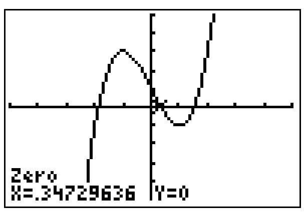
$$\mu = \int_0^1 x\, f(x)\,dx = \int_0^1 \left(\frac{3}{2}x - \frac{3}{2}x^3\right) dx = \left(\frac{3}{4}x^2 - \frac{3}{8}x^4\right)\Bigg]_0^1 = \frac{3}{4} - \frac{3}{8} = \frac{3}{8}$$

Let m be the median.

$$\int_0^m f(x)\,dx = \frac{1}{2} \Rightarrow \frac{3}{2}\int_0^m (1 - x^2)\,dx = \frac{1}{2} \Rightarrow \left(x - \frac{x^3}{3}\right)\Bigg]_0^m = \frac{1}{3} \Rightarrow m - \frac{m^3}{3} = \frac{1}{3} \Rightarrow m^3 - 3m + 1 = 0$$

We need a GDC to solve this cubic equation. Sometimes, cubic equations with rational solutions can be factorized, but this one has an irrational solution and therefore cannot be factorized on the set of rational numbers.





The cubic equation has three roots, but the only root that satisfies the condition set is 0.347. The other two roots are less than 0 and greater than 1.

$$\sigma^2 = \int_0^1 x^2 f(x)\,dx - \mu^2 = \int_0^1 \left(\frac{3}{2}x^2 - \frac{3}{2}x^4\right) dx - \mu^2 = \left(\frac{1}{2}x^3 - \frac{3}{10}x^5\right)\Bigg]_0^1 - \left(\frac{3}{8}\right)^2 = \frac{1}{5} - \frac{9}{64} = \frac{19}{320}$$

$$\sigma = \sqrt{\frac{19}{320}} = \frac{\sqrt{45}}{40} \approx 0.2437$$

2 $f(x) = \begin{cases} k(5-2x) & 0 \leqslant x \leqslant 2 \\ 0 & \text{otherwise} \end{cases}$

a) $\int_0^2 f(x)\,dx = 1$

```
EQUATION SOLVER
eqn:0=fnInt(K(5-
2X),X,0,2)-1
```

```
fnInt(K(5-2X)…=0
▪K=.16666666666…
 X=.34729635533…
 bound={-1ᴇ99,1…
▪left-rt=0
```

```
K▸Frac
             1/6
```

The value of x is not relevant to this calculation. It simply represents the old value that we calculated in the previous question.

b) $P(x > 1.5) = \int_{1.5}^{2} f(x)\,dx$

c) $P(0.5 < x < 1.5) = \int_{0.5}^{1.5} f(x)\,dx$

```
fnInt(K(5-2X),X,
1.5,2)▸Frac
             1/8
fnInt(K(5-2X),X,
0.5,1.5)▸Frac
             1/2
```

d) $\mu = \int_0^2 x\,f(x)\,dx$

$\sigma^2 = \int_0^2 x^2\,f(x)\,dx - \mu^2 \Rightarrow \sigma = \sqrt{\int_0^2 x^2\,f(x)\,dx - \mu^2}$

```
fnInt(KX(5-2X),X
,0,2)▸Frac
             7/9
√(fnInt(KX²(5-2X
),X,0,2)-(7/9)²)
      .5328701693
```

We have calculated the mean and standard deviation first since they have very similar inputs on a calculator.

$\int_0^m f(x)\,dx = \frac{1}{2}$

```
EQUATION SOLVER
eqn:0=fnInt(K(5-
2X),X,0,M)-1/2
```

```
fnInt(K(5-2X)…=0
 K=.16666666666…
 X=.34729635533…
▪M=.69722436226…
 bound={-1ᴇ99,1…
▪left-rt=0
```

The calculated values for the mean, median and standard deviation are $\frac{7}{9}$, 0.697 and 0.533 respectively.

3 $f(x) = \begin{cases} 2x - x^3 & 0 \leqslant x \leqslant k \\ 0 & \text{otherwise} \end{cases}$

a) $\int_0^k f(x)\,dx = 1 \Rightarrow \int_0^k (2x - x^3)\,dx = 1 \Rightarrow \left.\left(x^2 - \frac{x^4}{4}\right)\right]_0^k = 1 \Rightarrow k^2 - \frac{k^4}{4} = 1 \Rightarrow k^4 - 4k^2 + 4 = 0$

To find the value of k, we need to solve the biquadratic equation by using a simple substitution.

$k^2 = t \Rightarrow t^2 - 4t + 4 = 0 \Rightarrow (t-2)^2 = 0 \Rightarrow t = 2 \Rightarrow k = \sqrt{2}$

We have only taken the positive value of k since k cannot be negative.

b) $P(x > 0.5) = \int_{0.5}^{\sqrt{2}} f(x)\,dx = \int_{0.5}^{\sqrt{2}} (2x - x^3)\,dx = \left.\left(x^2 - \frac{x^4}{4}\right)\right]_{0.5}^{\sqrt{2}} = 2 - 1 - \frac{1}{4} + \frac{1}{64} = \frac{49}{64}$

c) This is the complementary event of the event in part **b**.

$P(0 < x < 0.5) = 1 - P(x > 0.5) = 1 - \frac{49}{64} = \frac{15}{64}$

d) $\mu = \int_0^{\sqrt{2}} x\,f(x)\,dx = \int_0^{\sqrt{2}} (2x^2 - x^4)\,dx = \left.\left(\frac{2x^3}{3} - \frac{x^5}{5}\right)\right]_0^{\sqrt{2}} = \frac{4\sqrt{2}}{3} - \frac{4\sqrt{2}}{5} = \frac{8\sqrt{2}}{15} \approx 0.754$

$\int_0^m f(x)\,dx = \frac{1}{2} \Rightarrow \int_0^m (2x - x^3)\,dx = \frac{1}{2} \Rightarrow \left.\left(x^2 - \frac{x^4}{4}\right)\right]_0^m = \frac{1}{2} \Rightarrow m^2 - \frac{m^4}{4} = \frac{1}{2} \Rightarrow m^4 - 4m^2 + 2 = 0$

Again, to find the value of m, we need to solve the biquadratic equation by using a simple substitution.

$m^2 = r \Rightarrow r^2 - 4r + 2 = 0 \Rightarrow r = \frac{4 \pm \sqrt{16-8}}{2} = 2 \pm \sqrt{2} \Rightarrow r = 2 - \sqrt{2} \Rightarrow m = \sqrt{2 - \sqrt{2}} \approx 0.765$

We have discarded the other possible solution for r, as well as the negative solution for m, since they exceed the interval $\left[0, \sqrt{2}\right]$.

$\sigma^2 = \int_0^2 x^2 f(x)\,dx - \mu^2 \Rightarrow \sigma^2 = \int_0^{\sqrt{2}} (2x^3 - x^5)\,dx - \left(\frac{8\sqrt{2}}{15}\right)^2 = \left.\left(\frac{x^4}{2} - \frac{x^6}{6}\right)\right]_0^{\sqrt{2}} - \frac{128}{225}$

$= 2 - \frac{4}{3} - \frac{128}{225} = \frac{22}{225} \Rightarrow \sigma = \sqrt{\frac{22}{225}} = \frac{\sqrt{22}}{15} \approx 0.3127$

4 $f(x) = \begin{cases} k(x+1) & 0 \leqslant x \leqslant 1 \\ 2kx^2 & 1 \leqslant x \leqslant 2 \\ 0 & \text{otherwise} \end{cases}$

a) $\int_0^1 f(x)\,dx + \int_1^2 f(x)\,dx = 1 \Rightarrow \int_0^1 (k(x+1))\,dx + \int_1^2 (2kx^2)\,dx = 1 \Rightarrow$

$\left.\left(k\left(\frac{x^2}{2} + x\right)\right)\right]_0^1 + \left.\left(\frac{2kx^3}{3}\right)\right]_1^2 = 1 \Rightarrow \frac{k}{2} + k + \frac{16k}{3} - \frac{2k}{3} = 1 \Rightarrow \frac{37k}{6} = 1 \Rightarrow k = \frac{6}{37}$

b) In this case, since the complementary event is just using one part of the piecewise function, it is simpler to calculate the probability by using the complementary event.

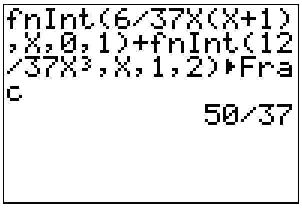
$P(x > 0.5) = 1 - P(x < 0.5) = 1 - \int_0^{0.5} \frac{6}{37}(x+1)\,dx = 1 - \left.\frac{6}{37}\left(\frac{x^2}{2} + x\right)\right]_0^{0.5} = 1 - \frac{6}{37}\left(\frac{1}{8} + \frac{1}{2}\right)$

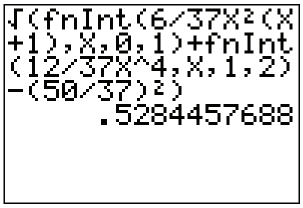
$= 1 - \frac{\cancel{6}\,3}{37} \times \frac{5}{\cancel{8}\,4} = \frac{148 - 15}{148} = \frac{133}{148}$

c) $\mathrm{P}(1 < x < 1.5) = \int_1^{1.5}\left(\frac{12}{37}x^2\right)dx = \left(\frac{4x^3}{37}\right)\Bigg]_1^{1.5} = \frac{27}{74} - \frac{4}{37} = \frac{19}{74}$

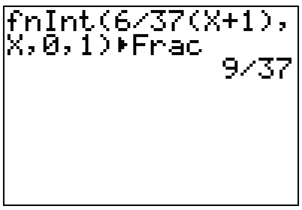
d) $\mu = \int_0^1 \frac{6}{37}x(x+1)\,dx + \int_1^2 \frac{12}{37}x^3 dx$

$\sigma = \sqrt{\left(\int_0^1 \frac{6}{37}x^2(x+1)\,dx + \int_1^2 \frac{12}{37}x^4 dx\right) - \mu^2}$

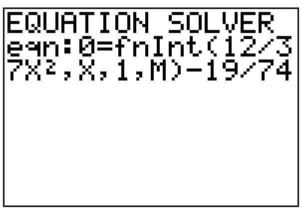


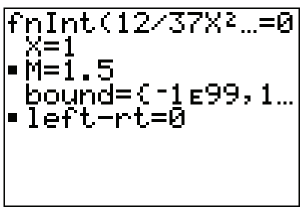


In order to find the median, we need to find the interval in which the value of m lies. Since the first part of the area under the probability distribution function is less than 0.5, we can conclude that the value of m is in the second interval.



The remaining part of the area is: $\frac{1}{2} - \frac{9}{37} = \frac{37-18}{74} = \frac{19}{74}$.





So, the calculated values of the mean, median and standard deviation are $\frac{50}{37}$, $\frac{3}{2}$ and 0.528 respectively.

5 $f(x) = \begin{cases} 2kx & 0 \leqslant x < 1 \\ 2kx^2 & 1 \leqslant x < 2 \\ k(8-2x) & 2 \leqslant x \leqslant 4 \\ 0 & \text{otherwise} \end{cases}$

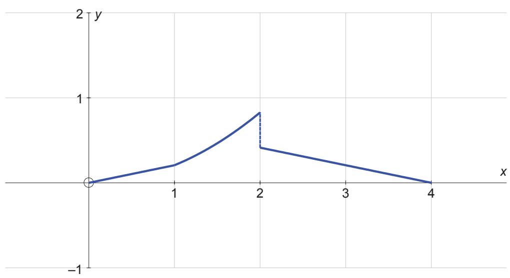
b) It is difficult to sketch the graph before we find the value of k; therefore, we are going to do part **b** first and then part **a**.

$\int_0^1 (2kx)\,dx + \int_1^2 (2kx^2)\,dx + \int_2^4 (k(8-2x))\,dx = 1 \Rightarrow$

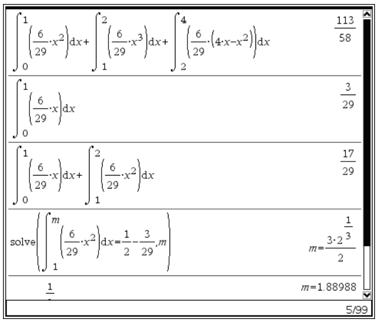
$(kx^2)\Big]_0^1 + \left(\frac{2kx^3}{3}\right)\Bigg]_1^2 + (k(8x - x^2))\Big]_2^4 = 1 \Rightarrow k + \frac{16k}{3} - \frac{2k}{3} + 16k - 12k = 1 \Rightarrow$

$\frac{29k}{3} = 1 \Rightarrow k = \frac{3}{29}$

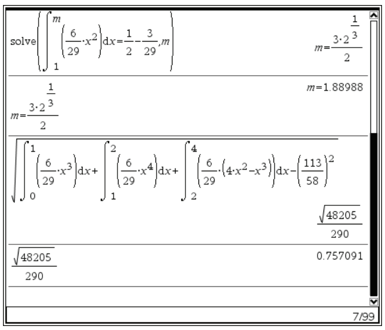
a)



c)

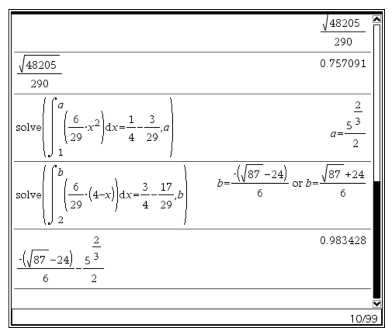


To find the median, we had to establish the interval in which the median lies. We saw that the area under the first interval is less than 0.5, whilst the area under the first two is greater than 0.5, and therefore the median must lie in the second interval. To calculate the mean value, we must not forget to subtract the area under the first interval.



The values of the mean, median and standard deviation are $\frac{113}{58}$, 1.89 and 0.757 respectively.

d) Looking at the previous calculation, since the area under the first interval is $\frac{3}{29} < \frac{1}{4}$, we can conclude that the first quartile (a) lies in the second interval. Additionally, since the area under the first two intervals is $\frac{17}{29} < \frac{3}{4}$, we can conclude that the third quartile (b) lies in the third interval. Again, we must not forget to subtract the areas of the previous intervals when calculating the quartiles.



We have taken only one solution for the third quartile since the other solution lies outside the interval $[2, 4]$. So, the interquartile range is 0.983 (correct to three significant figures).

6 $$f(x) = \begin{cases} \frac{15}{76}\left(x^4 - 2x^2 + 2\right) & 0 \leqslant x \leqslant 1 \\ -\frac{15}{8056}(15x - 121) & 1 < x \leqslant \frac{121}{15} \\ 0 & \text{otherwise} \end{cases}$$

a) $\mu = \int_0^1 x\, f(x)\, dx + \int_1^{\frac{121}{15}} x\, f(x)\, dx \approx 2.47$

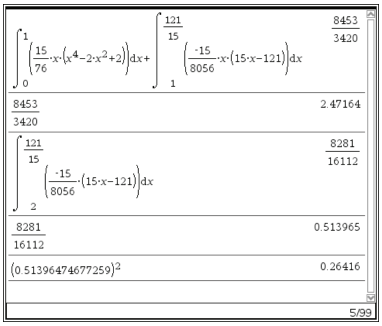
Since x is measured in tens of hours, the mean life is 24.7 hours.

b) Since the value of x is measured in tens of hours, $x = 2$.

$$P(x > 2) = \int_2^{\frac{121}{15}} \left(-\frac{15}{8056}(15x - 121) \right) dx$$

So, the probability that a battery will last at least 20 hours is 0.514.

c) Since both batteries have to work for the unit to work, we need to multiply the individual probabilities that each battery works for more than 20 hours. (The working of one battery is independent of the other battery.) The probability that the unit will work is 0.264.



7 $f(y) = \begin{cases} \frac{3}{500} y(10 - y) & 0 \leqslant y \leqslant 10 \\ 0 & \text{otherwise} \end{cases}$

a) $\mu = \frac{3}{500}\int_0^{10} (10y^2 - y^3)\, dy = \frac{3}{500}\left(\frac{10y^3}{3} - \frac{y^4}{4} \right)\Bigg]_0^{10} = \frac{3}{500} \times 10\,000\left(\frac{1}{3} - \frac{1}{4} \right) = 60 \times \frac{1}{12} = 5$

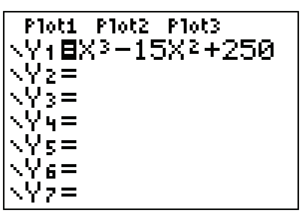
Since y is measured in tens of hours, the mean value is 50 hours.

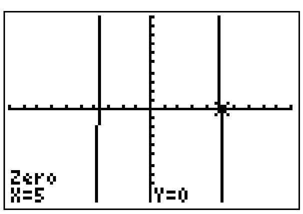
b) $\frac{3}{500}\int_0^{m} (10y - y^2)\, dy = \frac{1}{2} \Rightarrow \frac{3}{500}\left(5y^2 - \frac{y^3}{3} \right)\Bigg]_0^{m} = \frac{1}{2} \Rightarrow 5m^2 - \frac{m^3}{3} = \frac{250}{3} \Rightarrow m^3 - 15m^2 + 250 = 0$

This cubic equation can be factorized, for which we can use synthetic division (Horner's algorithm):

$m^3 - 15m^2 + 250 = 0 \Rightarrow (m - 5)(m^2 - 10m - 50) = 0 \Rightarrow m = 5$

However, it is easier to solve it by using a GDC.





So, the median value is 50 hours.

c) $\sigma^2 = \frac{3}{500}\int_0^{10} (10y^3 - y^4)\, dy - 5^2 = \frac{3}{500}\left(\frac{5}{2}y^4 - \frac{1}{5}y^5 \right)\Bigg]_0^{10} - 25 = \frac{3}{500} \times 10\,000\left(\frac{5}{2} - 2 \right) - 25$

$= 60 \times \frac{1}{2} - 25 = 5 \Rightarrow \sigma = \sqrt{5} \approx 2.24$

So, the standard deviation is 22.4 hours.

d) $P(y>8)=\dfrac{3}{500}\displaystyle\int_8^{10}\left(10y-y^2\right)dy=\dfrac{3}{500}\left(5y^2-\dfrac{y^3}{3}\right)\Bigg]_8^{10}$

$$=\frac{3}{500}\times\left(\left(500-\frac{1000}{3}\right)-\left(320-\frac{512}{3}\right)\right)=\frac{3}{500}\times\left(\frac{500-448}{3}\right)=\frac{52}{500}=\frac{13}{125}$$

e) **i)** Since the bulbs should last for more than 80 hours, the value of the variable y should be greater than 8.

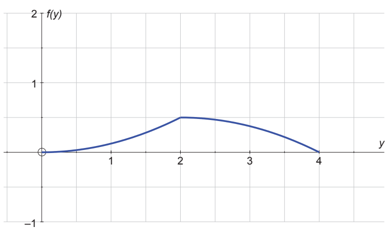
$$P(y>8)\times P(y>8)=\left(\frac{13}{125}\right)^2=\frac{169}{15\,625}$$

ii) 'At least one has to be replaced' is to be calculated using the complementary event. The complementary event is that no battery has to be replaced (both batteries will last for more than 8 hours).

$$1-P(y>8)\times P(y>8)=1-\frac{169}{15\,625}=\frac{15\,456}{15\,625}$$

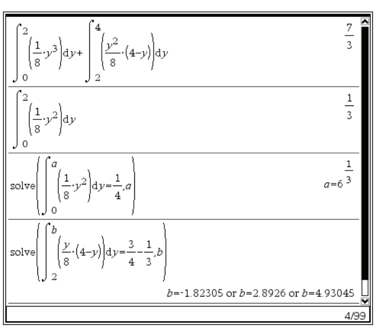
8 $f(y)=\begin{cases}\dfrac{1}{8}y^2 & 0\leqslant y<2\\ \dfrac{y}{8}(4-y) & 2\leqslant y\leqslant 4\\ 0 & \text{otherwise}\end{cases}$

a)

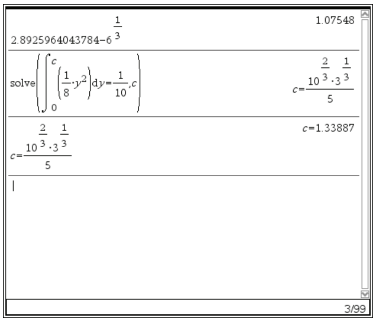


b) $\mu=\displaystyle\int_0^2 x\,f(x)\,dx+\int_2^4 x\,f(x)\,dx=\frac{7}{3}$

c) We need to find the area under the curve in the first integral. We notice that the area is $\dfrac{1}{3}>\dfrac{1}{4}$, so the first quartile (a) lies in the first interval. At the same time, since $\dfrac{1}{3}<\dfrac{3}{4}$, we can conclude that the third quartile (b) lies in the second interval. Notice that the GDC gives us three possible values of b, but only the middle one lies in the second interval. Finally, $b-a=2.8926-1.81712\approx 1.08$. So, the interquartile range is 108 barrels.



d) Since 10% is even less than 25%, we can conclude that the pumps will need replacing within the first interval.



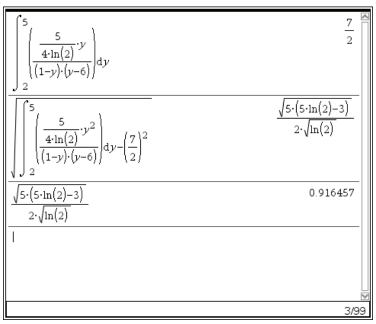
So, if the production falls below 134 barrels, we need to replace the pumps.

9 $f(y)=\begin{cases}\dfrac{c}{(1-y)(y-6)} & 2\leqslant y\leqslant 5\\ 0 & \text{otherwise}\end{cases}$

a) $\int_2^5 \frac{c}{(1-y)(y-6)}\,dy=1 \Rightarrow c\int_2^5\left(\frac{-\frac{1}{5}}{(1-y)}+\frac{-\frac{1}{5}}{(y-6)}\right)dy=1\Rightarrow -\frac{c}{5}\left(-\ln|1-y|+\ln|y-6|\right)\Big]_2^5=1\Rightarrow$

$-\frac{c}{5}\left(\left(-\ln 4+\underbrace{\ln 1}_{0}\right)-\left(-\underbrace{\ln 1}_{0}+\ln 4\right)\right)=1\Rightarrow\frac{2c\ln 4}{5}=1\Rightarrow c=\frac{5}{2\ln 4}=\frac{5}{4\ln 2}$

b) $\mu=\int_2^5 y\,f(y)\,dy=\frac{7}{2},\ \sigma=\sqrt{\int_2^5 y^2\,f(y)\,dy-\mu^2}\approx 0.916$



10 $f(y)=\begin{cases}a\left(by-y^2\right) & 0\leqslant y\leqslant 5\\ 0 & \text{otherwise}\end{cases}$

a) $\int_0^5 \left(a\left(by-y^2\right)\right)dy=1\Rightarrow a\left(\frac{by^2}{2}-\frac{y^3}{3}\right)\Big]_0^5=1\Rightarrow a=\frac{1}{\frac{25b}{2}-\frac{125}{3}}=\frac{6}{75b-250}=\frac{6}{25(3b-10)}$

b) Since we found a in terms of b in the previous part, we are going to use that substitution to solve this problem.

$\frac{6}{25(3b-10)}\int_0^5\left(by^2-y^3\right)dy=\frac{5}{2}\Rightarrow\frac{6}{25(3b-10)}\left(\frac{by^3}{3}-\frac{y^4}{4}\right)\Big]_0^5=\frac{5}{2}\Rightarrow$

$\left(\frac{125b}{3}-\frac{625}{4}\right)=\frac{125(3b-10)}{12}\Rightarrow\frac{125}{12}(4b-15)=\frac{125(3b-10)}{12}\Rightarrow 4b-15=3b-10$

So, $b=5$.

And, $a = \dfrac{6}{25(3\times 5-10)} = \dfrac{6}{125}$.

c) $\sigma^2 = \dfrac{6}{125}\int_0^5 (5y^3 - y^4)\,dy - \left(\dfrac{5}{2}\right)^2 = \dfrac{6}{125}\left(\dfrac{5y^4}{4} - \dfrac{y^5}{5}\right)\Big]_0^5 - \left(\dfrac{5}{2}\right)^2$

$= \dfrac{6}{125}\times 625\left(\dfrac{5}{4}-1\right) - \dfrac{25}{4} = \dfrac{30-25}{4} = \dfrac{5}{4}$

11 $f(x) = \begin{cases} k & a \leqslant x \leqslant b \\ 0 & \text{otherwise} \end{cases}, 0 < a < b$

a) $\int_a^b k\,dx = 1 \Rightarrow (kx)]_a^b = 1 \Rightarrow k(b-a) = 1 \Rightarrow k = \dfrac{1}{b-a}$

b) i) $\mu = \int_a^b \dfrac{1}{b-a}x\,dx = \dfrac{1}{b-a}\left(\dfrac{x^2}{2}\right)\Big]_a^b = \dfrac{1}{2(b-a)}(b^2-a^2) = \dfrac{\cancel{(b-a)}(b+a)}{2\cancel{(b-a)}} = \dfrac{a+b}{2}$

ii) $\int_a^m \dfrac{1}{b-a}\,dx = \dfrac{1}{2} \Rightarrow \dfrac{1}{b-a}(x)\Big]_a^m = \dfrac{1}{2} \Rightarrow \dfrac{1}{b-a}(m-a) = \dfrac{1}{2} \Rightarrow m-a = \dfrac{b-a}{2} \Rightarrow m = \dfrac{a+b}{2}$

iii) $\sigma^2 = \int_a^b \dfrac{1}{b-a}x^2\,dx - \left(\dfrac{a+b}{2}\right)^2 = \dfrac{1}{b-a}\left(\dfrac{x^3}{3}\right)\Big]_a^b - \left(\dfrac{a+b}{2}\right)^2 = \dfrac{1}{3(b-a)}(b^3-a^3) - \left(\dfrac{a+b}{2}\right)^2$

$= \dfrac{\cancel{(b-a)}(b^2+ba+a^2)}{3\cancel{(b-a)}} - \dfrac{a^2+2ab+b^2}{4} = \dfrac{1}{12}(4a^2+4ab+4b^2-3a^2-6ab-3b^2)$

$= \dfrac{1}{12}(a^2-2ab+b^2) = \dfrac{(a-b)^2}{12}$

12 $f(x) = \begin{cases} \dfrac{5x^4}{31} & 1 \leqslant x \leqslant 2 \\ 0 & \text{otherwise} \end{cases}$

a) i) $P(1.2 < x < 1.7) = \int_{\frac{6}{5}}^{\frac{17}{10}} \dfrac{5x^4}{31}\,dx = \dfrac{5}{31}\left(\dfrac{x^5}{5}\right)\Big]_{\frac{6}{5}}^{\frac{17}{10}} = \dfrac{1}{31}\left(\left(\dfrac{17}{10}\right)^5 - \left(\dfrac{12}{10}\right)^5\right) = \dfrac{1511}{4000}$

ii) $\int_1^m \dfrac{5x^4}{31}\,dx = \dfrac{1}{2} \Rightarrow \dfrac{5}{31}\left(\dfrac{x^5}{5}\right)\Big]_1^m = \dfrac{1}{2} \Rightarrow \dfrac{m^5-1}{31} = \dfrac{1}{2} \Rightarrow m^5 = \dfrac{33}{2} \Rightarrow m = \sqrt[5]{\dfrac{33}{2}} \approx 1.75$

iii) We will use the complementary event since this involves a slightly simpler calculation, as the event has 1 as a boundary.

$\int_k^2 \dfrac{5x^4}{31}\,dx = \dfrac{1}{4} \Rightarrow \int_1^k \dfrac{5x^4}{31}\,dx = \dfrac{3}{4} \Rightarrow \dfrac{5}{31}\left(\dfrac{x^5}{5}\right)\Big]_1^k = \dfrac{3}{4} \Rightarrow \dfrac{k^5-1}{31} = \dfrac{3}{4} \Rightarrow k^5 = \dfrac{97}{4} \Rightarrow k = \sqrt[5]{\dfrac{97}{4}} \approx 1.89$

b) 'At least one' of them is larger means that we are going to use the complementary event, which is no observation is larger than 1.5:

$P(B) = 1 - (P(x < 1.5))^2$

So, to find this, we need to find the probability that one observation is smaller than 1.5.

$$\mathrm{P}(x < 1.5) = \int_1^{\frac{3}{2}} \frac{5x^4}{31}\,dx = \frac{5}{31}\left(\frac{x^5}{5}\right)\Bigg]_1^{\frac{3}{2}} = \frac{\left(\frac{3}{2}\right)^5 - 1}{31} = \frac{243-32}{31\times 32} = \frac{211}{992} \approx 0.213$$

Therefore, we can find the probability: $\mathrm{P}(B) = 1 - (\mathrm{P}(x < 1.5))^2 = 1 - \left(\frac{211}{992}\right)^2 \approx 0.955$

13 $F(x) = \begin{cases} 0 & 0 \leqslant x < 5 \\ k\left(x^3 - 21x^2 + 147x - 335\right) & 5 \leqslant x \leqslant 7 \\ 1 & x > 7 \end{cases}$

a) The cumulative distribution function is such that:

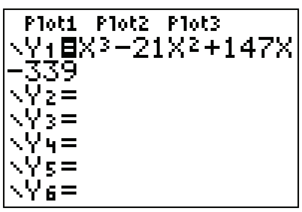
$F(7) = 1 \Rightarrow k\left(7^3 - 21\times 7^2 + 147\times 7 - 335\right) = 1 \Rightarrow 8k = 1 \Rightarrow k = \frac{1}{8}$

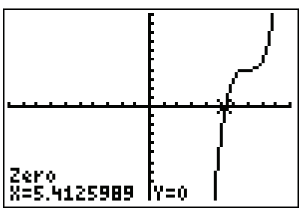
b) To find the probability density function, we need to differentiate the cumulative distribution function with respect to the variable x.

$$f(x) = F'(x) = \begin{cases} \frac{3}{8}x^2 - \frac{21}{4}x^2 + \frac{147}{8} & 5 \leqslant x \leqslant 7 \\ 0 & \text{otherwise} \end{cases} = \begin{cases} \frac{3}{8}(x-7)^2 & 5 \leqslant x \leqslant 7 \\ 0 & \text{otherwise} \end{cases}$$

c) The median can be found by using the probability density function, or the cumulative distribution function. Since the first method has been demonstrated in the previous question, we are going to use the second method this time.

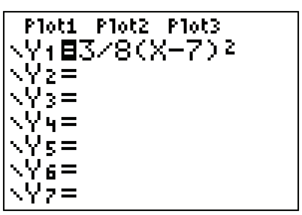
$F(m) = \frac{1}{2} \Rightarrow \frac{1}{8}\left(m^3 - 21m^2 + 147m - 335\right) = \frac{1}{2} \Rightarrow m^3 - 21m^2 + 147m - 339 = 0$

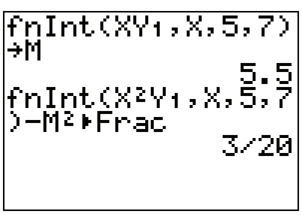




So, the median value is 5.41 (correct to three significant figures).

d) To find the variance, we need to find the mean value first. We are going to use a GDC to speed up the calculation.





$\mathrm{Var}(X) = \frac{3}{20}$

14 $f(y) = \begin{cases} 4y^k & 0 \leqslant y \leqslant 1 \\ 0 & \text{otherwise} \end{cases}, k > 0$

a) $\int_0^1 4y^k\, dy = 1 \Rightarrow 4\left(\dfrac{y^{k+1}}{k+1}\right)\Bigg]_0^1 = 1 \Rightarrow \dfrac{4}{k+1} = 1 \Rightarrow 4 = k+1 \Rightarrow k = 3$

b) $\mu = \int_0^1 4y \times y^3\, dy = \int_0^1 4y^4\, dy = 4\left(\dfrac{y^5}{5}\right)\Bigg]_0^1 = \dfrac{4}{5}$

c) Since the probability is 0.5, the value of a is actually the median value. Therefore, we are going to use the complementary event to calculate the value of a in a simpler manner.

$\mathrm{P}(y > a) = 0.5 \Rightarrow \mathrm{P}(y < a) = 0.5 \Rightarrow \int_0^a 4y^3\, dy = \dfrac{1}{2} \Rightarrow 4\left(\dfrac{y^4}{4}\right)\Bigg]_0^a = \dfrac{1}{2} \Rightarrow a^4 = \dfrac{1}{2} \Rightarrow$

$a = \sqrt[4]{\dfrac{1}{2}} = \dfrac{\sqrt[4]{8}}{2} \approx 0.841$

15 $f(y) = \begin{cases} 2ye^{-y^2} & y \geqslant 0 \\ 0 & \text{otherwise} \end{cases}$, where y is measured in thousands of hours.

a) Firstly, we need to see that the probability density function is always non-negative.

$f(y) = 2\underbrace{y}_{\geqslant 0}\underbrace{e^{-y^2}}_{>0} \geqslant 0$

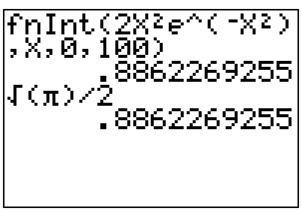
Then we have to confirm that the area under the curve is equal to 1.

$\lim\limits_{y\to\infty} \int_0^y 2te^{-t^2}\, dt = \lim\limits_{y\to\infty}\left(-e^{-t^2}\right)\Big]_0^y = \lim\limits_{y\to\infty}\left(-e^{-y^2} - \left(-e^0\right)\right) = 0 + 1 = 1$

b) $\mathrm{P}(y > 2) = 1 - \mathrm{P}(y \leqslant 2) = 1 - \int_0^2 2te^{-t^2}\, dt = 1 - \left(-e^{-t^2}\right)\Big]_0^2 = 1 - \left(-e^{-4} + 1\right) = e^{-4} \approx 0.0183$

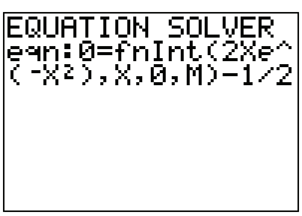
c) $\mu = \lim\limits_{y\to\infty} \int_0^y 2t^2e^{-t^2}\, dt$

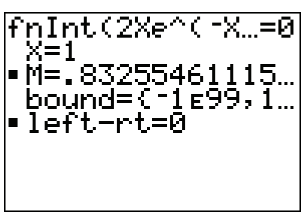
Notice that with respect to the values of the function, 100 can 'play' the role of positive infinity.



d) $\mathrm{P}(y \leqslant m) = \dfrac{1}{2} \Rightarrow \int_0^m 2te^{-t^2}\, dt = \dfrac{1}{2} \Rightarrow \left(-e^{-t^2}\right)\Big]_0^m = \dfrac{1}{2} \Rightarrow \left(-e^{-m^2} + 1\right) = \dfrac{1}{2} \Rightarrow \dfrac{1}{2} = e^{-m^2} \Rightarrow$

$-m^2 = \ln\dfrac{1}{2} \Rightarrow m^2 = \ln 2 \Rightarrow m = \sqrt{\ln 2} \approx 0.832\,56$





e) $P(y \leqslant a) = \frac{1}{4} \Rightarrow \int_0^a 2te^{-t^2}\,dt = \frac{1}{4} \Rightarrow \left(-e^{-t^2}\right)\Big]_0^a = \frac{1}{4} \Rightarrow \left(-e^{-a^2}+1\right) = \frac{1}{4} \Rightarrow \frac{3}{4} = e^{-a^2} \Rightarrow$

$-a^2 = \ln\frac{3}{4} \Rightarrow a^2 = \ln\frac{4}{3} = 2\ln 2 - \ln 3 \Rightarrow a = \sqrt{2\ln 2 - \ln 3} \approx 0.536\,36$

$P(y \leqslant b) = \frac{3}{4} \Rightarrow \int_0^b 2te^{-t^2}\,dt = \frac{3}{4} \Rightarrow \left(-e^{-t^2}\right)\Big]_0^b = \frac{3}{4} \Rightarrow \left(-e^{-b^2}+1\right) = \frac{3}{4} \Rightarrow \frac{1}{4} = e^{-b^2} \Rightarrow$

$-b^2 = \ln\frac{1}{4} \Rightarrow b^2 = \ln 4 = 2\ln 2 \Rightarrow b = \sqrt{2\ln 2} \approx 1.177\,41$

Therefore, the interquartile range is: $b - a \approx 0.641\,05$.

f) $P(y \leqslant 0.2) = \int_0^{0.2} 2te^{-t^2}\,dt = \left(-e^{-t^2}\right)\Big]_0^{0.2} = \left(-e^{-0.2^2}+1\right) = 1 - e^{-0.04} \approx 0.039\,21$

Since we would like not to have servicing of an engine valve, we use the complementary event, which has a probability of 0.960 79. So, to find the probability that the engine (at least one valve) needs servicing before 200 hours of work, we calculate: $1 - (0.960\,79)^2 \approx 0.076\,88$.

```
fnInt(2Xe^(-X²),
X,0,.2)
          .0392105608
1-Ans
          .9607894392
1-Ans²
          .0768836536
```

16 $f(y) = \begin{cases} \frac{1}{2}\left(cy + \frac{y^2}{3}\right) & 0 \leqslant y \leqslant 2 \\ 0 & \text{otherwise} \end{cases}$

a) $\int_0^2 \frac{1}{2}\left(cy + \frac{y^2}{3}\right)dy = 1 \Rightarrow \frac{1}{2}\left(\frac{cy^2}{2} + \frac{y^3}{9}\right)\Bigg]_0^2 = 1 \Rightarrow 2c + \frac{8}{9} = 2 \Rightarrow c = \frac{5}{9}$

b) $P(y < 1) = \int_0^1 \left(\frac{5}{18}y + \frac{y^2}{6}\right)dy = \left(\frac{5y^2}{36} + \frac{y^3}{18}\right)\Bigg]_0^1 = \frac{5}{36} + \frac{1}{18} = \frac{7}{36}$

c) Now, in this problem, we have a binomial random variable: $X \sim B\left(n = 10, p = \frac{7}{36}\right)$.

$P(x = 3) = \binom{10}{3}\left(\frac{7}{36}\right)^3\left(\frac{29}{36}\right)^7 \approx 0.1942$

d) Let H be the event that a student needs at least one hour to complete the exam.

$P(H) = 1 - \frac{7}{36} = \frac{29}{36}$

Next, let's assume that a student needs at least 90 minutes to complete the exam. Again, for ease of calculation, we are going to use the complementary event.

$P(Y \geqslant 1.5) = 1 - P(Y < 1.5) = 1 - \int_0^{\frac{3}{2}} \left(\frac{5}{18}y + \frac{y^2}{6}\right)dy = 1 - \left(\frac{5y^2}{36} + \frac{y^3}{18}\right)\Bigg]_0^{\frac{3}{2}}$

$= 1 - \frac{45}{144} - \frac{27}{144} = \frac{72}{144} = \frac{1}{2}$

Therefore: $P(Y \geqslant 1.5 \mid H) = \dfrac{\frac{1}{2}}{\frac{29}{36}} = \frac{18}{29} \approx 0.6207$.

17 $f(y) = \begin{cases} ky^2(5-y) & 0 \leqslant y \leqslant 5 \\ 0 & \text{otherwise} \end{cases}$

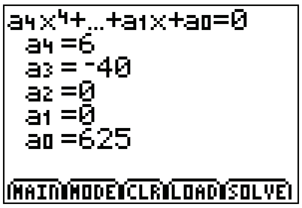
a) $\int_0^5 ky^2(5-y)\,dy = 1 \Rightarrow k\left[\left(5\frac{y^3}{3} - \frac{y^4}{4}\right)\right]_0^5 = 1 \Rightarrow k \times 625\left(\frac{1}{3} - \frac{1}{4}\right) = 1 \Rightarrow \frac{625}{12}k = 1 \Rightarrow k = \frac{12}{625}$

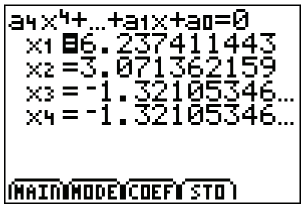
b) $\mu = \int_0^5 \frac{12}{625} y^3(5-y)\,dy = \frac{12}{625}\left[\left(5\frac{y^4}{4} - \frac{y^5}{5}\right)\right]_0^5 = \frac{12}{625} \times 625\left(\frac{5}{4} - 1\right) = 3$

$\int_0^m \frac{12}{625} y^2(5-y)\,dy = \frac{1}{2} \Rightarrow \frac{12}{625}\left[\left(5\frac{y^3}{3} - \frac{y^4}{4}\right)\right]_0^{5m} = \frac{1}{2} \Rightarrow \frac{12}{625}\left(\frac{5m^3}{3} - \frac{m^4}{4}\right) = \frac{1}{2} \Rightarrow$

$\frac{\cancel{12}}{625} \times \frac{20m^3 - 3m^4}{\cancel{12}} = \frac{1}{2} \Rightarrow 6m^4 - 40m^3 + 625 = 0$

To solve this equation, we need to use a calculator and the PolySmlt2 application.





The only solution in the given interval is x_2; therefore, the median is 3.1. To find the mode, we just need to find the maximum point of the probability density function.

$f(y) = \frac{12}{625} y^2(5-y) \Rightarrow f'(y) = \frac{12}{625}\left(2y(5-y) - y^2\right) = \frac{12y}{625}(10 - 3y) \Rightarrow$

$f'(y) = 0 \Rightarrow y = 0 \text{ or } y = \frac{10}{3}$

The first value gives a minimum value, whilst the second value gives the maximum value; therefore, the mode is 3.3.

c) $P(Y < 3) = \int_0^3 \frac{12}{625} y^2(5-y)\,dy = \frac{12}{625}\left[\left(5\frac{y^3}{3} - \frac{y^4}{4}\right)\right]_0^3 = \frac{12}{625} \times 9\left(5 - \frac{9}{4}\right) = \frac{\cancel{12}\,3}{625} \times 9 \times \frac{11}{\cancel{4}} = \frac{297}{625}$

d) $\sigma^2 = \int_0^5 \frac{12}{625} y^4(5-y)\,dy - 3^3 = \frac{12}{625}\left[\left(5\frac{y^5}{5} - \frac{y^6}{6}\right)\right]_0^5 - 9 = \frac{12}{625} \times 625\left(5 - \frac{25}{6}\right) - 9$

$= 12 \times \frac{5}{6} - 9 = 1 \Rightarrow \sigma = \sqrt{1} = 1$

e) $\mu \pm \sigma \in [2, 4] \Rightarrow P(2 \leqslant Y \leqslant 4) = P(Y < 3) = \int_2^4 \frac{12}{625} y^2(5-y)\,dy = \frac{12}{625}\left[\left(5\frac{y^3}{3} - \frac{y^4}{4}\right)\right]_2^4$

$= \frac{12}{625}\left(\left(5 \times \frac{64}{3} - 64\right) - \left(5 \times \frac{8}{3} - 4\right)\right) = \frac{12}{625} \times \left(\frac{280}{3} - 60\right)$

$= \frac{12}{625} \times \frac{100}{3} = \frac{16}{25} = 0.64$

We notice that this is fairly close to the empirical rule of about 68% within one standard deviation of the mean value, so we cannot say that it contradicts it.

18 $f(t) = \begin{cases} \frac{4}{625}(5t^3 - t^4) & 0 \leqslant t \leqslant 5 \\ 0 & \text{otherwise} \end{cases}$

a) $\mu = \int_0^5 \frac{4}{625} t\left(5t^3 - t^4\right) dt = \frac{4}{625}\left(t^5 - \frac{t^6}{6}\right)\Bigg]_0^5 = \frac{4}{\cancel{5^4}} \times 5^{\cancel{5}}\left(1 - \frac{5}{6}\right) = \frac{4 \times 5}{6} = \frac{10}{3}$

To find the mode, we need to find the maximum point of the probability density function on the given interval.

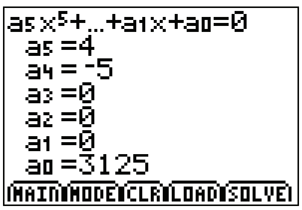
$f(t) = \frac{4}{625}\left(5t^3 - t^4\right) \Rightarrow f\,'(t) = \frac{4}{625}\left(15t^2 - 4t^3\right) = \frac{4t^2(15 - 4t)}{625}$

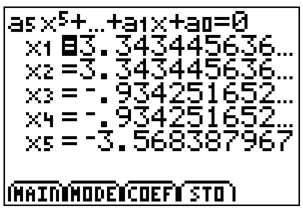
$f\,'(t) = 0 \Rightarrow \frac{4t^2(15 - 4t)}{625} = 0 \Rightarrow t = 0 \text{ or } t = \frac{15}{4}$

The first value gives us the minimum probability; therefore, the mode is $\frac{15}{4}$.

b) $\int_0^m \frac{4}{625}\left(5t^3 - t^4\right) dt = 1 \Rightarrow \frac{4}{625}\left(\frac{5t^4}{4} - \frac{t^5}{5}\right)\Bigg]_0^m = 1 \Rightarrow \frac{5m^4}{4} - \frac{m^5}{5} = \frac{625}{4} \Rightarrow 4m^5 - 5m^4 + 3125 = 0$

To solve this equation, we need to use a calculator and the PolySmlt2 application.





We have multiple roots but only one root is within the given interval; therefore, the median value is 3.34 (correct to three significant figures).

c) $P(1 < t < 2) = \int_1^2 \frac{4}{625}\left(5t^3 - t^4\right) dt = \frac{4}{625}\left(\frac{5t^4}{4} - \frac{t^5}{5}\right)\Bigg]_1^2 = \frac{4}{625} \times \left(20 - \frac{32}{5} - \frac{5}{4} + \frac{1}{5}\right)$

$= \frac{4}{625} \times \left(\frac{75}{4} - \frac{31}{5}\right) = \frac{\cancel{4}}{625} \times \frac{375 - 124}{\cancel{20}\,5} = \frac{251}{3125} \approx 0.0803$

d) $\sigma^2 = \int_0^5 \frac{4}{625} t^2\left(5t^3 - t^4\right) dt - \left(\frac{10}{3}\right)^2 = \frac{4}{625}\left(\frac{5t^6}{6} - \frac{t^7}{7}\right)\Bigg]_0^5 - \frac{100}{9} = \frac{4}{\cancel{5^4}} \times 5^{\cancel{7}\,3}\left(\frac{1}{6} - \frac{1}{7}\right) - \frac{100}{9}$

$= 500 \times \frac{1}{42} - \frac{100}{9} = \frac{250}{21} - \frac{100}{9} = \frac{750 - 700}{63} = \frac{50}{63} \Rightarrow \sigma = \sqrt{\frac{50}{63}} \approx 0.891$

e) Firstly, we need to calculate the probability that one plane has been delayed for more than one hour.

$P(t > 1) = 1 - P(t < 1) = 1 - \int_0^1 \frac{4}{625}\left(5t^3 - t^4\right) dt = 1 - \frac{4}{625}\left(\frac{5t^4}{4} - \frac{t^5}{5}\right)\Bigg]_0^1 = 1 - \frac{4}{625}\left(\frac{5}{4} - \frac{1}{5}\right)$

$= 1 - \frac{4}{625} \times \frac{21}{20} = 1 - \frac{21}{3125} = \frac{3104}{3125}$

i) $P(t > 1) \times P(t > 1) = \left(\frac{3104}{3125}\right)^2 \approx 0.987$

ii) Again, in this case, we are going to use the complementary event:

$$1-(\mathrm{P}(t<1))^2 = 1-\left(\frac{21}{3125}\right)^2 \approx 0.999\,95$$

iii) We can even apply the binomial distribution in this case:

$$X \sim \mathrm{B}\left(n=2,\ p=\frac{3104}{3125}\right) \Rightarrow \mathrm{P}(x=1) = 2\times\frac{3104}{3125}\times\frac{21}{3125} \approx 0.013\,35$$

Solution Paper 1 type

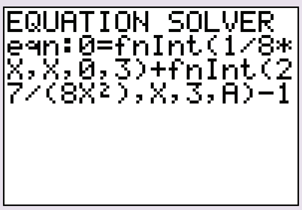
19 $f(x)=\begin{cases}\frac{1}{8}x & 0\leqslant x<3\\ \frac{27}{8x^2} & 3<x\leqslant a\end{cases}$

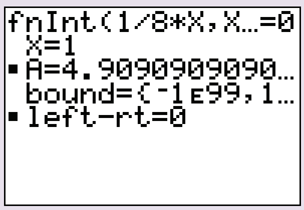
$$\int_0^3\left(\frac{1}{8}x\right)dx+\int_3^a\left(\frac{27}{8x^2}\right)dx=1\Rightarrow\frac{1}{8}\left[\left(\frac{x^2}{2}\right)\right]_0^3+\frac{27}{8}\left[\left(-\frac{1}{x}\right)\right]_3^a=1\Rightarrow\frac{9}{16}-\frac{27}{8a}+\frac{9}{8}=1\Rightarrow$$

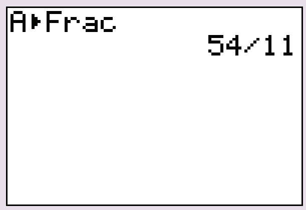
$$\frac{27}{16}-1=\frac{27}{8a}\Rightarrow\frac{11}{16}=\frac{27}{8a}\Rightarrow a=\frac{16\times27}{11\times8}=\frac{54}{11}$$

Solution Paper 2 type

19



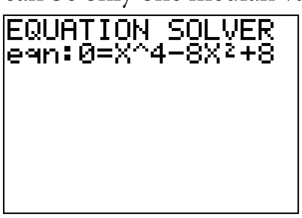


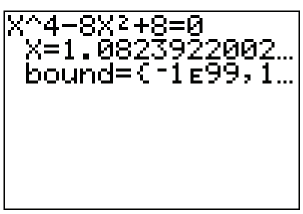


20 $f(x)=\begin{cases}\frac{1}{4}x(4-x^2) & 0\leqslant x\leqslant 2\\ 0 & \text{otherwise}\end{cases}$

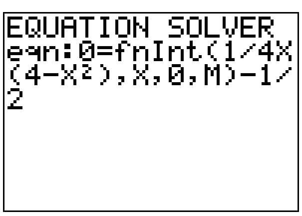
$$\int_0^m\frac{1}{4}x(4-x^2)\,dx=\frac{1}{2}\Rightarrow\frac{1}{4}\left[\left(2x^2-\frac{x^4}{4}\right)\right]_0^m=\frac{1}{2}\Rightarrow 2m^2-\frac{m^4}{4}=2\Rightarrow m^4-8m^2+8=0$$

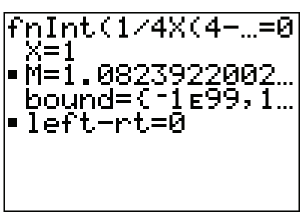
This quartic equation has no rational solution, so we are going to use a GDC to solve it. Notice that there can be only one median value and its value should be in the neighbourhood of 1.





Another way to find the median is to put the integral value directly as the finite integral.





So, the median value is 1.08 (correct to three significant figures).

Exercise 17.5

Solution Paper 1 type

1 Let X be the number of hours it takes to change the batteries. $X \sim N(\mu = 50, \sigma^2 = 7.5^2)$.

We are going to use the tables to standardize the variable. $Z = \dfrac{X - \mu}{\sigma}$.

a $P(X \leqslant 50) = 0.5$, since 50 is the mean value.

b $P(50 \leqslant X \leqslant 75) = P(0 \leqslant Z \leqslant 3.33) = 0.9996 - 0.5 = 0.4996$

c $P(X < 42.5) = P(Z \leqslant -1) = 1 - P(Z \leqslant 1) = 1 - 0.8413 = 0.1587$

d $P(42.5 \leqslant X \leqslant 57.5) = P(-1 \leqslant Z \leqslant 1) = 2 \times 0.8413 - 1 = 0.6826$

e $P(X > 65) = P(Z > 2) = 1 - P(Z < 2) = 1 - 0.9773 = 0.0227$

f $P(X = 47.5) = 0$, since X is the continuous variable and the probability of obtaining an exact value is 0.

Note: The accuracy of the answers is not as given in the textbook since we were using the tables. See below for the results obtained using a calculator.

Solution Paper 2 type

1

normCdf(-∞,50,50,7.5)	0.5
normCdf(50,75,50,7.5)	0.499571
normCdf(-∞,42.5,50,7.5)	0.158655
normCdf(42.5,57.5,50,7.5)	0.682689
normCdf(65,∞,50,7.5)	0.02275
normCdf(47.5,47.5,50,7.5)	0.
	6/99

2 a) $P(|z| < 1.2) = P(-1.2 < z < 1.2) = 2 \times 0.8849 - 1 = 0.7698$

b) $P(|z| > 1.4) = P(z < -1.4) + P(z > 1.4) = 2 \times P(z > 1.4) = 2(1 - P(z < 1.4))$

$= 2(1 - 0.9192) = 2 \times 0.0808 = 0.1616$

Let X be the normal variable, where $X \sim N(\mu = 3, \sigma^2 = 3)$.

c) $P(x < 3.7) = P\left(z < \dfrac{3.7 - 3}{\sqrt{3}}\right) = P(z < 0.404) = 0.6700$

d) $P(x > -3.7) = P\left(z > \dfrac{-3.7 - 3}{\sqrt{3}}\right) = P(z > -3.868) = P(z < 3.868) \approx 1$

Using a calculator gives slightly different answers:

normCdf(-1.2,1.2,0,1)	0.769861
2·normCdf(1.4,∞,0,1)	0.161513
normCdf(-∞,3.7,3,1.732050808)	0.656947
normCdf(-3.7,∞,3,1.732050808)	0.999945
	4/99

3 Let X be the mileage of a car, where $X \sim N\left(\mu = 11.4, \sigma^2 = 1.26^2\right)$.

a) $P(X < 8.4) = P\left(Z < \dfrac{8.4 - 11.4}{1.26}\right) = P(Z < -2.38) = 1 - P(Z < 2.38) = 1 - 0.9911 = 0.009$

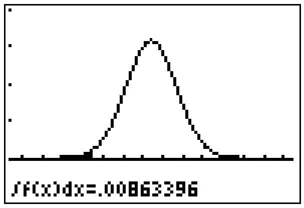
b) In solving this question, we notice that both values can be written as 11.4 ± 3, so we can use the previous result and this symmetry.

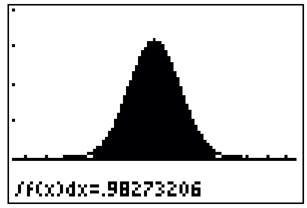
$$P(8.4 < X < 14.4) = P(-2.38 < Z < 2.38) = 1 - 2 \times P(Z < -2.38) = 2 \times P(Z < 2.38) - 1$$
$$= 1.9822 - 1 = 0.9822$$

Here, we are going to use the function mode and calculate the definite integral under the normal curve. Notice that we need to adopt the window.

```
 Plot1  Plot2  Plot3
\Y1■normalpdf(X,
11.4,1.26)
\Y2=
\Y3=
\Y4=
\Y5=
\Y6=
```

```
WINDOW
 Xmin=4.4
 Xmax=18.4
 Xscl=1
 Ymin=-.1
 Ymax=.4
 Yscl=.1
 Xres=1
```





4 $P(Z > z) = 0.1 \Rightarrow P(Z < z) = 0.9 \Rightarrow z = 1.2816$

5 $P(|Z| > z_0) = 0.95 \Rightarrow P(-z_0 < Z < z_0) = 0.95 \Rightarrow 2 \times P(Z < z_0) - 1 = 0.95 \Rightarrow$

$P(Z < z_0) = 0.975 \Rightarrow z_0 = 1.96$

Solution Paper 1 type

6 Let X be the scores on the examination, where $X \sim N\left(\mu = 550, \sigma^2 = 100^2\right)$.

a $P(X < 400) = P\left(Z < \dfrac{400 - 550}{100}\right) = P(Z < -1.5) = 1 - P(Z < 1.5) = 1 - 0.9332 = 0.0668$

b $P(450 < X < 650) = P\left(\dfrac{450 - 550}{100} < Z < \dfrac{650 - 550}{100}\right) = P(-1 < Z < 1) = 2 \times P(Z < 1) - 1 = 2 \times 0.8413 - 1 = 0.6826$

c $P(X \leqslant a) = 0.9 \Rightarrow P\left(Z \leqslant \dfrac{a - 550}{100}\right) = 0.9 \Rightarrow \dfrac{a - 550}{100} = 1.2816 \Rightarrow a = 678.16$; therefore, we need to score at least 679.

d $P(|X| \leqslant a) = 0.5 \Rightarrow P\left(-\dfrac{a - 550}{100} \leqslant Z \leqslant \dfrac{a - 550}{100}\right) = 0.5 \Rightarrow P\left(Z \leqslant \dfrac{a - 550}{100}\right) = 0.75 \Rightarrow$

$\dfrac{a - 550}{100} = 0.6745 \Rightarrow a = 627.45$

So, a is the upper quartile, whilst to find the IQR we need to double the distance to the mean value. Hence, $IQR = 2 \times (627.45 - 550) = 134.9$.

Solution Paper 2 type

6 **a–b**

```
normalcdf(-450,4
00,550,100)
     .0668072287
normalcdf(450,65
0,550,100)
     .6826894809
```

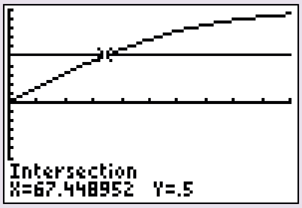
Note: −450 takes the role of −∞.
It is sufficient to tke 10 standard deviations from the mean value.

c

```
invNorm(.9,550,1
00)
     678.1551567
```

d Here, we will demonstrate a solution by using the function feature on the calculator. We are going to find the distance between the mean value and the upper or lower quartile. The IQR will then be calculated by doubling that distance.

```
 Plot1  Plot2  Plot3
\Y1=normalcdf(55
0-X,550+X,550,10
0)
\Y2=.5
\Y3=
\Y4=
\Y5=
```



IQR = 2 · 67.45 = 134.9

7 In this problem, we are going to use Solver on a GDC. Solver has the option of using multiple variables, changing them according to the information given in the question, yet solving the equation for one variable at a time.

a)

```
EQUATION SOLVER
eqn:0=normalcdf(
-1000,500,M,S)-P
```

```
normalcdf(-10…=0
 M=512
 S=5.7
•P=.01763414104…
 bound={-1E99,1…
• left-rt=0
```

So, 1.8% of the bags would be underweight.

b)

```
normalcdf(-10…=0
•M=509.97890772…
 S=5.7
 P=.04
 bound={-1E99,1…
• left-rt=-5E-14
```

c)

```
normalcdf(-10…=0
 M=510
•S=5.7120480063…
 P=.04
 bound={-1E99,1…
• left-rt=0
```

So, the mean setting (part **b**) must be 509.979. And, the standard deviation (part **c**) should be 5.712.

8 Let X be the heights of the students, where $X \sim \mathrm{N}\left(\mu = 151, \sigma^2 = 64\right)$.

a)–b)

```
normalcdf(71,166
,151,8)
     .9696037028
normalcdf(145,15
7,151,8)
     .5467454411
```

9 Let X be the number of minutes, where $X \sim \mathrm{N}\left(\mu = 12, \sigma^2 = 4\right)$. To find the number of days, the calculated probabilities will be multiplied by 180 school days.

a), **b)** & **c)**

```
180·normCdf(17,∞,12,2)       1.11774
180·normCdf(-∞,10,12,2)      28.5579
180·normCdf(9,13,12,2)       112.438
                                3/99
```

So, the answers are 1 day (part **a**), 29 days (part **b**) and 112 days (part **c**).

Solution Paper 1 type

10 $\mathrm{P}(X < 16.56) = 0.64 \Rightarrow \mathrm{P}\left(Z < \dfrac{16.56 - 16}{\sigma}\right) = 0.64 \Rightarrow \dfrac{0.56}{\sigma} = 0.3585 \Rightarrow \sigma = \dfrac{0.56}{0.3585} = 1.56$

Solution Paper 2 type

10 Again, we will use Solver on a GDC.

```
EQUATION SOLVER
eqn:0=normalcdf(
-100,16.56,16,S)
-.64
```

```
normalcdf(-10…=0
▪S=1.5622430393…
 bound={-1E99,1…
▪left-rt=0
```

Solution Paper 1 type

11 $\mathrm{P}(X > 104) = 0.246 \Rightarrow \mathrm{P}(X < 104) = 0.754 \Rightarrow \mathrm{P}\left(Z < \dfrac{104 - 91}{\sigma}\right) = 0.754 \Rightarrow \dfrac{13}{\sigma} = 0.687\,13 \Rightarrow \sigma = \dfrac{13}{0.687\,13} \approx 18.92$

Solution Paper 2 type

11 Again, we will use Solver.

```
EQUATION SOLVER
eqn:0=normalcdf(
104,10000,91,S)-
.246
```

```
normalcdf(104…=0
▪S=18.919243877…
 bound={-1E99,1…
▪left-rt=0
```

12 Using Solver on a GDC:

```
EQUATION SOLVER
eqn:0=normalcdf(
36.5,1000,M,3)-.
029
```

```
normalcdf(36.…=0
▪M=30.812909249…
 bound={-1E99,1…
▪left-rt=-2E-14
```

Solution Paper 1 type

13 $P(X > 63) = 0.878 \Rightarrow P(X < 63) = 0.122 \Rightarrow P\left(Z < \frac{63 - \mu}{32}\right) = 0.122 \Rightarrow$

$\frac{63 - \mu}{32} = -1.165 \Rightarrow 63 - \mu = -37.2815 \Rightarrow \mu \approx 100.3$

Solution Paper 2 type

13

```
EQUATION SOLVER
eqn:0=normalcdf(
63,10000,M,32)-.
878
```

```
normalcdf(63,…=0
▪M=100.28151036…
 bound={-1E99,1…
▪left-rt=0
```

14 Given that the variance is 25, we have to be careful and input the standard deviation of 5 into the calculator.

```
EQUATION SOLVER
eqn:0=normalcdf(
-1000,27.5,M,5)-
.312
```

```
normalcdf(-10…=0
▪M=29.950946141…
 bound={-1E99,1…
▪left-rt=0
```

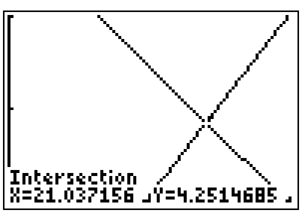
15 To solve this problem, we need to set up a system of two equations with two unknowns.

$$\left.\begin{aligned} P(X > 14.6) &= 0.935 \\ P(X > 29.6) &= 0.022 \end{aligned}\right\} \Rightarrow \left.\begin{aligned} P\left(Z > \frac{14.6 - \mu}{\sigma}\right) &= 0.935 \\ P\left(Z > \frac{29.6 - \mu}{\sigma}\right) &= 0.022 \end{aligned}\right\} \Rightarrow \left.\begin{aligned} P\left(Z < \frac{14.6 - \mu}{\sigma}\right) &= 0.065 \\ P\left(Z < \frac{29.6 - \mu}{\sigma}\right) &= 0.978 \end{aligned}\right\} \Rightarrow$$

$$\left.\begin{aligned} \frac{14.6 - \mu}{\sigma} &= \text{invNorm}(0.065) \\ \frac{29.6 - \mu}{\sigma} &= \text{invNorm}(0.978) \end{aligned}\right\} \Rightarrow \left.\begin{aligned} \frac{14.6 - \mu}{\text{invNorm}(0.065)} &= \sigma \\ \frac{29.6 - \mu}{\text{invNorm}(0.978)} &= \sigma \end{aligned}\right\}$$

We are going to use a graphical method on our GDC to solve this system.

```
Plot1 Plot2 Plot3
\Y1■(14.6-X)/inv
Norm(.065)
\Y2■(29.6-X)/inv
Norm(.978)
\Y3=
\Y4=
\Y5=
```

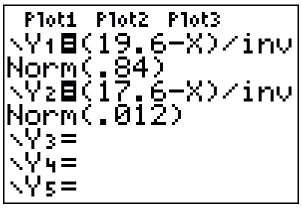


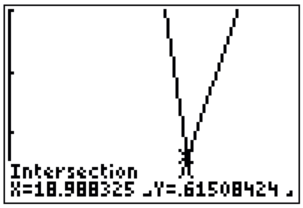
The variables x and y are the mean value and the standard deviation respectively.

16 Again we need to set up a system of two equations with two unknowns.

$$\left.\begin{array}{l} P(X > 19.6) = 0.16 \\ P(X < 17.6) = 0.012 \end{array}\right\} \Rightarrow \left.\begin{array}{l} P\left(Z > \dfrac{19.6-\mu}{\sigma}\right) = 0.16 \\ P\left(Z < \dfrac{17.6-\mu}{\sigma}\right) = 0.012 \end{array}\right\} \Rightarrow \left.\begin{array}{l} P\left(Z < \dfrac{19.6-\mu}{\sigma}\right) = 0.84 \\ P\left(Z < \dfrac{17.6-\mu}{\sigma}\right) = 0.012 \end{array}\right\} \Rightarrow$$

$$\left.\begin{array}{l} \dfrac{19.6-\mu}{\sigma} = \text{invNorm}(0.84) \\ \dfrac{17.6-\mu}{\sigma} = \text{invNorm}(0.012) \end{array}\right\} \Rightarrow \left.\begin{array}{l} \dfrac{19.6-\mu}{\text{invNorm}(0.84)} = \sigma \\ \dfrac{17.6-\mu}{\text{invNorm}(0.012)} = \sigma \end{array}\right\}$$

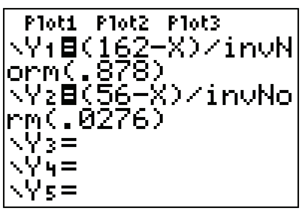


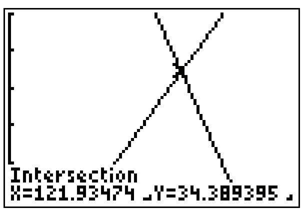


17 Again, we need to set up a system of two equations with two unknowns.

$$\left.\begin{array}{l} P(X > 162) = 0.122 \\ P(X < 56) = 0.0276 \end{array}\right\} \Rightarrow \left.\begin{array}{l} P\left(Z > \dfrac{162-\mu}{\sigma}\right) = 0.122 \\ P\left(Z < \dfrac{56-\mu}{\sigma}\right) = 0.0276 \end{array}\right\} \Rightarrow \left.\begin{array}{l} P\left(Z < \dfrac{162-\mu}{\sigma}\right) = 0.878 \\ P\left(Z < \dfrac{56-\mu}{\sigma}\right) = 0.0276 \end{array}\right\} \Rightarrow$$

$$\left.\begin{array}{l} \dfrac{162-\mu}{\sigma} = \text{invNorm}(0.878) \\ \dfrac{56-\mu}{\sigma} = \text{invNorm}(0.0276) \end{array}\right\} \Rightarrow \left.\begin{array}{l} \dfrac{162-\mu}{\text{invNorm}(0.878)} = \sigma \\ \dfrac{56-\mu}{\text{invNorm}(0.0276)} = \sigma \end{array}\right\}$$

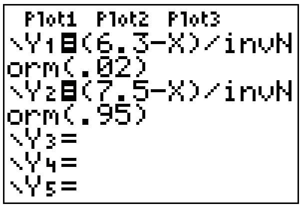


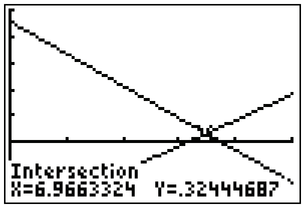


18 **a)** We need to set up a system of two equations with two unknowns.

$$\left.\begin{array}{l} P(X < 6.3) = 0.02 \\ P(X > 7.5) = 0.05 \end{array}\right\} \Rightarrow \left.\begin{array}{l} P\left(Z < \dfrac{6.3-\mu}{\sigma}\right) = 0.02 \\ P\left(Z > \dfrac{7.5-\mu}{\sigma}\right) = 0.05 \end{array}\right\} \Rightarrow \left.\begin{array}{l} P\left(Z < \dfrac{6.3-\mu}{\sigma}\right) = 0.02 \\ P\left(Z < \dfrac{7.5-\mu}{\sigma}\right) = 0.95 \end{array}\right\} \Rightarrow$$

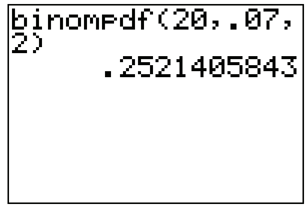
$$\left.\begin{array}{l} \dfrac{6.3-\mu}{\sigma} = \text{invNorm}(0.02) \\ \dfrac{7.5-\mu}{\sigma} = \text{invNorm}(0.95) \end{array}\right\} \Rightarrow \left.\begin{array}{l} \dfrac{6.3-\mu}{\text{invNorm}(0.02)} = \sigma \\ \dfrac{7.5-\mu}{\text{invNorm}(0.95)} = \sigma \end{array}\right\}$$





b) The probability that a pole is rejected is the sum of the probabilities that a pole was too short or too long: $p = 0.02 + 0.05 = 0.07$, and, since there are 20 poles, we have a binomial distribution with the following parameters: $Y \sim \mathrm{B}(n = 20,\ p = 0.07)$.

$\mathrm{P}(y = 2) \approx 0.252$



19 Let X be the number of millilitres of water in a bottle, where $X \sim \mathrm{N}(\mu = 1012,\ \sigma^2 = 25)$.

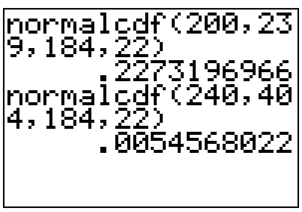
a) $\mathrm{P}(X > 1010) = \mathrm{P}\left(Z > \dfrac{1010 - 1012}{5}\right) = \mathrm{P}(Z > -0.4) = \mathrm{P}(Z < 0.4) = 0.6554$

b) $\mathrm{P}(X < 1000) = \mathrm{P}\left(Z < \dfrac{1000 - 1012}{5}\right) = \mathrm{P}(Z < -2.4) = 1 - \mathrm{P}(Z < 2.4) = 1 - 0.9918 = 0.0082$

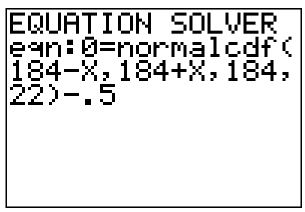
c) $n = 10\,000 \cdot\ \mathrm{P}(X < 1000) = 10\,000 \cdot\ 0.0082 = 82$

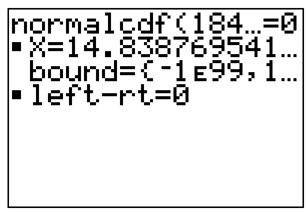
20 Let X be the cholesterol level, where $X \sim \mathrm{N}(\mu = 184,\ \sigma^2 = 22^2)$.

a)–b)



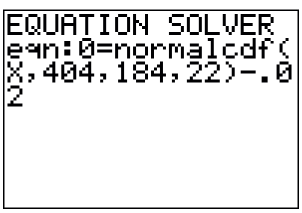
c)

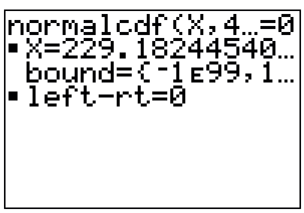




So, the IQR $= 2 \times 14.838\,7695 \approx 29.678$.

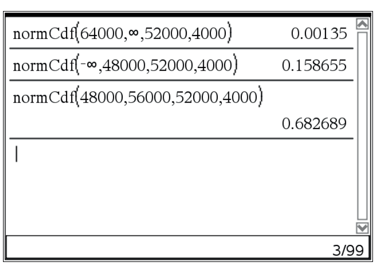
d)





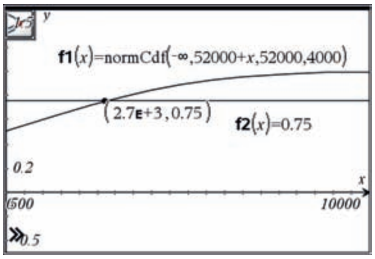
21 Let X be the treadlife of the tyres, where $X \sim \mathrm{N}(\mu = 52\,000,\ \sigma^2 = 4000^2)$.

a)–c)



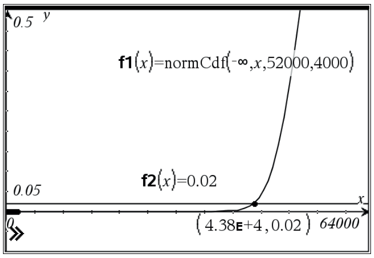
So, it is not likely (0.14% chance) that a set of tyres last more than 64 000 km (part **a**); you would expect 15.87% of the tyres to last less than 48 000 km (part **b**); and you would expect 68.27% to last between 48 000 km and 56 000 km (part **c**).

d)



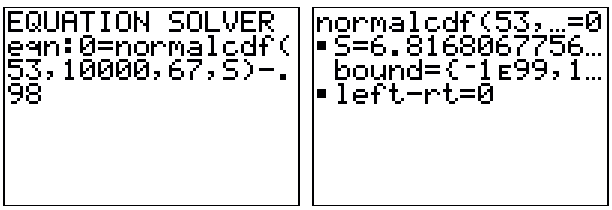
So, the IQR $= 2 \cdot 2700 = 5400$ (correct to 3 s.f.).

e)

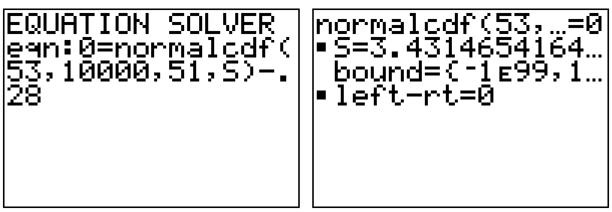


So, the minimum life the company should guarantee is 43 800 km (correct to 3 s.f.).

22 a)



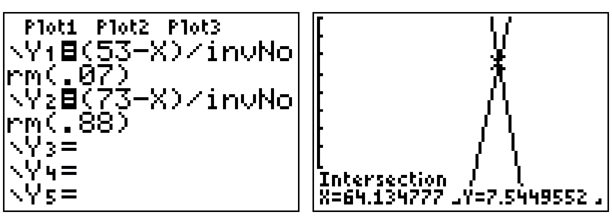
b)



c)

$$\left.\begin{array}{l} \mathrm{P}(X < 53) = 0.07 \\ \mathrm{P}(X \geqslant 73) = 0.12 \end{array}\right\} \Rightarrow \left.\begin{array}{l} \mathrm{P}\left(Z < \dfrac{53-\mu}{\sigma}\right) = 0.07 \\ \mathrm{P}\left(Z \geqslant \dfrac{73-\mu}{\sigma}\right) = 0.12 \end{array}\right\} \Rightarrow \left.\begin{array}{l} \mathrm{P}\left(Z < \dfrac{53-\mu}{\sigma}\right) = 0.07 \\ \mathrm{P}\left(Z \leqslant \dfrac{73-\mu}{\sigma}\right) = 0.88 \end{array}\right\} \Rightarrow$$

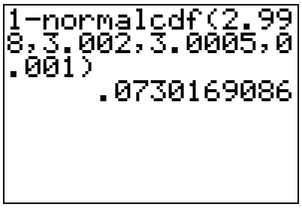
$$\left.\begin{array}{l} \dfrac{53-\mu}{\sigma} = \mathrm{invNorm}(0.07) \\ \dfrac{73-\mu}{\sigma} = \mathrm{invNorm}(0.88) \end{array}\right\} \Rightarrow \left.\begin{array}{l} \dfrac{53-\mu}{\mathrm{invNorm}(0.07)} = \sigma \\ \dfrac{73-\mu}{\mathrm{invNorm}(0.88)} = \sigma \end{array}\right\}$$



So, the mean value and the standard deviation are 64.13 and 7.54 respectively.

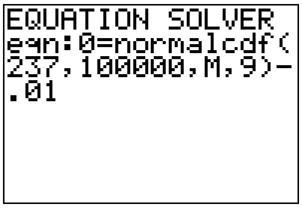
23 Let X be the diameter of a bearing, where $X \sim \mathrm{N}\left(\mu = 3.0005, \sigma^2 = 0.001^2\right)$.

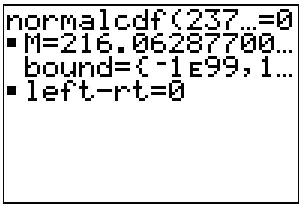
We are going to use the complementary event that the bearing is accepted.



$1 - \mathrm{P}(2.998 < X < 3.002) \approx 0.0730$

24 Let X be the amount of fill in a can, where $X \sim \mathrm{N}\left(\mu, \sigma^2 = 9^2\right)$.

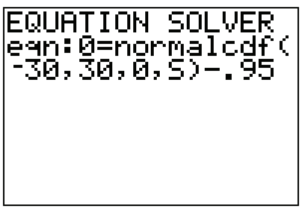


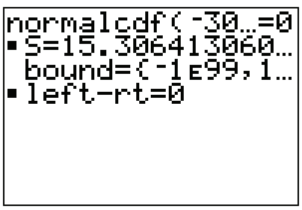


So, the mean value is 216 cc.

25 Since we can translate the normal curve horizontally along the x-axis without changing the standard deviation, we can write: $X \sim \mathrm{N}\left(\mu = 0, \sigma^2\right)$.

$\mathrm{P}(-30 < x < 30) = 0.95$

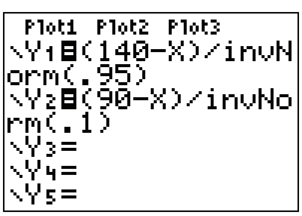


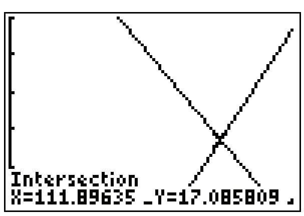


26 **a)** Let X be the speeds of cars on the main highway, where $X \sim \mathrm{N}\left(\mu, \sigma^2\right)$.

$$\left.\begin{array}{r}\mathrm{P}(X < 140) = 0.95\\ \mathrm{P}(X < 90) = 0.1\end{array}\right\} \Rightarrow \left.\begin{array}{l}\mathrm{P}\left(Z < \dfrac{140-\mu}{\sigma}\right) = 0.95\\ \mathrm{P}\left(Z < \dfrac{90-\mu}{\sigma}\right) = 0.1\end{array}\right\} \Rightarrow \left.\begin{array}{l}\dfrac{140-\mu}{\sigma} = \mathrm{invNorm}(0.95)\\ \dfrac{90-\mu}{\sigma} = \mathrm{invNorm}(0.1)\end{array}\right\} \Rightarrow$$

$$\left.\begin{array}{l}\dfrac{140-\mu}{\mathrm{invNorm}(0.95)} = \sigma\\ \dfrac{90-\mu}{\mathrm{invNorm}(0.1)} = \sigma\end{array}\right\}$$





So, the mean speed is 112 km/h and the standard deviation is 17.1 km/h.

b) We can use the stored values of x and y in the calculator for the following calculation.

```
normalcdf(110,10
00,X,Y)
          .544187711
```

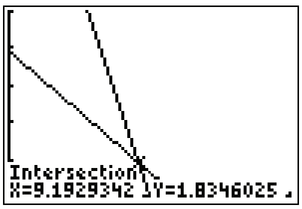
So, 54.4% of the cars exceed the speed of 110 km/h.

27 $X \sim \mathrm{N}\left(\mu, \sigma^2\right)$

$$\left.\begin{array}{r}\mathrm{P}(X \leqslant 10) = 0.67 \\ \mathrm{P}(X \leqslant 12) = 0.937\end{array}\right\} \Rightarrow \left.\begin{array}{r}\mathrm{P}\left(Z \leqslant \dfrac{10-\mu}{\sigma}\right) = 0.67 \\ \mathrm{P}\left(Z \leqslant \dfrac{12-\mu}{\sigma}\right) = 0.937\end{array}\right\} \Rightarrow \left.\begin{array}{r}\dfrac{10-\mu}{\sigma} = \mathrm{invNorm}\,(0.67) \\ \dfrac{12-\mu}{\sigma} = \mathrm{invNorm}\,(0.937)\end{array}\right\} \Rightarrow$$

$$\left.\begin{array}{r}\dfrac{10-\mu}{\mathrm{invNorm}\,(0.67)} = \sigma \\ \dfrac{12-\mu}{\mathrm{invNorm}\,(0.937)} = \sigma\end{array}\right\}$$

```
Plot1 Plot2 Plot3
\Y1■(10-X)/invNo
rm(.67)
\Y2■(12-X)/invNo
rm(.937)
\Y3=
\Y4=
\Y5=
```



So, the expected value is actually the mean value which is 9.19.

28 **a)** In this case, we can use Solver on a GDC since we will be keeping some values and recalculating others with respect to the calculated values.

i)

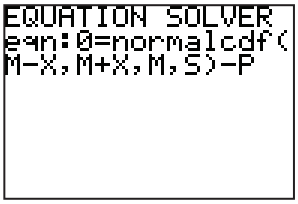
```
EQUATION SOLVER
eqn:0=normalcdf(
100,108,M,S)-P
```

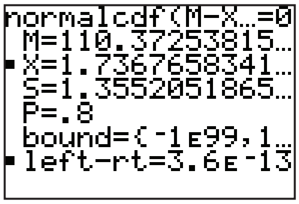
```
normalcdf(100…=0
 M=110
■S=1.3552051865…
 P=.07
 bound={-1E99,1…
■left-rt=-1E-14
```

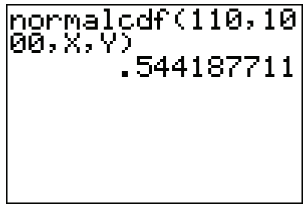
ii)

```
normalcdf(100…=0
■M=110.37253815…
 S=1.3552051865…
 P=.04
 bound={-1E99,1…
■left-rt=1.2E-13
```

b) Since A and B are symmetrical with respect to the mean, we can use one variable, x, to find both values.


EQUATION SOLVER
eqn:0=normalcdf(
M-X,M+X,M,S)-P


normalcdf(M-X…=0
M=110.37253815…
X=1.7367658341…
S=1.3552051865…
P=.8
bound={-1E99,1…
left-rt=3.6E-13


normalcdf(110,10
00,X,Y)
.544187711